Springer theory and the geometry of quiver flag varieties

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Abstract

The thesis consists of the following chapters:

1. *Springer theory.*

   For any projective map $E \to V$, Chriss and Ginzburg defined an algebra structure on the (Borel-Moore) homology $\mathcal{Z} := H_*(E \times_V E)$, which we call Steinberg algebra. (Graded) Projective and simple $\mathcal{Z}$-modules are controlled by the BBD-decomposition associated to $E \to V$. We restrict to collapsings of unions of homogeneous vector bundles over homogeneous spaces because we have the cellular fibration technique and for equivariant Borel-Moore homology we can use localization to torus-fixed points. Examples of Steinberg algebras include group rings of Weyl groups, Khovanov-Lauda-Rouquier algebras, nil Hecke algebras.

2. *Steinberg algebras.*

   We choose a class of Steinberg algebras and give generators and relations for them. This fails if the homogeneous spaces are partial and not complete flag varieties, we call this the parabolic case.

3. *The parabolic case.*

   In the parabolic cases, we realize the Steinberg algebra $\mathcal{Z}^P$ as corner algebra in a Steinberg algebra $\mathcal{Z}^B$ associated to Borel groups (this means $\mathcal{Z}^P = e\mathcal{Z}^B e$ for an idempotent element $e \in \mathcal{Z}^B$).

4. *Monoidal categories.*

   We explain how to construct monoidal categories from families of collapsings of homogeneous bundles.

5. *Construct collapsings.*

   We construct collapsing maps over given loci which are resolutions of singularities or generic Galois coverings. For closures of homogeneous decomposition classes of the Kronecker quiver these maps are new.

6. *Quiver flag varieties.*

   Quiver flag varieties are the fibres of certain collapsings of homogeneous bundles. We investigate when quiver flag varieties have only finitely many orbits and we describe the category of flags of quiver representations as a Δ-filtered subcategory for the quasi-hereditary algebra $KQ \otimes KA_n$.

7. *A_n-equioriented.*

   For the A_n-equioriented quiver we find a cell decompositions of the quiver flag varieties, which are parametrized by certain multi-tableaux.

Contents

Chapter 1

A survey on Springer theory

Summary. A **Springer map** is for us a union of collapsings of (complex) homogeneous vector bundles and a **Steinberg variety** is the cartesian product of a Springer map with itself. Chriss and Ginzberg constructed on the (equivariant) Borel-Moore homology and on the (equivariant) K-theory of a Steinberg variety a convolution product making it an associative algebra, we call this a **Steinberg algebra** (cp. [CG97],2.7, 5.2 for the nonequivariant case). The decomposition theorem for perverse sheaves gives the indecomposable, projective graded modules over the Steinberg algebra. Also this convolution yields a module structure on the respective homology groups of the fibres under the Springer maps, which we call **Springer fibre modules**. In short, *for us* a **Springer theory** is the study of a Steinberg algebra together with its graded modules.
We give two examples: Classical Springer theory and quiver-graded Springer theory.

(1) Definitions and basic properties.

(2) Examples

  (a) Classical Springer theory.

  (b) Quiver-graded Springer theory.

(3) We discuss literature on the two examples.

1.1 Definition of a Springer theory

Roughly, following the introduction of Chriss and Ginzburg's book ([CG97])[1], Springer theory is a uniform geometric construction for a wide class of (non-commutative) algebras together with families of modules over these algebras. Examples include

(1) Group algebras of Weyl groups together with their irreducible representations,

(2) affine Hecke algebras together with their standard modules and irreducible representations,

[1] We take a more general approach, what usually is considered as Springer theory you find in the example *classical Springer theory*. Nevertheless, our approach is still only a special case of [CG97], chapter 8.

(3) Hecke algebras with unequal parameters,

(4) Khovanov-Lauda-Rouquier-algebras (or shortly KLR-algebras) and alternatively called quiver Hecke algebras

(5) Quiver Schur algebras

For an algebraic group G and closed subgroup P (over $\mathbb{C}$) we call $G \to G/P$ a **principal homogeneous bundle**. For a given P-variety F we have the **associated bundle** defined by the quotient

$$G \times^P F := G \times F/\sim, \quad (g,f) \sim (g',f')\colon \iff \text{ there is } p \in P\colon (g,f) = (g'p, p^{-1}f')$$

and $G \times^P F \to G/P, (g,f) \mapsto gP$. Given a representation $\rho\colon P \to \mathbf{Gl}(F)$, i.e. a morphism of algebraic groups, we call associated bundles of the form $G \times^P F \to G/P$ **homogeneous vector bundles** (over a homogeneous space).

Definition 1. The uniform geometric construction in all cases is given by the following: Given $(G, P_i, V, F_i)_{i \in I}$ with I some finite set,

(∗) G a connetcted reductive group with parabolic subgroups P_i.
We also assume there exists a maximal torus $T \subset G$ which is contained in every P_i.
(∗) V a finite dimensional G-representation, $F_i \subset V$ a P_i-subrepresentation of V, $i \in I$.

We identify V, F_i with the affine spaces having the vector spaces as $\mathbb{C}$-valued points. Let $E_i := G \times^{P_i} F_i, i \in I$ and consider the following morphisms of algebraic varieties[2]:

$$V \xleftarrow{\pi} E := \bigsqcup_{i \in I} E_i \xrightarrow{\mu} \bigsqcup_{i \in I} G/P_i \qquad gf_i \mapsfrom [(g,f_i)] \mapsto gP_i$$

Then, $E \to V \times \bigsqcup_{i \in I} G/P_i, [(g,f_i)] \mapsto (gf_i, gP_i)$ is a closed embedding (see [Slo80b], p.25,26), it follows that π is projective. We call the algebraic correspondence[3] (E, π, μ) **Springer triple**, the map π **Springer map**, its fibres **Springer fibres**. Via restriction of $E \to V \times \bigsqcup_{i \in I} G/P_i$ to $\pi^{-1}(s) \to \{x\} \times \bigsqcup_{i \in I} G/P_i$ one sees that all Springer fibres are via μ closed subschemes of $\bigsqcup_{i \in I} G/P_i$.

[2] algebraic variety = separated integral scheme of finite type over a field

[3] two scheme morpisms $X \xleftarrow{p} Z \xrightarrow{q} Y$ are called algebraic correspondence, if p is proper and q is flat

We also have another induced roof-diagram

$$\begin{array}{ccc} & Z := E\times_V E & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle m} \\ V & & (\bigsqcup_{i\in I} G/P_i)\times(\bigsqcup_{i\in I} G/P_i) \end{array}$$

with $p\colon E\times_V E \xrightarrow{pr_E} E \xrightarrow{\pi} V$ projective and $m\colon E\times_V E \xrightarrow{(pr_E,pr_E)} E\times E \xrightarrow{\mu\times\mu} (\bigsqcup_{i\in I} G/P_i)\times (\bigsqcup_{i\in I} G/P_i)$. Observe, by definition

$$Z = \bigsqcup_{i,j\in I} Z_{i,j}, \quad Z_{i,j} = E_i \times_V E_j.$$

We call the roof-diagram (Z, p, m) **Steinberg triple**, the scheme Z **Steinberg variety** (even though as a scheme Z might be neither reduced nor irreducible). But in view of our (co-)homology choice below we only study the underlying reduced scheme and look at its $\mathbb{C}$-valued points endowed with the analytic topology.

If all parabolic groups P_i are Borel groups, the Steinberg variety Z is a cellular fibration over $\bigsqcup_{i\in I} G/P_i$ via the map

$$Z \xrightarrow{m} \bigsqcup_{i\in I} G/P_i \times \bigsqcup_{i\in I} G/P_i \xrightarrow{\mathrm{pr}_1} \bigsqcup_{i\in I} G/P_i$$

(see definition of cellular fibration in [CG97], 5.5 or subsection 8.3.5 in the Appendix.) We choose a (co-)homology theory which can be calculated for spaces with cellular fibration property and which has a localization to the T-fixed points theory. Let $H_*^A,\ A \in \{pt, T, G\}$ be **(A-equivariant) Borel-Moore homology**. We could also choose (equivariant) K-theory, but we just give some known results about it.
There is a natural product $*$ on $H_*^A(Z)$ called convolution product constructed by Chriss and Ginzburg in [CG97].

$$\begin{aligned} *\colon H_*^A(Z)\times H_*^A(Z) &\to H_*^A(Z) \\ (c,c') &\mapsto c * c' := (q_{1,3})_*(p_{1,2}^*(c)\cap p_{2,3}^*(c')) \end{aligned}$$

where $p_{a,b}\colon E\times E\times E \to E\times E$ is the projection on the a,b-th factors, $q_{a,b}$ is the restriction of $p_{a,b}$ to $E\times_V E\times_V E$, then $p_{a,b}^*(c)\in H_*^A(p_{a,b}^{-1}(E\times_V E))$ and $\cap\colon H_p^A(X)\times H_q^A(Y)\to H_{p+q-2d}^A(X\cap Y)$ is the intersection pairing which is induced by the $\cup$-product in relative singular cohomology for $X, Y \subset M$ two A-equivariant closed subsets of a d-dimensional complex manifold M (cp. [CG97], p.98, (2.6.16)).

