REPRESENTATION THEORY
EXERCISES 8

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1. Let Λ be a ring. We will denote by $\operatorname{Mod}\Lambda$ the category of Λ-modules and by $\operatorname{fg}\Lambda$ and $\operatorname{mod}\Lambda$, respectively, its full subcategories consisting of finitely generated and finitely presented modules.

(a) Prove the following:

   (i) The category $\operatorname{fg}\Lambda$ is abelian iff $\operatorname{fg}\Lambda = \operatorname{mod}\Lambda$ iff Λ is right noetherian.

   (ii) The category $\operatorname{mod}\Lambda$ is abelian iff Λ is right coherent.

   (iii) If Λ is right coherent, then the embedding $\operatorname{mod}\Lambda \hookrightarrow \operatorname{Mod}\Lambda$ is homological.

(b) Show similarly that the canonical embedding $\operatorname{Filt}(\Delta) \hookrightarrow \mathcal{A}$ is homological for any highest weight category $\mathcal{A}$ with standard objects Δ_i.

2. For the highest weight categories $\mathcal{A} = \operatorname{mod}\Lambda$ in Exercises 4.2 (b) and 4.3 (a) do the following:

(a) Compute the Auslander-Reiten quiver of $\mathcal{A}$.

(b) Determine all indecomposable objects in $\mathcal{A}$ belonging to $\operatorname{Filt}(\Delta)$.

(c) Determine all indecomposable objects in $\mathcal{A}$ belonging to $\operatorname{Filt}(\nabla)$.

(d) Find the indecomposable objects T_i in $\mathcal{A}$ such that $T = \bigoplus_i T_i$ is the characteristic tilting object and determine the exact sequences

$$0 \longrightarrow V_i \longrightarrow T_i \longrightarrow \nabla_i \longrightarrow 0$$

$$0 \longrightarrow \Delta_i \longrightarrow T_i \longrightarrow W_i \longrightarrow 0$$

satisfying $V_i \in \operatorname{Filt}(\{\nabla_j : j < i\})$ and $W_i \in \operatorname{Filt}(\{\Delta_j : j < i\})$.

(e) Describe the Ringel dual $\operatorname{End}_{\mathcal{A}}(T)$ as a path algebra $\mathbb{C}Q$ modulo an admissible ideal I.

3. Let k be a field of characteristic p and let $V = k^n$ for some positive integer n.

Verify the following claims in the case that p is not a prime less than or equal to d:

(i) The canonical projection $(V^{\otimes d})^{\mathfrak{S}_d} \to S^d V$ is invertible.

(ii) The Schur algebra $S_k(n,d)$ is semisimple.

(iii) Explicitly, for $n = d = 2$, we have $S_k(2,2) \cong k \times M_3(k)$.

4. Describe the basic version of the group algebra $k\mathfrak{S}_p$ and the Schur algebra $S_{\mathbb{F}_p}(p,p)$ each as a path algebra modulo an admissible ideal, both for $p = 2$ and for $p = 3$.

To be handed in by December 12, 2019, 2 p.m. into post box 30.