Universität Bielefeld WS 2019/20

REPRESENTATION THEORY
EXERCISES 2

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1. Show for objects X in abelian categories:

(a) Every ascending chain $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ of subobjects with semisimple quotients X_{i+1}/X_i satisfies $X_i \subseteq \operatorname{soc}^i(X)$ for all $i \in \mathbb{N}$.

(b) If X is artinian and $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq 0$ is a descending chain of subobjects with semisimple quotients X_i/X_{i+1}, then $\operatorname{rad}^i(X) \subseteq X_i$ for all $i \in \mathbb{N}$.

(c) If X is artinian and there is a smallest $r \in \mathbb{N}$ with $\operatorname{rad}^r(X) = 0$ as well as a smallest $s \in \mathbb{N}$ with $\operatorname{soc}^s(X) = X$, then $r = s$. This integer is then referred to as the *Loewy length* of X.

Hint for (b): For artinian objects X the radical $\operatorname{rad}(X)$ is characterized as the smallest subobject Y of X such that the quotient X/Y is semisimple.

2. Let Γ and Λ be unitary rings and $\phi\colon \Gamma \to \Lambda$ a not necessarily unitary homomorphism of rings. Write e for the idempotent $\phi(1)$ in Λ. Consider the *restriction of scalars* along ϕ, i.e. the functor

$$\operatorname{Mod}\Lambda \xrightarrow{\phi_*} \operatorname{Mod}\Gamma$$

that associates with each Λ-module M the Γ-module Me with right Γ-operation $(m, x) \mapsto mf(x)$ and that is given on morphisms by restriction. Prove the following:

(a) ϕ_* is isomorphic to $- \otimes_\Lambda \Lambda e$ with right adjoint $\phi^! = \operatorname{Hom}_\Gamma(\Lambda e, -)$.

(b) ϕ_* is isomorphic to $\operatorname{Hom}_\Lambda(e\Lambda, -)$ with left adjoint $\phi^* = - \otimes_\Gamma e\Lambda$.

(c) ϕ_* is exact, $\phi^!$ left exact and ϕ^* right exact.

Assume from now on that ϕ is the canonical inclusion $e\Lambda e \to \Lambda$ with an idempotent e in Λ.

(d) $\phi_* \circ \phi^!$ and $\phi_* \circ \phi^*$ are both isomorphic to the identity.

(e) ϕ_* surjectively maps submodules of $e\Lambda$ to right ideals of $e\Lambda e$.

(f) ϕ_* maps simple Λ-modules that are not annihilated by the element e to simple $e\Lambda e$-modules. On the level of isomorphism classes, this assignment is bijective.

(g) $\phi^!$ maps injective Γ-modules to injective Λ-modules and ϕ^* maps projective Γ-modules to projective Λ-modules. On the level of isomorphism classes, this assignment is injective.

(h) Explicitly describe the assignments in (f) and (g) for $\Lambda = \mathbb{C}Q/(ab)$ and $e = e_1 + e_3$ and

$$Q \;=\; 1 \xrightarrow{\;a\;} 2 \xrightarrow{\;b\;} 3 \;.$$

To be handed in by October 31, 2019, 2 p.m. into post box 30.

3. Let $\mathcal{X}$ be a class of objects in an abelian category $\mathcal{A}$. Show that $\mathrm{Filt}(\mathcal{X})$ is the smallest full subcategory of $\mathcal{A}$ that contains $\mathcal{X}$ and is closed under extensions.

4 (*). Find examples M of finite-length modules whose endomorphism ring is ...

(a) ... not left artinian.

(b) ... not right artinian.

(c) ... neither left nor right artinian.

Hint for (b): Consider the field F of rational functions in infinitely many indeterminates X_i over a field K and its K-endomorphism σ given by $\sigma(X_i) = X_i^2$. Let G be the vector space over $\sigma(F)$ spanned by the monomials $X_{i_1} \cdots X_{i_\ell}$ with $i_1 < \cdots < i_\ell$ and $\ell \in \mathbb{N}_+$. Take $M = R/I$ where

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in F,\ b \in {}_F F_\sigma \right\}, \qquad I = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}.$$

Source: Gupta and Singh: *Ring of endomorphisms of a finite length module*, Proc. AMS, 1985.