It holds

$$H_p^A(Z_{i,j}) * H_q^A(Z_{k,\ell}) \subset \delta_{j,k} H_{p+q-2e_k}^A(Z_{i,\ell}), \quad e_k = \dim_{\mathbb{C}} E_k.$$

We call $(H_*^A(Z), *)$ the (A-equivariant) **Steinberg algebra** for $(G, P_i, V, F_i)_{i\in I}$. It is naturally an graded module over $H_*^A(pt)$, see Appendix section **??**, (6). We denote by

$D^b_A(V)$ the A-equivariant derived category of V defined by Bernstein and Lunts in [BL94].

There is a the following identification.

Theorem 1.1.1. *([CG97], chapter 8) Let $A \in \{pt, T, G\}$ we write $e_i = \dim_{\mathbb{C}} E_i$. There is an isomorphism of $\mathbb{C}$-algebras*

$$H^A_*(Z) \to \mathrm{Ext}^*_{D^b_A(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}[e_i], \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}[e_i]),$$

where $\underline{\mathbb{C}}$ is the constant sheaf associated to $\mathbb{C}$ on the appropriate spaces. If we set

$$H^A_{[p]}(Z) := \bigoplus_{i,j\in I} H^A_{e_i+e_j-p}(Z_{i,j})$$

then $H^A_{[*]}(Z)$ is a graded module over $H^*_A(pt) = H^A_{-*}(pt)$. It is even a graded algebra over $H^*_A(pt)$. The right hand side is naturally a graded algebra over $H^*_A(pt) = \mathrm{Ext}^*_{D^b_A(pt)}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ and the isomorphism is an isomorphism of graded $H^*_A(pt)$-algebras. Furthermore, the Verdier duality on $D^b_A(V)$ induces an anti-involution on the algebra on the right hand side.

On the left hand side the anti-involution is given by pullback along the swapping-the-two-factors map. The proof is only given for $A = pt$, but as Varagnolo and Vasserot in [Var09] observed, the same proof can be rewritten for the A-equivariant case.

1.2 Convolution modules

Compare [CG97], section 2.7. Given two subsets $S_{1,2} \subset M_1 \times M_2$, $S_{2,3} \subset M_2 \times M_3$ the set-theoretic convolution is defined as

$$S_{1,2} \circ S_{2,3} := \{(m_1, m_3) \mid \exists\, m_2 \in M_2 \colon (m_1, m_2) \in S_{1,2}, (m_2, m_3) \in S_{2,3}\} \subset M_1 \times M_3.$$

Now, let $S_{i,j} \subset M_i \times M_j$ be A-equivariant locally closed subsets of smooth complex A-varieties, let $p_{i,j} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ be projection on the (i, j)-th factors and assume $q_{1,3} := p_{1,3}|_{p^{-1}_{12}(S_{1,2})\cap p^{-1}_{2,3}(S_{2,3})}$ is proper. Then we get a map

$$\begin{aligned} *\colon H^A_p(S_{1,2}) \times H^A_q(S_{2,3}) &\to H^A_{p+q-2\dim_{\mathbb{C}} M_2}(S_{1,2} \circ S_{2,3}) \\ c_{1,2} * c_{2,3} &:= (q_{1,3})_*(p^*_{1,2}c_{1,2} \cap p^*_{2,3}c_{2,3}). \end{aligned}$$

This way we defined the algebra structure on the Steinberg algebra, but it also gives a left module stucture on $H^A_*(S)$ for any A-variety S with $Z \circ S = S$ and a right module structure when $S \circ Z = S$.

(a) We choose $M_1 = M_2 = M_3 = E$ and embed $Z = E \times_V E \subset E \times E, E = E \times pt \subset E \times E$, then it holds $Z \circ E = E$. If we regrade the Borel-Moore homology (and the

Poincare dual A-equivariant cohomology) of E as follows

$$H^A_{[p]}(E) := \bigoplus_{i\in I} H^A_{e_i-p}(E_i) \quad (= \bigoplus_{i\in I} H_A^{e_i+p}(E_i) =: H_A^{[p]}(E))$$

then $H^A_{[*]}(E)$ and $H^{[*]}(E)$ carry the structure of a graded left $H^A_{[*]}(Z)$-module.

(b) We choose $M_1 = M_2 = M_3 = E$ and embed $E \subset E \times E$ diagonally, then $E \circ E = E$, it holds $H^A_{(*)}(E) = H^*_A(E)$ as graded algebras where $H^A_{(p)}(E) := \bigoplus_i H^A_{2e_i-p}(E_i)$ and the ring structure on the cohomology is given by the cup product. If we take now $Z = E \times_V E \subset E \times E$ then $E \circ Z = Z$ and we get a structure as graded left $H^*_A(E)$-module on $H^A_{[*]}(Z)$.

(c) We choose $M_1 = M_2 = M_3 = E$, $A = pt$ and embed $Z = E \times_V E \subset E \times E, \pi^{-1}(s) = \pi^{-1}(s) \times pt \subset E \times E$, then it holds $Z \circ \pi^{-1}(s) = E$. If we regrade the Borel-Moore homology and singular cohomology of $\pi^{-1}(s)$ as follows

$$H_{[p]}(\pi^{-1}(s)) := \bigoplus_{i\in I} H_{e_i-p}(\pi_i^{-1}(s)), \quad H^{[p]}(\pi^{-1}(s)) := \bigoplus_{i\in I} H^{e_i+p}(\pi_i^{-1}(s))$$

then $H_{[*]}(\pi^{-1}(s))$ and $H^{[*]}(\pi^{-1}(s))$ are graded left $H_{[*]}(Z)$-module.
We call these the **Springer fibre modules**.

Similarly in all examples one can obtain a right module structure (the easy swaps are left to the reader). Independently, one can define the same graded module structure on $H_*(\pi^{-1}(s)), H^*(\pi^{-1}(s))$ using the description of the Steinberg algebra as Ext-algebra and a Yoneda operation (for this see [CG97], 8.6.13, p.448).
There is also a result of Joshua (see [Jos98]) saying that all hypercohomology groups $\mathbb{H}^*_A(Z, F^\bullet), F^\bullet \in D^b_A(Z)$ carry the structure of a left (and right) $H^A_*(Z)$-module.

1.3 The Steinberg algebra

1.3.1 The Steinberg algebra $H^A_{[*]}(Z)$ as module over $H^{-*}_A(pt)$.

We set $\widetilde{W} := \bigsqcup_{i,j\in I} W_{i,j}$ with $W_{i,j} := W_i \setminus W/W_j$ where W is the Weyl group for (G,T) and $W_i \subset W$ is the Weyl group for (L_i, T) with $L_i \subset P_i$ the Levi subgoup. We will fix representatives $w \in G$ for all elements $w \in \widetilde{W}$.
Let $C_w = G \cdot (eP_i, wP_j)$ be the G-orbit in $G/P_i \times G/P_i$ corresponding to $w \in W_{i,j}$.

Lemma 1. *(1) $p\colon C_w \subset G/P_i \times G/P_j \xrightarrow{pr_1} G/P_i$ is G-equivariant, locally trivial with fibre $p^{-1}(eP_i) = P_i w P_j/P_j$.*

(2) $P_i w P_j/P_j$ admits a cell decomposition into affine spaces via Schubert cells

$$xB_j x^{-1} vwP_j/P_j, \quad v \in W_i$$

where $B_j \subset P_j$ *is a Borel subgroup and* $x \in W$ *such that* ${}^xB_j \subset P_i$. *In particular,* $H_{odd}(P_iwP_j/P_j) = 0$ *and*

$$H_*(P_iwP_j/P_j) = \bigoplus_{v \in W_i} \mathbb{C}b_{i,j}(v), \quad b_{i,j}(v) := [\overline{xB_jx^{-1}vwP_j/P_j}].$$

It holds that $\deg b_{i,j}(v) = 2\ell_{i,j}(v)$ *where* $\ell_{i,j}(v)$ *is the length of a minimal coset representative in* W *for* $x^{-1}vwW_j \in W/W_j$.

