REPRESENTATION THEORY
EXERCISES 7

HENNING KRAUSE
JAN GEUENICH

1. Let k be a field and let Q and Q' be two finite acyclic quivers whose underlying graphs coincide. Prove that there exists an equivalence of triangulated categories:

$$\mathbf{D}^b(\operatorname{mod} kQ) \;\simeq\; \mathbf{D}^b(\operatorname{mod} kQ')$$

Hint: Let Λ be an artin algebra and let $e_1, \ldots, e_n$ be a complete set of representatives of local idempotents of Λ up to isomorphism. If $S = e_i\Lambda$ is a simple projective non-injective Λ-module, then $T_S = \tau^{-1}S \oplus \bigoplus_{j\neq i} e_j\Lambda$ is a tilting object in $\operatorname{Thick}(\Lambda_\Lambda)$.

Remark: The algebra $\operatorname{End}_\Lambda(T_S)$ is known as the ***APR (Auslander–Platzeck–Reiten) tilt*** of Λ at S.

2. Verify the following facts about abelian categories $\mathcal{A}$:

(a) $\mathcal{A}$ is hereditary iff for every morphism f in $\mathcal{A}$ there is an exact sequence in $\mathcal{A}$ of the form

$$0 \longrightarrow X \xrightarrow{\binom{\alpha}{\beta}} \operatorname{Im} f \oplus Z \xrightarrow{(\gamma\ \delta)} Y \longrightarrow 0$$

such that $\gamma\alpha = f$ where α is an epimorphism and γ is a monomorphism.

Assume from now on that $\mathcal{A}$ is hereditary and linear over some commutative ring k such that the k-modules $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ and $\operatorname{Ext}_{\mathcal{A}}(X, Y)$ have finite length for all objects $X, Y \in \mathcal{A}$.

(b) If $X, Y \in \mathcal{A}$ are indecomposable with $\operatorname{Ext}^1_{\mathcal{A}}(Y, X) = 0$, then every nonzero map $X \to Y$ in $\mathcal{A}$ is either a monomorphism or an epimorphism.

(c) For every tilting object T in $\mathcal{A}$ the *tilted algebra* $\operatorname{End}_{\mathcal{A}}(T)$ has global dimension at most 2.

3. Let k be a field. Prove that the canonical embedding of $k[x]$ into its field of fractions $k(x)$ induces an equivalence of categories

$$\frac{\operatorname{mod} k[x]}{\operatorname{mod}_0 k[x]} \xrightarrow{\sim} \operatorname{mod} k(x)$$

where $\operatorname{mod}_0 k[x]$ denotes the Serre subcategory of $\operatorname{mod} k[x]$ consisting of all torsion modules.

How can this be regarded as an affine analog of Serre's theorem about coherent sheaves on $\mathbb{P}^1_k$?

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