REPRESENTATION THEORY
EXERCISES 6

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1. Let $\mathcal{A}$ be a highest weight category with standard modules $(\Delta_i)_{i\in I}$ ordered by a finite poset I. Recall that this means that $\mathcal{A}$ is a length category admitting exact sequences

$$0 \to U_i \to P_i \to \Delta_i \to 0$$

such that the object $\bigoplus_{i\in I} P_i$ is a projective generator of $\mathcal{A}$, the rings $\operatorname{End}_{\mathcal{A}}(\Delta_i)$ are division rings, each U_i belongs to the subcategory $\operatorname{Filt}(\{\Delta_j : j > i\})$ and $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for all $i \not\leq j$.

Verify the following claims, where h denotes the height of the poset I:

(a) The projective dimension of each object in $\operatorname{Filt}(\{\Delta_i : i \in I\})$ is at most $h - 1$.

(b) The objects $S_i = P_i/\operatorname{rad} P_i \cong \Delta_i/\operatorname{rad}\Delta_i$ form a complete set of representatives of simple objects in $\mathcal{A}$ and each of the objects $\operatorname{rad}\Delta_i$ belongs to the subcategory $\operatorname{Filt}(\{S_j : j < i\})$.

(c) The global dimension of $\mathcal{A}$ is at most $2(h - 1)$.

2. Let Λ be a semiprimary ring. Define $\mathfrak{H}(\Lambda)$ to be the simply-laced quiver whose vertices are the idempotent ideals of Λ with an arrow $I \to J$ iff $I \subseteq J$ and J/I is an heredity ideal in Λ/I.

Convince yourself of the following facts:

(a) The vertices of $\mathfrak{H}(\Lambda)$ correspond to the Serre subcategories of $\operatorname{mod}\Lambda$. There are $2^{n(\Lambda)}$.

(b) The paths from 0 to Λ in $\mathfrak{H}(\Lambda)$ correspond to the heredity chains of Λ. Therefore the global dimension of Λ is at most $2(\ell - 1)$ where ℓ is the minimal length of such a path.

(c) There is an arrow $I \to J$ between two vertices I, J in $\mathfrak{H}(\Lambda)$ iff $I \subseteq J$ and the embedding $\operatorname{mod}\Lambda/J \to \operatorname{mod}\Lambda/I$ of Serre subcategories is homological with semisimple quotient.

Choose any of the algebras $\Lambda = \mathbb{C}Q/R$ below for which you compute the quiver $\mathfrak{H}(\Lambda)$ and then compare the value of the global dimension of Λ with the bounds implied by 1. (b) and 2. (b):

(i) $Q = {}_{a}\!\circlearrowleft 1 \xrightarrow{\;b\;} 2$

$R = (a^2)$

(ii) $Q = 1 \underset{a}{\overset{b}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3$

$R = (ab - cd, dc)$

(iii) $Q = 1 \xrightarrow{\;a\;} 2 \xrightarrow{\;b\;} 3$

$R = 0$ or $R = (ab)$

(iv) $Q = \begin{array}{ccc} 1 & \underset{a}{\overset{b}{\rightleftarrows}} & 2 \\ {\scriptstyle d}\nwarrow & & \swarrow{\scriptstyle c} \\ & 3 & \end{array}$

$R = (ab, db)$

To be handed in by November 28, 2019, 2 p.m. into post box 30.

3. Prove that every semiprimary ring Λ of global dimension at most two is quasi-hereditary.

Hint: Consider a local idempotent e in Λ minimizing the Loewy length of the module $e\Lambda$.

4. Consider the regular module $X = \Lambda_\Lambda$ over the algebra $\Lambda = \mathbb{C}Q/(ba, ab - cd, dc)$ defined by

$$Q = 1 \underset{a}{\overset{b}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3\,.$$

Determine its *Iyama complement* $X' = \bigoplus_{t>0} \mathfrak{r}^t X$ and compute $\operatorname{gldim} \operatorname{End}_\Lambda(X \oplus X')$.