(3) *For* $A \in \{pt, T, G\}$ *it holds* $H^A_{odd}(C_w) = 0$ *and since* G/P_i *is simply connected*

$$H^A_n(C_w) = \bigoplus_{p+q=n} H^p_A(G/P_i) \otimes H_q(P_iwP_j/P_i),$$
$$H^A_*(C_w) = \bigoplus_{u \in W/W_i, v \in W_i} \mathbb{C}b_i(u) \otimes b_{i,j}(v),$$

where $b_i(u) = [\overline{B_iuP_i/P_i}]^*$ *is of degree* $2\dim_{\mathbb{C}} G/P_i - 2\ell_i(u)$ *with* $\ell_i(u)$ *is the length of a minimal coset representative for* $u \in W/W_i$ *and* $b_{i,j}(v)$ *as in (2).*

Proof: See lemma 80 in the Appendix.

This implies using degeneration of Serre cohomology spectral sequences (see section in the Appendix) the following properties for the homology of Z.

Corollary 1.3.0.1. *(1)* Z *has a filtration by closed G-invariant subvarieties such that the successive complements are* $Z_w := m^{-1}(C_w), w \in \widetilde{W}$ *and the restriction of* m *to* Z_w *is a vector bundle over* C_w *of rank* d_w *(as complex vector bundle). Furthermore,*

$$H^A_n(Z) = \bigoplus_{w \in \widetilde{W}} H^A_n(Z_w) = \bigoplus_{w \in \widetilde{W}} H^A_{n-2d_w}(C_w)$$
$$= \bigoplus_{i,j \in I} \bigoplus_{w \in W_{i,j}} \bigoplus_{u,v} \mathbb{C}b_i(u) \otimes b_{i,j}(v)$$

where the index set of the last direct sum is

$$\{u \in W/W_i, v \in W_i \mid 2\dim G/P_i - 2\ell_i(u) + 2\ell_{i,j}(v) = n - 2d_w\}.$$

(2) *We have* $H_{odd}(Z) = 0, H^{odd}(Z) = 0$.

(3) Z *is equivariantly formal (for* T *and* G*, for Borel-Moore homology and cohomology). In particular, for* $A \in \{T, G\}$ *the forgetful maps* $H^A_*(Z) \twoheadrightarrow H_*(Z)$ *and* $H^*_A(Z) \to H^*(Z)$ *are surjective algebra homomorphisms. It even holds the stronger isomorphism of* $\mathbb{C}$*-algebras*

$$H_*(Z) = H^A_*(Z)/H^A_{<0}(pt)H^A_*(Z)$$
$$H^*(Z) = H^*_A(Z)/H^{>0}_A(pt)H^*_A(Z)$$

As a consequence we get the following isomorphisms.

$$1)\ H_*^A(Z) = H_*(Z) \otimes_{\mathbb{C}} H_*^A(pt) \quad \text{of } H_*^A(pt)\text{-modules}$$
$$2)\ H_A^*(Z) = H^*(Z) \otimes_{\mathbb{C}} H_A^*(pt) \quad \text{of } H_A^*(pt)\text{-modules}$$

We can see that $H_{[*]}^A(Z)$ has finite dimensional graded pieces and the graded pieces are bounded from below in negative degrees.

1.3.2 The Steinberg algebra $H_*^A(Z)$ and $H_A^*(E)$

Recall from a previous section that $H_A^*(E)$ is a graded left (and right) $H_{[*]}^A(Z)$-module and that $H_A^*(E)$ has a $H_A^*(pt)$-algebra structure with respect to the cup product, the $H_{[*]}^A(Z)$-operation is $H_A^*(pt)$-linear.

Remark. Let $q_i\colon E_i \to pt,\ i \in I$, there is an isomorphism of algebras

$$\mathrm{End}_{H_A^*(pt)}(H_A^*(E)) = H_*^A(E \times E) = \mathrm{Ext}^*_{D^A(pt)}(\bigoplus_{i\in I}(q_i)_*\underline{\mathbb{C}}[e_i], \bigoplus_{i\in I}(q_i)_*\underline{\mathbb{C}}[e_i]),$$

the first equality follows from [CG97], Ex. 2.7.43, p.123, for the second: Use the Thom isomorphism to replace $E \times E$ by a union of flag varieties, then use theorem 1.1.1 for the Springer map given by the projection to a point.
Furthermore, under the identifications, the following three graded $H_*^A(pt)$-algebra homomorphisms are equal.

(1) The map $H_*^A(Z) \to \mathrm{End}_{H_A^*(pt)}(H_A^*(E)), c \mapsto (e \mapsto c * e)$.

(2) $i_*\colon H_*^A(Z) \to H_*^A(E \times E)$ where $i\colon Z \to E \times E$ is the natural embedding.

(3) Set $\mathcal{A}_\pi := \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}[e_i]$.

$$\mathrm{Ext}^*_{D^A(V)}(\mathcal{A}_\pi, \mathcal{A}_\pi) \to \mathrm{Ext}^*_{D^A(pt)}(a_*(\mathcal{A}_\pi), a_*(\mathcal{A}_\pi)),$$
$$f \mapsto a_*(f)$$

where $a\colon V \to pt$.

We do not prove this here.

Lemma 2. *([VV11], remark after Prop.3.1, p.12) Assume that $T \subset \bigcap_i P_i$ is a maximal torus and $Z^T = E^T \times E^T, E^T = \bigsqcup_{i\in I}(G/P_i)^T$. Let $A \in \{T, G\}$. The map from (1)*

$$H_*^A(Z) \to \mathrm{End}_{H_A^*(pt)}(H_A^*(E)), \quad c \mapsto (e \mapsto c * e)$$

is an injective homomorphism of $H_A^*(pt)$-algebras. Let $\mathfrak{t}$ be the Lie algebra of T, then it holds $H_G^*(E) \cong \mathbb{C}[\mathfrak{t}]^{\oplus I}$, where $\mathbb{C}[\mathfrak{t}]$ is the ring of regular functions on the affine space $\mathfrak{t}$.

Proof: For G-equivariant Borel-Moore homology we claim that the following diagram is commutative

$$\begin{array}{ccc} H_*^T(Z^T)\otimes_{\mathbb{C}} K & \longrightarrow & H_*^T(E^T\times E^T)\otimes_{\mathbb{C}} K \\ \uparrow & & \uparrow \\ H_*^T(Z) & \longrightarrow & H_*^T(E\times E) \\ \uparrow & & \uparrow \\ H_*^G(Z) & \longrightarrow & H_*^G(E\times E) \end{array}$$

where $K = Quot(H_T^*(pt))$ The commutativity of the lowest square uses functoriality of the forgetful maps. By assumption $Z^T = (E\times E)^T$, the highest horizontal map is an isomorphsim. Now, since $H_*^T(Z), H_*^T(E\times E)$ are free $H_T^*(pt)$-modules, we get that the maps $H_*^T(Z)\to H_*^T(Z)\otimes K$, $H_*^T(E\times E)\to H_*^T(E\times E)\otimes K$ are injective. By the localization theorem see Appendix, theorem 8.3.1 or [Bri00], lemma 1, we get the isomorphisms $H_*^T(Z)\otimes K\cong H_*^T(Z^T)\otimes K$, $H_*^T(E\times E)\otimes K\cong H_*^T(E^T\times E^T)\otimes K$. That implies that the middle horizontal map has to be injective, together with (2) from the previous remark it implies the claim for T-equivariant Borel-Moore homology. But by the splitting principle, i.e. the identification of the G-equivariant Borel-Moore homology with the W-invariant subspace in the T-equivariant Borel-Moore homology, the forgetful maps become the inclusion of the W-invariant subspace. This means the two vertical maps in the lower square are injective. This implies that the lowest horizontal map is injective. Together, with (2) of the previous remark the claim follows for $A = G$. □

The main ingredient to the previous lemma is a weak version of Goretzky's, Kottwitz' and MacPherson's localization theorem (see [GKM98]). Similar methods are currently developed by Gonzales for K-theory in [Gon].

