REPRESENTATION THEORY
EXERCISES 4

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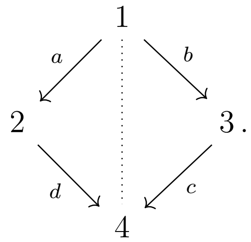
A semiperfect ring Λ is said to be *triangular* if there is a complete set $e_1, \dots, e_n$ of representatives of the conjugacy classes of local idempotents in Λ such that $e_i J(\Lambda) e_j = 0$ for all $i \geq j$.

1. Let Λ be a right artinian ring.

(a) Show that, whenever Λ is triangular, then the category $\operatorname{mod} \Lambda$ can be regarded as a highest weight category with a canonical choice of standard objects.

(b) Find an example where $\operatorname{mod} \Lambda$ is a highest weight category but Λ is not triangular.

2. Let $\mathcal{A} = \operatorname{mod} \Lambda$ where $\Lambda = \mathbb{C}Q/(ad - bc)$ is the algebra defined by



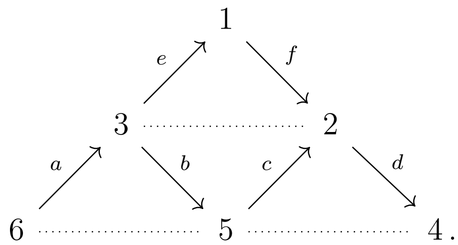
Show that $\mathcal{A}$ can be regarded as a highest weight category with standard objects ...

(a) ... $\Delta_1 = P_4$, $\Delta_2 = P_3$, $\Delta_3 = P_2$ and $\Delta_4 = P_1$.

(b) ... $\Delta_1 = S_1$, $\Delta_2 = S_2$, $\Delta_3 = S_4$ and $\Delta_4 = P_3$.

Describe in either case a $\mathbb{C}$-algebra Λ' such that $\operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(\Delta_4, -) \simeq \operatorname{mod} \Lambda'$.

3. Let $\mathcal{A} = \operatorname{mod} \Lambda$ where $\Lambda = \mathbb{C}Q/(ab, cd, bc - ef)$ is the algebra defined by



(a) Verify that the choice $\Delta_1 = S_1$, $\Delta_2 = S_2$, $\Delta_3 = P_3/b\Lambda$, $\Delta_4 = P_4$, $\Delta_5 = P_5$ and $\Delta_6 = P_6$ of standard objects turns $\mathcal{A}$ into a highest weight category.

(b) For which permutations σ is $\mathcal{A}$ a highest weight category with standard objects $\Delta_i = S_{\sigma(i)}$?

(c) For which permutations σ is $\mathcal{A}$ a highest weight category with standard objects $\Delta_i = P_{\sigma(i)}$?

To be handed in by November 14, 2019, 2 p.m. into post box 30.

4. Prove that the full subcategory $\mathcal{F} = \mathrm{Filt}(\Delta_1, \ldots, \Delta_n)$ of a highest weight category with standard objects $\Delta_1, \ldots, \Delta_n$ is closed under direct summands.

Hint: Every $X \in \mathcal{F}$ fits into a short exact sequence $0 \to \tilde{X} \to X \to \bar{X} \to 0$ with $\tilde{X} \in \mathrm{Filt}(\Delta_n)$ and $\bar{X} \in \mathrm{Filt}(\Delta_1, \ldots, \Delta_{n-1})$ where the assignments $X \mapsto \tilde{X}$ and $X \mapsto \bar{X}$ are functorial.