REPRESENTATION THEORY
EXERCISES 3

HENNING KRAUSE
JAN GEUENICH

For (small) abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, an exact functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and a Serre subcategory $\mathcal{C}$ of $\mathcal{B}$ we will denote by $\operatorname{Ex}_{\mathcal{C}}^{F}(\mathcal{B}, \mathcal{D})$ the category whose objects are the exact functors $\mathcal{B} \to \mathcal{D}$ that annihilate $\mathcal{C}$ and whose morphisms $G \to G'$ are the natural transformations $GF \Rightarrow G'F$.

1. Fix an exact functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ of abelian categories that annihilates a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$.

(a) Show that F is a quotient functor of $\mathcal{A}$ by $\mathcal{C}$ iff the map $\operatorname{Ex}_0^F(\mathcal{B}, \mathcal{D}) \xrightarrow{F^{\mathcal{D}}} \operatorname{Ex}_{\mathcal{C}}^{\mathrm{id}}(\mathcal{A}, \mathcal{D})$ given by precomposition with F on objects and as the identity on morphisms is an isomorphism of categories for all $\mathcal{D}$.

We call F a *weak quotient functor* of $\mathcal{A}$ by $\mathcal{C}$ if $F^{\mathcal{D}}$ is an equivalence of categories for all $\mathcal{D}$.

(b) Explain why the existence of a quotient functor of $\mathcal{A}$ by $\mathcal{C}$ is equivalent to the existence of a weak quotient functor of $\mathcal{A}$ by $\mathcal{C}$ and how to obtain one from the other.

(c) Let P be a projective module in the length category $\mathcal{A} = \operatorname{mod} \Lambda$ and let $\mathcal{C}$ be the kernel of the functor $F = \operatorname{Hom}_{\mathcal{A}}(P, -) \colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{B} = \operatorname{mod} \operatorname{End}_{\mathcal{A}}(P)$. In the lectures, F was shown to be a weak quotient functor of $\mathcal{A}$ by $\mathcal{C}$. In which situations is F a quotient functor?

Given a class S of morphisms in an abelian category $\mathcal{A}$, an exact functor $\mathcal{A} \xrightarrow{F} \mathcal{A}[S^{-1}]$ of abelian categories is said to be the *localization* of $\mathcal{A}$ at S if it satisfies the following universal property:

(1) F maps morphisms in S to isomorphisms.

(2) If $\mathcal{A} \xrightarrow{H} \mathcal{B}$ is another exact functor between abelian categories mapping morphisms in S to isomorphisms, then there is a unique exact functor $\mathcal{A}[S^{-1}] \xrightarrow{G} \mathcal{B}$ with $GF = H$.

Generalizing from rings to abelian categories and leaving set-theoretic issues aside, it is not hard to construct the localization granted that S is a *multiplicative system* in the following sense:

(i) *Multiplicativity:* S contains all identities in $\mathcal{A}$ and is closed under composition.

(ii^L) *Left Ore condition:* For all morphisms s in S and r in $\mathcal{A}$ with common source there exists morphisms s' in S and r' in $\mathcal{A}$ with common target such that $s'r = r's$.

(ii^R) *Right Ore condition:* For all morphisms s in S and r in $\mathcal{A}$ with common target there exists morphisms s' in S and r' in $\mathcal{A}$ with common source such that $rs' = sr'$.

(iii^L) *Left reversibility:* If $rs = 0$ for morphisms s in S and r in $\mathcal{A}$, there is s' in S with $s'r = 0$.

(iii^R) *Right reversibility:* If $sr = 0$ for morphisms s in S and r in $\mathcal{A}$, there is s' in S with $rs' = 0$.

2. Let $\mathcal{A}$ be an abelian category. Prove for every Serre subcategory $\mathcal{C}$ of $\mathcal{A}$:

(a) The morphisms s in $\mathcal{A}$ with $\operatorname{Ker}(s)$ and $\operatorname{Coker}(s)$ in $\mathcal{C}$ form a multiplicative system $S_{\mathcal{C}}$.

(b) The localization functor $\mathcal{A} \to \mathcal{A}\big[S_{\mathcal{C}}^{-1}\big]$ is the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$.

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3. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be an exact functor of abelian categories with a fully faithful left or right adjoint. Prove that F is a weak quotient functor of $\mathcal{A}$ by the Serre subcategory $\operatorname{Ker} F$.

Hint: In an adjoint pair (L, R) the functor L (resp. R) is fully faithful iff the unit $\mathrm{id} \to RL$ (resp. the counit $LR \to \mathrm{id}$) is an isomorphism.

4. (a) Prove $\operatorname{mod} \mathbb{Z} \,/ \operatorname{tors} \mathbb{Z} \simeq \operatorname{mod} \mathbb{Q}$ where $\operatorname{tors} \mathbb{Z}$ is the full subcategory of torsion groups.

(b) Decide for each pair $\mathcal{C} \subseteq \mathcal{D}$ of Serre subcategories of $\mathcal{A} = \operatorname{mod} \Lambda$ if the canonical inclusion $\mathcal{C} \to \mathcal{D}$ is a homological embedding, where $\Lambda = \mathbb{C}Q/(ab)$ is the algebra defined by

$$Q \;=\; 1 \xrightarrow{\;a\;} 2 \xrightarrow{\;b\;} 3 \;.$$