The previous lemma is false for not equivariant Borel-Moore homology as the following example shows.

Example. Let G be a reductive group with a Borel subgroup B and $\mathfrak{u}$ be the Lie algebra of its unipotent radical. $Z := (G\times^B \mathfrak{u})\times_{\mathfrak{g}} (G\times^B \mathfrak{u})$, then it holds that the algebra $H_*(Z)$ can under the isomorphism in Kwon (see [Kwo09]) be identified with $\mathbb{C}[\mathfrak{t}]/I_W\#\mathbb{C}[W]$ where $I_W\subset \mathbb{C}[\mathfrak{t}]$ is the ideal generated by the kernel of the map $\mathbb{C}[\mathfrak{t}]^W\to\mathbb{C},\ f\mapsto f(0)$. The *skew ring* $\mathbb{C}[\mathfrak{t}]/I_W\#\mathbb{C}[W]$ is defined as the $\mathbb{C}$-vector space $\mathbb{C}[\mathfrak{t}]/I_W\otimes_{\mathbb{C}}\mathbb{C}[W]$ with the multiplication $(f\otimes w)\cdot(g\otimes v) := fw(g)\otimes wv$. Furthermore, we can identify $\mathrm{End}_{\mathbb{C}}(H^*(E))$ via the Thom-isomorphism and the Borel map with $\mathrm{End}_{\mathbb{C}-lin}(\mathbb{C}[\mathfrak{t}]/I_W)$. The canonical map identifies with

$$\mathbb{C}[\mathfrak{t}]/I_W\#\mathbb{C}[W]\to \mathrm{End}_{\mathbb{C}-\mathrm{lin}}(\mathbb{C}[\mathfrak{t}]/I_W)$$
$$f\otimes w\mapsto (p\mapsto fw(p))$$

This map is neither injective nor surjective. For example $\sum_{w\in W} 1\otimes w\neq 0$ in $\mathbb{C}[\mathfrak{t}]/I_W\#\mathbb{C}[W]$ but its image $(p\mapsto\sum_{w\in W} w(p))$ is zero because $\sum_{w\in W} w(p)\in I_W$. Because both spaces have the same $\mathbb{C}$-vector space dimension, it is clear that it is also not surjective.

Furthermore, $H_*^A(Z)$ is naturally a $H_A^*(E)$-module. In fact let $e_{i,j} := \overline{e} \in W_{i,j}$ be the double coset of the neutral element, then $H_A^*(E) \cong H_*^A(\bigsqcup_{i,j\in I} Z^{e_{i,j}})$ is even a subalgebra of $H_*^A(Z)$.

Corollary 1.3.0.2. *In the situation of the previous lemma, i.e. $T \subset \bigcap_i P_i$ is a maximal torus and $Z^T = E^T \times E^T, E^T = \bigsqcup_{i\in I}(G/P_i)^T$ and let $A \in \{T, G\}$. There are injective homomorphism of $H_G^*(pt)$-algebras*

$$H_*^A(pt) \subset H_*^A(E) \subset H_*^A(Z) \to \operatorname{End}_{H_A^*(pt)}(H_A^*(E)),$$

where the first inclusion is given by the pullback along the map $E \to pt$. In particular, $H_A^*(pt)$ is contained in the centre of $H_*^A(Z)$ (we only know examples where it is equal to the centre).

Let $w \in \widetilde{W}$. Observe, that $H_A^*(E)$ already operates on $H_*^A(Z^w)$ and the composition $H_*^A(Z) = \bigoplus_w H_*^A(Z^w)$ is a direct sum composition of $H_A^*(E)$-modules. Using the Thom-isomorphism (see Appendix, subsection **??**, (5)), up to a degree shift we can also study $H_*^A(C^w)$ as module over $H_A^*(\bigsqcup_{i\in I} G/P_i)$. Now, let e_i be the idempotent in $H_A^*(\bigsqcup_{i\in I} G/P_i) = \bigoplus_{i\in I} H_A^*(G/P_i)$ which corresponds to the projection on the i-th direct summand. Since for $w \in W_{i,j}$ it holds $H_*^A(C_w) = H_A^*(G/P_i) \otimes_{\mathbb{C}} H_*(P_i w P_j/P_j)$ also as $H_A^*(G/P_i)$-module, we conclude that $H_*^A(C_w)$ is always a projective module over $H_A^*(\bigsqcup_{i\in I} G/P_i)$. In other words this discussion yields.

Lemma 3. *(1) Let $w \in W_{i,j}$. Each $H_*^A(Z^w)$ is a projective graded $H_A^*(E)$-module of the form*

$$\bigoplus_{v\in W_i} (H_A^*(E)e_i)[2d_w + \deg b_{i,j}(v)],$$

where $[d]$ denotes the degree shift by d. In particular, $H_*^A(Z)$ is a projective graded $H_A^*(E)$-module.

(2) If all $P_i = B_i$ are Borel subgroups of G, then

$$H_*^A(Z) = \bigoplus_{w,j\in W\times I} (\bigoplus_{i\in I} (H_A^*(E)e_i)[d_{w,i,j}])$$

as graded $H_A^*(E)$-module for certain $d_{w,i,j} \in \mathbb{Z}$. In particular, if we forget the grading $H_*^A(Z)$ is a free $H_A^*(E)$-module of rank $\#W \cdot \#I$.

1.4 Indecomposable projective graded modules over $H_{[*]}^A(Z)$ and their tops for a different grading

Let X be an irreducible algebraic variety, we call a decomposition $X = \bigcup_{a\in\mathcal{A}} S_a$ into finitely many irreducible smooth locally closed subsets a *weak* stratification. Since $\pi\colon E = \bigsqcup_{i\in I} E_i \to V$ is a G-equivariant projective map, there exists (and we fix it) a *weak* stratification into G-invariant subsets $V = \bigsqcup_{a \in \mathcal{A}} S_a$ such that $\pi^{-1}(S_a) \xrightarrow{\pi} S_a$ is a locally trivial[4] fibration with constant fibre $F_a := \pi^{-1}(s_a)$ where $s_a \in S_a$ one fixed point, for every $a \in \mathcal{A}$. (For projective maps of complex algebraic varieties one can always find such a weak stratification, see [Ara01], 4.4.1-4.4.3)[5]

Recall that for any G-equivariant projective map of complex varieties, the decomposition theorem holds (compare [BBD82] for the not equivariant version and [BL94] for the equivariant version). Let t run over all simple[6] G-equivariant local systems $\mathcal{L}_t$ on some stratum $S_t = S_{a_t}$, $a_t \in \mathcal{A}$, we write $IC_t^A := (i_{\overline{S_t}})_*(\mathcal{IC}^A(S_t, \mathcal{L}_t)[d_{S_t}]$ with $d_{S_t} = \dim_{\mathbb{C}} S_t$ for the simple perverse sheaf in the category of A-equivariant perverse sheaves $Perv_A(V) \subset D_A^b(V)$, see again [BL94], p. 41. Let $e_i = \dim_{\mathbb{C}} E_i, i \in I$, then $\underline{\mathbb{C}}_{E_i}[e_i]$ is a simple perverse sheaf in $D_A^b(E)$. For a graded vector space $L = \bigoplus_{d \in \mathbb{Z}} L_d$ we define $L(n)$ to be the graded vector space with $L(n)_d := L_{n+d}$, $n \in \mathbb{Z}$. We see $\mathbb{C}$ as the graded vector space concentrated in degree zero. For an element $F^\bullet \in D_A^b(X)$ for an A-variety X we write $F^\bullet[n]$ for the (class of the) complex $(F^\bullet[n])_d := F^{d+n}$, $n \in \mathbb{Z}$. Now given $F^\bullet \in D_A^b(X)$ and a finite dimensional graded vector space $L := \bigoplus_{i=1}^r \mathbb{C}(d_i)$ we define

$$L \otimes_{gr} F^\bullet := \bigoplus_{i=1}^r F^\bullet[d_i] \in D_A^b(X)$$

The A-equivariant decomposition theorem applied to π gives

$$\bigoplus_{i \in I} (\pi_i)_* \underline{\mathbb{C}}_{E_i}[e_i] = \bigoplus_t L_t \otimes_{gr} IC_t^A \quad \in D_A^b(V)$$

where the $L_t := \bigoplus_{d \in \mathbb{Z}} L_{t,d}$ are complex finite dimensional graded vector spaces.

