REPRESENTATION THEORY
EXERCISES 9

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Let k be a commutative ring and let $n, d \in \mathbb{N}$. We begin with some notations:

- For compositions λ in $\Lambda = \Lambda(n, d)$ denote by λ_* the corresponding partition.
- The symmetric group $\mathfrak{S}_d$ acts from the right on the set $I = \underline{n}^{\underline{d}}$ where $\underline{m} = \{1, \ldots, m\}$.
- Let $\Omega = (I \times I)/\mathfrak{S}_d$ be the set of orbits under the induced diagonal action.
- For $\omega \in \Omega$ let $s(\omega), t(\omega)$ be the elements in Λ with $j^* = s(\omega)$ and $i^* = t(\omega)$ for $(i, j) \in \omega$.
- For $\lambda \in \Lambda$ let $\omega_\lambda \in \Omega$ be the orbit $\{(i, i) \in I \times I : i^* = \lambda\}$.
- Let $e \in V^{\underline{n}}$ be the standard basis of the free k-module $V = k^{\underline{n}}$.
- Let $(e_i)_{i \in I}$ with $e_i := e_{i_1} \otimes \cdots \otimes e_{i_d}$ be the induced k-basis of $V^{\otimes d}$.
- Let $(e_{ij})_{(i,j) \in I \times I}$ be the induced k-basis of $\operatorname{End}_k(V^{\otimes d})$ determined by $e_{ij}(e_\ell) = \delta_{\ell j} e_i$.
- For $\lambda \in \Lambda$ let V^λ be the $k\mathfrak{S}_d$-submodule of $V^{\otimes d}$ with k-basis $(e_i)_{i \in I,\, i^* = \lambda}$.
- For $\omega \in \Omega$ set $e_\omega := \sum_{(i,j) \in \omega} e_{ij}$ and for $\lambda \in \Lambda$ set $e_\lambda := e_{\omega_\lambda}$.

1. Verify the following facts:

(a) $(e_\omega)_{\omega \in \Omega}$ is a k-basis of the Schur algebra $S_k(n, d)$.

(b) $(e_\lambda)_{\lambda \in \Lambda}$ is a complete set of orthogonal idempotents of $S_k(n, d)$.

(c) $V^{\otimes d} = \bigoplus_{\lambda \in \Lambda} V^\lambda$ and $V^\lambda \cong k \otimes_{k\mathfrak{S}_\lambda} k\mathfrak{S}_d$ as $k\mathfrak{S}_d$-modules.

(d) $e_\omega(V^{s(\omega)}) \subseteq V^{t(\omega)}$ and $e_\omega(V^\lambda) = 0$ for $\lambda \neq s(\omega)$.

(e) $V^{s(\omega)} \xrightarrow{e_\omega} V^{t(\omega)}$ is invertible iff $s(\omega)_* = t(\omega)_*$.

(f) $V^\lambda \cong V^\mu$ iff $\lambda_* = \mu_*$.

The *power sums* p_i and *elementary symmetric polynomials* $s_i \in k[x_1, \ldots, x_d]$ are defined by

$$p_i = \sum_{j=1}^{d} x_j^i \qquad \text{and} \qquad \prod_{j=1}^{d} (T + x_j) = \sum_{i=0}^{d} s_i T^{d-i}.$$

2. Let $\mathfrak{S}_d$ act on the polynomial ring $k[x_1, \ldots, x_d]$ by k-algebra automorphisms via $\sigma x_i = x_{\sigma(i)}$.

(a) Prove the *fundamental theorem of symmetric polynomials*, i.e. verify that the morphism

$$k[y_1, \ldots, y_d] \longrightarrow k[x_1, \ldots, x_d]^{\mathfrak{S}_d}$$

of k-algebras induced by $y_i \mapsto s_i$ is an isomorphism.

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(b) Verify *Newton's identity*

$$d \cdot s_d \;=\; \sum_{j=1}^{d} (-1)^{j-1} s_{d-j} p_j \,.$$

Hint for (a): Use induction on the lexicographic order on the set of monomials.

From now on let k be a field of characteristic p such that p is not a prime less than or equal to d.

3. Show that the symmetric power $S^d V$ is generated by elements of the form $v^{\otimes d}$ with $v \in V$.

Hint: Use the identity

$$d! \cdot s_d \;=\; \sum_{J \subseteq \underline{d}} (-1)^{d-|J|} \left(\sum_{j \in J} x_j \right)^d \,.$$

4. Verify the following version of *Schur-Weyl duality*: The image of the k-linear map

$$\begin{array}{ccc} \mathrm{End}_k(V) & \longrightarrow & \mathrm{End}_k(V^{\otimes d}) \\ f & \longmapsto & \partial f^{\otimes d} \end{array}$$

which is defined on pure tensors as

$$\partial f^{\otimes d}(v_1 \otimes \cdots \otimes v_d) \;=\; \sum_{i=1}^{d} v_1 \otimes \cdots \otimes v_{i-1} \otimes f(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_d$$

generates the Schur algebra $S_k(n,d) = \mathrm{End}_{k\mathfrak{S}_d}(V^{\otimes d})$ as a k-algebra.

If k is infinite, deduce the surjectivity of the canonical k-algebra homomorphism

$$k\,\mathrm{GL}(n,k) \xrightarrow{\;\phi\;} S_k(n,d) \,.$$

Hint: $\Gamma^d \, \mathrm{End}_k(V)$ is spanned over k by elements of the form $f^{\otimes d}$ with $f \in \mathrm{End}_k(V)$.