Let $\mathbb{D}$ be the Verdier-duality on V, it holds $\mathbb{D}(\pi_*(\underline{\mathbb{C}}[d])) = \pi_*(\underline{\mathbb{C}}[d]), \mathbb{D}(IC_t^A) = IC_{t^*}^A$ where we define for $t = (S, \mathcal{L})$ the associated dual local system as $t^* = (S, \mathcal{L}^*)$, $\mathcal{L}^* := \mathcal{H}om(\mathcal{L}, \underline{\mathbb{C}})$. This implies $L_t = L_{t^*}$ for all t.

1.4.1 Indecomposable projectives in the category of graded left $H_{[*]}^A(Z)$-modules

We set

$$P_t^A := \operatorname{Ext}^*_{D_A^b(V)}(IC_t^A, \bigoplus_{i \in I} (\pi_i)_* \underline{\mathbb{C}}[e_i]).$$

[4]with respect to the analytic topology

[5]If the image of π is irreducible, by [Ara01], theorem 1.9.10 we can refine this stratification to a (finite) Whitney stratification, but it is not clear if we can find a Whitney stratification into G-invariant subsets.

[6]a local system is simple if the by monodromy associated representation of the fundamental group has no nontrivial subrepresentation. Usually this is called irreducible.

It is a graded (left) $H^A_{[*]}(Z)$-module. It is indecomposable because IC^A_t is simple. Clearly it holds as left graded $H^A_{[*]}(Z)$-modules

$$\begin{aligned}
H^A_{[*]}(Z) &= \bigoplus_{d\in\mathbb{Z},t} L_{t,d} \otimes [\bigoplus_{n\in\mathbb{Z}} \operatorname{Ext}^{n+d}_{D^b_A(V)}(IC^A_t, \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}[e_i])] \\
&= \bigoplus_{d\in\mathbb{Z},t} L_{t,d} \otimes_{\mathbb{C}} P^A_t[d] \\
&= \bigoplus_t L_t \otimes_{gr} P^A_t
\end{aligned}$$

that implies that P^A_t is a projective module and that $(P^A_t)_t$ is a complete set of isomorphism classes up to shift of indecomposable projective graded $H^A_{[*]}(Z)$-modules.

Lemma 4. *Assume that $H^*_A(pt)$ is a graded subalgebra of the centre of $H^A_{[*]}(Z)$. The elements $H^{>0}_A(pt)$ operate on any graded simple $H^A_{[*]}(Z)$-module S by zero. In particular, by lemma 1.3.0.1 we see that S is a graded simple modules over $H_{[*]}(Z)$. Any graded simple module is finite-dimensional and there exists up to isomorphism and shift only finitely many graded simple modules.*
For any graded simple module S there is no nonzero degree zero homomorphism $S \to S(a),\ a \neq 0$.

Proof: By assumption that $H^*_A(pt)$ is central, we obtain that $H^{>0}_A(pt) \cdot S$ is a graded left $H^A_{[*]}(Z)$-module, clearly it is a submodule of S. Since S is simple it holds $H^{>0}_A(pt) \cdot S$ is zero or S. Assume it is S, then there exists $x \in H^d_A(pt)$ for a $d > 0$ such that $x \cdot S \neq 0$. Since x is central, this is a submodule of S and we have $x \cdot S = S$. Let $y \in S, y \neq 0$, homogeneous. Then, it holds $S = H^A_{[*]}(Z) \cdot y = H^A_{[*]}(Z) \cdot xy$ contradicting the fact that there is a uniquely determined minimal nonzero degree for S. Therefore $H^{>0}_A(pt) \cdot S = 0$. By [NO82], II.6, p.106, we know that the graded simple modules considered as modules over the ungraded rings $H^A_*(Z), H_*(Z)$ are still simple modules. Since the finite-dimensional algebra $H_*(Z)$ has up to isomorphism only finitetly many simples, the claim follows.
Any nonzero degree 0 homomorphism $\phi\colon S \to S(a)$ has to be an isomorphism. Let $S = H^A_{[*]}(Z) \cdot y$ as before, set $\deg y = m$. Then $S(a) = H^A_{[*]}(Z) \cdot \phi(y)$, deg $\phi(y) = m$ which gives a contradiction when considering the minimal nonzero degrees of S and $S(a)$. □

Corollary 1.4.0.3. *There is a bijection between isomorphism classes up to shift of*

(1) indecomposable projective graded $H^A_{[*]}(Z)$-modules

(2) indecomposable projective graded $H_{[*]}(Z)$-modules

(3) simple graded $H_{[*]}(Z)$-modules.

The bijection between (1) and (2) is clear from the decomposition theorem, it maps $P \mapsto P/H^{>0}_A(pt)P$. We pass from (3) to (2) by taking the projective cover and we pass from (2) to (3) by taking the top (which is graded because for a finite dimensional graded algebra the radical is given by a graded ideal).

Example. (due to Khovanov and Lauda, [KL09]) Let $G \supset B \supset T$ be a reductive group containing a Borel subgroup containing a maximal torus, $Z = G/B \times G/B$. Then, it is known that $H_*^G(Z) = \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W}(\mathbb{C}[\mathfrak{t}]) =: NH$ where W is the Weyl group associated to (G,T) and $\mathfrak{t} = \mathrm{Lie}(T)$.
The G-equivariant pushforward (to the point) of the shift of the constant sheaf is a direct sum of shifts of copies of the constant sheaves on the point, therefore there exist precisely one indecomposable projective graded $H_{[*]}^G(Z)$-module up to isomorphism and shift. It is easy to see that $P := \mathbb{C}[\mathfrak{t}]$ is an indecomposable projective module and $P/H_G^{>0}(pt)P = \mathbb{C}[\mathfrak{t}]/I_W$ is the only graded simple NH-module which is the top of P. Also, one checks that $H_{[*]}(Z) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{t}]/I_W)$ is a semi-simple algebra which has up to isomorphism and shift only the one graded simple module $\mathbb{C}[\mathfrak{t}]/I_W$.

In the following subsectionwe equip the Steinberg algebra with a grading by positive integers which leads to a description of graded simple modules in terms of the multiplicity vector spaces L_t in the BBD-decomposition theorem.

1.4.2 Simples in the category of graded finitely generated left $H^A_{<*>}(Z)$-modules

Given a graded vector space L, we write $\langle L\rangle := \bigoplus_{d\in\mathbb{Z}} L_d$ for the underlying (ungraded) vector space. If we regrade $H_*^A(Z)$ as follows

$$H^A_{<n>}(Z) := \bigoplus_{s,t} \mathrm{Hom}_{\mathbb{C}}(\langle L_t\rangle, \langle L_s\rangle) \otimes_{\mathbb{C}} \mathrm{Ext}^n_{D^b_A(V)}(IC^A_t, IC^A_s)],$$

in other words

$$H^A_{<*>}(Z) = \mathrm{Ext}^*(\bigoplus_t \langle L_t\rangle \otimes_{\mathbb{C}} IC^A_t, \bigoplus_t \langle L_t\rangle \otimes_{\mathbb{C}} IC^A_t)$$

as graded algebra. This is as an ungraded algebra isomorphic to $H^A_*(Z)$. With the same arguments as in the previous section one sees that $P^A_t := \mathrm{Ext}^*_{D^b_A(V)}(IC^A_t, \pi_*\underline{\mathbb{C}})$ are a complete representative system for the isomorphism classes of the indecomposable projective graded $H^A_{<*>}(Z)$-modules.
We claim that there is a graded $H^A_{<*>}(Z)$-module structure on the (multiplicity-)vector space $\langle L_t\rangle$ such that the family $\{\langle L_t\rangle\}_t$ is a complete set of the isomorphism classes up to shift of graded simple modules. Using $\mathrm{Hom}(IC^A_t, IC^A_s) = \mathbb{C}\delta_{s,t}$, $\mathrm{Ext}^n(IC^A_t, IC^A_s) = 0$ for $n < 0$ we get

$$H^A_{<*>}(Z) = \underbrace{\bigoplus_t \mathrm{End}(\langle L_t\rangle)}_{\deg=0} \oplus \bigoplus_{s,t} \mathrm{Hom}(\langle L_t\rangle, \langle L_s\rangle) \otimes_{\mathbb{C}} \mathrm{Ext}^{>0}(IC^A_t, IC^A_s).$$

Now, the *second* summand is the graded radical, i.e. the elements of degree > 0 (with respect to the new grading). It follows

$$H^A_{<*>}(Z) \twoheadrightarrow H^A_{<*>}(Z)/(H^A_{<*>}(Z))_{>0} = \bigoplus_t \mathrm{End}_{\mathbb{C}}(\langle L_t \rangle).$$

This gives $\langle L_t \rangle$ a natural graded $H^A_{<*>}(Z)$-module structure concentrated in degree zero (the positive degree elements in $H^A_{<*>}(Z)$ operate by zero). Observe, that $\langle L_t \rangle$ does not depend on A, i.e. in fact they are modules over $H^A_{<*>}(Z)$ via the forgetful morphism $H^A_{<*>}(Z) \to H_{<*>}(Z)$.
That means we can instead look for the simple graded modules of $H_{<*>}(Z)$.

Remark. Let H_* be a finite dimensional positively graded algebra such that

$$H_0 = H_*/H_{>0} = \bigoplus_t \mathrm{End}(L_t)$$

is a semi-simple algebra. Then $H_{>0}$ is the set of nilpotent elements, i.e. Jacobson radical of H_*. Furthermore all simple and projective H_*-modules are graded modules.

* $(L_t)_t$ is the tuple of (pairwise distinct isomorphism classes of all) simple modules.
* For each t pick an $e_t \in \mathrm{End}(L_t) \subset H_0$ which corresponds to projection and then inclusion of a one dimensional subspace of L_t.
  $(P_t := H_* \cdot e_t)_t$ is the tuple of (pairwise distinct isomorphism classes of all) indecomposable projective modules.

We can apply this remark to $H = H_{<*>}(Z)$. As a consequence we see that up to shift $(\langle L_t \rangle)_t$ is the tuple of (pairwise distinct isomorphism classes of all) simple graded $H^A_*(Z)$-modules.

From now on, the case where the two gradings coincide will play a special role.

Remark. The following conditions are equivalent

(1) $H^A_{[*]}(Z) = H^A_{<*>}(Z)$ as graded algebra for every $A \in \{pt, T, G\}$.

(1)′ $H^A_{[*]}(Z) = H^A_{<*>}(Z)$ as graded algebra for at least one $A \in \{pt, T, G\}$.

(2) $(\pi_i)_* \underline{\mathbb{C}}[e_i]$ is A-equivariant perverse for every $i \in I$ for every $A \in \{pt, T, G\}$.

(2)′ $(\pi_i)_* \underline{\mathbb{C}}[e_i]$ is A-equivariant perverse for every $i \in I$ for at least one $A \in \{pt, T, G\}$.

(3) $\pi_i \colon E_i \to V$ is **semi-small** for every $i \in I$, this means by definition $\dim Z_{i,i} = e_i$ for every $i \in I$.

In this case, we say the Springer map is semi-small. Also, π semi-small is equivalent to $H_{top}(Z_{i,i}) = H_{[0]}(Z_{i,i}), i \in I$. Observe, that $H_{[0]}(Z)$ is always a subalgebra of $H_{[*]}(Z)$ and in the semi-small case isomorphic to the quotient algebra $H_{[*]}(Z)/(H_{[*]}(Z))_{>0}$. Assume π semi-small, then it holds $2\dim \pi_i^{-1}(s) \leq e_i - d_S,\ i \in I$ where $x \in S$ belongs to the stratification and $H_{top}(\pi^{-1}(s)) := \bigoplus_{i\colon 2\dim \pi_i^{-1}(s) = e_i - d_S} H_{2\dim \pi_i^{-1}(s)}(\pi_i^{-1}(s))$ is a left $H_{[0]}(Z)$-module via the restriction of the convolution. If I consists of a single element, $H_{top}(Z) = H_{[0]}(Z)$ and $H_{2\dim \pi^{-1}(s)}(\pi^{-1}(s))$ is a $H_{[0]}(Z)$-module.

Remark. If one applies the decomposition theorem to $\pi_i, i \in I$ one gets that $L_t = \bigoplus_{i\in I} L_t^{(i)}$ (as graded vector space) where $L_t^{(i)}$ is the multiplicity vector space for IC_t in $(\pi_i)_*\underline{\mathbb{C}}[e_i]$. It holds $\{L_t^{(i)} \mid L_t^{(i)} \neq 0\}$ is the complete set of isomorphism classes of simple $H_*(Z_{i,i})$-modules.

Remark. In fact, Syu Kato pointed out that the categories of finitely generated graded modules over $H^A_{[*]}(Z)$ and $H^A_{<*>}(Z)$ are equivalent. This has been used in [Kat13].

Remark. Now, we know that the forgetful (=forgetting the grading) functor from finite dimensional graded $H_{[*]}(Z)$-modules to finite dimensional $H_*(Z)$-modules maps graded simple modules to simple modules. We can use the fact that we know that simples and graded simples are parametrized by the same set, to see: Every simple $H_*(Z)$-module L_t has a grading such that it becomes a graded simple $H_{[*]}(Z)$-module and every graded simple is of this form.

For the decomposition matrix for the finite dimensional algebra $H_*(Z)$, there is the following result of Chriss and Ginzburg.

Theorem 1.4.1. *([CG97], thm 8.7.5) Assume $H_{odd}(\pi^{-1}(s)) = 0$ for all $x \in V$. Then, the following matrix multiplication holds*

$$[P\colon L] = IC \cdot D \cdot IC^t$$

where all are matrices indexed by $s = (S, \mathcal{L}), t = (S', \mathcal{L}')$ such that $L_t \neq 0, L_s \neq 0$ and $()^t$ denote the transposed matrix.

$$[P\colon L]_{s,t} := [P_s\colon L_t] = \sum_k \dim \operatorname{Ext}^k(IC_t, IC_s)$$

$$IC_{s,t} := \sum_k [\mathcal{H}^k(i_S^*(IC_t))\colon \mathcal{L}]$$

$$D_{s,t} := \delta_{S,S'} \sum_k (-1)^k \dim H^k(S, (\mathcal{L}')^* \otimes \mathcal{L})$$

According to Kato in [Kat13], the whole theory of these algebras is reminiscent of quasi-hereditary algebras (but we have infinite dimensional algebras). He introduces standard and costandard modules for $H^G_{<*>}(Z)$ in [Kat13], thm 1.3, under some assumptions[7].He shows that under these assumptions, $H^G_{<*>}(Z)$ has finite global dimension (see [Kat13], thm 3.5).

1.4.3 Springer fibre modules in the category of graded $H^A_?(Z)$-modules

Recall, that Springer fibre modules $H_{[*]}(\pi^{-1}(s)), H^{[*]}(\pi^{-1}(s)), s \in V$ are naturally graded modules over $H_{[*]}(Z)$, but if we forget about the grading and we can show that they are actually semi-simple in $H_*(Z)$-mod, then, we can see them as semi-simple graded $H^A_{<*>}(Z)$-modules for $A \in \{G, T, pt\}$ by the previous section.

[7]= finitely many orbits with connected stabilizer groups in the image of the Springer map, $H^G_{<*>}(Z)$ and the in the decomposition theorem occurring IC_t are *pure of weight zero*

Let $A = pt$. Since the map π is locally trivial over $S := S_a$ we find that

$$i_S^*(\bigoplus_{i\in I} R^k(\pi_i)_*\underline{\mathbb{C}}[e_i]), \quad i_S^!(\bigoplus_{i\in I} R^k(\pi_i)_*\underline{\mathbb{C}}[e_i]$$

are local systems on S, via monodromy they correspond to the $\pi_1(S,s)$-representations

$$H^{[k]}(\pi^{-1}(s)) = \bigoplus_{i\in I} H^{e_i+k}(\pi_i^{-1}(s)), \quad \bigoplus_{i\in I} H_{e_i-k}(\pi_i^{-1}(s)) = H_{[k]}(\pi^{-1}(s))$$

with $e_i := \dim_{\mathbb{C}} E_i$ respectively (for a fixed point $s = s_a \in S$, cp. [CG97], Lemma 8.5.4). Now, let us make the extra assumption that the image of the Springer map is irreducible and the stratification $\{S_a\}_{a\in\mathcal{A}}$ is a Whitney stratification (every algebraic stratification of an irreducible variety can be refined to a Whitney stratification see [Ara01], thm 1.9.10, p.30), which is totally ordered by inclusion into the closure. Let $S \subset \overline{S'}$ be an incusion for two strata S, S', we write $\mathrm{Ind}_{S'}^S(\mathcal{L}) := i_S^* \circ \mathcal{H}^*(IC_{(S',\mathcal{L})})$, i.e. we consider the functors for $k \in [-d_{S'}, -d_S]$

$$\begin{aligned} \mathrm{Ind}_{S'}^S(-)_k \colon LocSys(S') &\to LocSys(S) \\ \mathcal{L} &\to \mathrm{Ind}_{S'}^S(\mathcal{L})_k := i_S^* \circ \mathcal{H}^k(IC_{(S',\mathcal{L})}) \end{aligned}$$

where $LocSys(S)$ is the category of local systems on S, i.e. locally constant sheaves on S of finite dimensional vector spaces (for other $k \in \mathbb{Z}$ this is the zero functor). If we apply the functor $i_S^* \circ \mathcal{H}^k$ on the right hand side of the decomposition theorem we notice the following (for the cohomology groups of IC-sheaves, see [Ara01], section 4.1, p.41), let $t = (S', \mathcal{L})$.

$$i_S^*\mathcal{H}^k(IC_t) = \begin{cases} \mathcal{L}, & \text{if } d_S = d_{S'}, k = -d_S \\ \mathrm{Ind}_{S'}^S(\mathcal{L})_k & \text{if } d_S < d_{S'}, k \in [-d_{S'}, -d_S - 1] \\ 0 & \text{else.} \end{cases}$$

and

$$i_S^!\mathcal{H}^k(IC_t[d]) = \mathcal{H}^{k+d}(\mathbb{D}_S i_S^* IC_{t^*}) = i_S^*\mathcal{H}^{-k-d-2d_S}(IC_{t^*})$$

implies

$$i_S^!\mathcal{H}^k(IC_t[d]) = \begin{cases} \mathcal{L}^*, & \text{if } d_S = d_{S'}, k + d = -d_S \\ \mathrm{Ind}_{S'}^S(\mathcal{L}^*)_{-k-d-2d_S} & \text{if } d_S < d_{S'}, -k-d-2d_S \in [-d_{S'}, -d_S - 1] \\ 0 & \text{else.} \end{cases}$$

where $d_S = \dim_{\mathbb{C}} S$. This implies

$$H^{[k]}(\pi^{-1}(s)) = \bigoplus_t \bigoplus_{d\in\mathbb{Z}} L_{t,d} \otimes_{\mathbb{C}} i_S^* \mathcal{H}^{k+d}(IC_t)$$

$$= \bigoplus_{t=(S,\mathcal{L})} L_{t,-d_S-k} \otimes_{\mathbb{C}} \mathcal{L} \oplus \underbrace{\bigoplus_{t=(S',\mathcal{L}),d_S<d_{S'}} \bigoplus_{r=-d_{S'}}^{-d_S-1} L_{t,r-k} \otimes_{\mathbb{C}} \mathrm{Ind}_{S'}^S(\mathcal{L})_r}_{=:H^{[k]}(\pi^{-1}(s))_{>S}}$$

and

$$H_{[k]}(\pi^{-1}(s)) = \bigoplus_t \bigoplus_{d\in\mathbb{Z}} L_{t,d} \otimes_{\mathbb{C}} i_S^! \mathcal{H}^{i+d}(IC_{t^*})$$

$$= \bigoplus_{t=(S,\mathcal{L})} L_{t,-d_S-k} \otimes_{\mathbb{C}} \mathcal{L}^* \oplus \underbrace{\bigoplus_{t=(S',\mathcal{L}),d_S<d_{S'}} \bigoplus_{r=-d_{S'}}^{-d_S-1} L_{t,-r-2d_S-k} \otimes_{\mathbb{C}} \mathrm{Ind}_{S'}^S(\mathcal{L}^*)_r}_{=:H_{[k]}(\pi^{-1}(s))_{>S}}$$

as $\pi_1(S,s)$-representations. We call the direct summands isomorphic to $\mathrm{Ind}_{S'}^S(\mathcal{L})_r, r \in [-d_{S'}, -d_S - 1]$ the *unwanted summands*. Now we can explain how you can recover from the $\pi_1(S,s)$-representations $H^{[k]}(\pi^{-1}(s)), k \in \mathbb{Z}$ the data for the decomposition theorem (i.e. the local sytems and the graded multiplicity spaces). If d_S is the maximal one, it holds

$$H^{[*]}(\pi^{-1}(s)) = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{t=(S,\mathcal{L})} L_{t,-d_S-k} \otimes_{\mathbb{C}} \mathcal{L}$$

and we can recover the graded multiplicity spaces L_t with $t = (S, ?)$ for the dense stratum ocurring in the decomposition theorem. For arbitrary S we consider

$$H^{[*]}(\pi^{-1}(s))/H^{[*]}(\pi^{-1}(s))_{>S} \cong \bigoplus_{k\in\mathbb{Z}} \bigoplus_{t=(S,\mathcal{L})} L_{t,-d_S-k} \otimes_{\mathbb{C}} \mathcal{L}$$

and by induction hypothesis we know the $\pi_1(S,s)$-representation $H^{[*]}(\pi^{-1}(s))_{>S}$, therefore we can recover the L_t with $t = (S, ?)$ from the above representation.

Now assume that π is semi-small. Then, we know that $L_{t,d} = 0$ for all $t = (S,\mathcal{L})$ whenever $d \neq 0$. We can also restrict our attention on a direct summand $(\pi_i)_*\underline{\mathbb{C}}[e_i]$ for one $i \in I$ and find the decomposition into simple perverse sheaves. That means we only need $H^{e_i-d_S}(\pi_i^{-1}(s))$ to recover the data for the decomposition theorem. It also holds $2\dim \pi_i^{-1}(s) \leq e_i - d_S, i \in I$ and since $H^{e_i-d_S}(\pi_i^{-1}(s)) = 0$ whenever $2\dim \pi_i^{-1}(s) < e_i - d_S$, we only need to consider the strata S with $2\dim \pi_i^{-1}(s) = e_i - d_S$, then

$$H^{e_i-d_S}(\pi_i^{-1}(s)) = H^{top}(\pi_i^{-1}(s)) \neq 0$$

and we call S a **relevant stratum** for i $(\in I)$. We call a stratum relevant if it is relevant for at most one $i \in I$.
Analogously, one can replace $H^{[k]}(\pi^{-1}(s))$ by $H_{[-k]}(\pi^{-1}(s))$ and stalk by costalk.

Let $s \in V$ be arbitrary. By a previous section we know that $H_{[*]}(\pi^{-1}(s))$ and $H^{[*]}(\pi^{-1}(s))$ are left (and right) graded $H_{[*]}(Z)$-modules. The following lemma explains their special role. Unfortunately, the following statement is only known if all strata S contain a G-orbit $G \cdot s := \mathcal{O} \subset S$ such that $\pi_1(\mathcal{O}, s) = \pi_1(S, s)$. For local systems on the strata this is by monodromy the same as the assumption that all strata are G-orbits. Let C be a finite group, we write $\mathrm{Simp}(C)$ for the set of isomorphism classes of simple $\mathbb{C}C$-modules and denote by $\mathbf{1} \in \mathrm{Simp}(\mathbf{C})$ the trivial representation[8].

Lemma 5. *([CG97], Lemma 8.4.11, p.436, Lemma 3.5.3, p.170) Assume that the image of the Springer map contains only finitely many G-orbits.*

(a) Let $\mathcal{O} = Gs \subset V$ be a G-orbit. There is an equivalence of categories between

$$\{G\text{-equivariant local systems on } \mathcal{O}\} \leftrightarrow C(s)-\mathrm{mod}$$

*where $C(s) = \mathrm{Stab}_G(s)/(\mathrm{Stab}_G(s))^o$ is the **component group** of the stabilizer of s. In particular, via monodromy also the $\pi_1(\mathcal{O}, s)$-representations which correspond to G-equivariant local systems on $\mathcal{O}$ are equivalent to $C(s)-\mathrm{mod}$.*

(b) The $C(s)$-operation and the $H_{[*]}(Z)$-operation on $H_{[*]}(\pi^{-1}(s))$ (and on $H^{[*]}(\pi^{-1}(s))$) commute.

The semi-simplicity of $C(s)-\mathrm{mod}$ implies that

$$H_{[*]}(\pi^{-1}(s)) = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{\chi \in \mathrm{Simp}(C(s))} (H_{[k]}(\pi^{-1}(s)))_\chi \otimes_{\mathbb{C}} \chi$$

where $\mathrm{Simp}(C(s))$ is the set of isomorphism classes of simple $C(s)$-modules and for any $C(s)$-module M we call $M_\chi := \mathrm{Hom}_{C(s)-\mathrm{mod}}(\chi, M)$ an **isotypic component**. Since the two operation commute it holds $(H_{[*]}(\pi^{-1}(s))_\chi$ naturally has the structure of a graded $H_{[*]}(Z)$-module. But we will from now just see it as a module over $H_*(Z)$. As $H_*(Z)$–$C(s)$-bimodule decomposition we can write the previous decomposition as

$$H_{[*]}(\pi^{-1}(s)) = \bigoplus_{\chi \in \mathrm{Simp}(C(s))} H_*(\pi^{-1}(s))_\chi \boxtimes \chi$$

where $H_*(\pi^{-1}(s))_\chi \boxtimes \chi$ is the obvious bimodule $H_*(\pi^{-1}(s))_\chi \otimes \chi$. As an immediate consequence of this we get, if Gs is a dense orbit in the image of the Springer map, then

$$L_{t,-*}(-d_{Gs}) = H_{[*]}(\pi^{-1}(s))_\chi, \text{ for } t = (s, \chi), \chi \in \mathrm{Simp}(C(s)),$$

in particular, $H_{[*]}(\pi^{-1}(s))$ is a semisimple $H_*(Z)$-module (graded and not graded), even a semisimple $H_*(Z) - C(s)$-bimodule. For more general orbits, we do not know if it is semisimple. In the case of a semi-small Springer map we have the following result.

[8] In the literature this is called $Irr(C)$, we use the word irreducible only for a property of topological spaces

Theorem 1.4.2. *Assume the image of the Springer map π has only finitely many orbits and π is semi-small. There is a bijection between the following sets*

(1) $\{(s,\chi) \mid \mathcal{O} = Gs, \chi \in \mathrm{Simp}(C(s)), H_{[d_{\mathcal{O}}]}(\pi^{-1}(s))_\chi \neq 0\}$ *where the s in V are in a finite set of points representing the G-orbits in the image of the Springer map.*

(2) $\mathrm{Simp}(H_{<0>}(Z)-\mathrm{mod}) :=$ *simple $H_{<0>}(Z)$-modules up to isomorphism*

(3) $\mathrm{Simp}(H^A_{<*>}(Z)-\mathrm{mod}^{\mathbb{Z}})$ *:= simple graded $H^A_{<*>}(Z)$-modules up to isomorphism and shift for any $A \in \{pt, T, G\}$.*

Between (1) and (2), it is given by $(s,\chi) \mapsto H_{[d_{\mathcal{O}}]}(\pi^{-1}(s))_\chi$. We call this bijection the ***Springer correspondence****.*
For a relevant orbit $\mathcal{O}$ (for at least one $i \in I$) it holds

$$H_{[d_{\mathcal{O}}]}(\pi^{-1}(s))_{\mathbf{1}} = \bigoplus_{i:\ 2\dim \pi_i^{-1}(s) = e_i - d_{\mathcal{O}}} H_{top}(\pi_i^{-1}(s))^{C(s)} \neq 0$$

and $C(s)$ operates on the top-dimensional irreducible components of $\pi_i^{-1}(s)$ by permutation. This implies we get an injection

$$\{\textit{relevant } G\textit{-orbits in } Im(\pi)\} \hookrightarrow \mathrm{Simp}(H_{<0>}(Z)-\mathrm{mod})$$
$$\mathcal{O} = Gs \mapsto H_{[d_{\mathcal{O}}]}(\pi^{-1}(s))^{C(s)}$$

sketch of proof: For $k = d_{\mathcal{O}}$ look at the decomposition for $H_{[k]}(\pi^{-1}(s))$ and use that $L_{t,d} = 0$ whenever $d \neq 0$ to see that the unwanted summands vanish. Then show that the decomposition coincides with the second decomposition (with respect to the irreducible characters of $C(s)$) of $H_{[k]}(\pi^{-1}(s))$ which gives the identification of the L_t with the $H_{[d_{\mathcal{O}}]}(\pi^{-1}(s))_\chi$. □

It is an open question to understand Springer fibre modules more generally. Also, Springer correspondence hints at a hidden equivalence of categories. This functorial point of view we investigate in the next subsection.

1.5 The Springer functor

We consider $H^A_{[*]}(Z)$ again with the grading from the theorem 1.1.1. Let $proj^{\mathbb{Z}} H^A_*(Z)$ be the category of finitely generated projective $\mathbb{Z}$-graded left $H^A_{[*]}(Z)$-modules, morphisms are the module homomorphisms which are homogeneous of degree 0. Let $\mathcal{P}^A \subset D^b_A(X)$ be the full subcategory closed under direct sums and shifts generated by IC^A_t, $t = (S, \mathcal{L})$ be the tuple of strata with simple local system on it which occur in the decomposition theorem (with nonzero multiplicity spaces L_t).

The following lemma is in a special case due to Stroppel and Webster, see [SW11].

Lemma 6. *The functor*

$$\begin{aligned} proj^{\mathbb{Z}} H^A_{[*]}(Z) &\to \mathcal{P}_A \\ M &\mapsto \bigoplus_{i\in I} (\pi_i)_* \underline{\mathbb{C}}_{E_i}[e_i] \otimes_{H^A_{[*]}(Z)} M \end{aligned}$$

*is an equivalence of semisimple categories mapping $P^A_t \mapsto IC^A_t$. We call this the **Springer functor*** [9].

Proof: By theorem 1.1.1 we know $H^A_{[*]}(Z) \cong \mathrm{Ext}^*_{D^b_A(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i], \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i])$ is an isomorphism of graded algebras. This makes the functor well-defined. The direct sum decomposition of $\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i]$ by the decomposition theorem in $\mathcal{P}_A$ corresponds to idempotent elements in $H^A_{[0]}(Z)$, which correspond (up to isomorphism and shift) to the indecomposable projective graded modules, let for example $P_t = H^A_{[*]}(Z)e_t$. Shifts of graded modules are mapped to shifts in $\mathcal{P}_A$, therefore the functor is essentially surjective. It is fully faithful because of the mentioned equality

$$\mathrm{Hom}_{proj^{\mathbb{Z}} H^A_{[*]}(Z)}(P_t, P_s(n)) = e_s H^A_{[n]}(Z) e_t = \mathrm{Hom}_{D^b_A(V)}(IC_t, IC_s[n])$$

□

Let $\mathcal{P}^A(V) \subset D^b_A(V)$ be the category of A-equivariant perverse sheaves on V. Assume for a moment that the map π is semi-small. Then, we know that $\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i]$ is an object of $\mathcal{P}^A(V)$. In this situation the two gradings of the Steinberg algebra coincide. The top-dimensional Borel-Moore homology $H_{top}(Z_{i,i})$ coincides with the degree zero subalgebra $H_{[0]}(Z_{i,i})$. We want the Springer functor to go to a category of perverse sheaves, i.e. we do not want to allow shifts of the grading for modules. Therefore, we pass to

$$H_{[0]}(Z) = H_{<*>}(Z)/(H_{<*>}(Z))_{>0} = H^A_{<*>}(Z)/(H^A_{<*>}(Z))_{>0}, \quad A \in \{pt, T, G\}$$

and replace projective graded modules over $H^A_{[*]}(Z)$ by the additive category of simple modules over $H_{[0]}(Z)$, this equals the category $H_{[0]}(Z)-\mathrm{mod}$ of finite dimensional (ungraded) modules over $H_{[0]}(Z)$ because the algebra is semi-simple.
In particular, it holds
$H_{[0]}(Z) = \mathrm{Ext}^0_{D^b_A(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[d_i], \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[d_i]) = \mathrm{End}_{\mathcal{P}^A(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[d_i])$, $A \in \{pt, T, G\}$.

The following lemma is for classical Springer theory due to Dustin Clausen, cp. Thm 1.2 in [Cla08].

Lemma 7. *If the Springer map π is semi-small, we have the following version of the*

[9] This name is due to Dustin Clausen in his thesis.

Springer functor

$$\mathcal{S}\colon H_{[0]}(Z)-\mathrm{mod} \to \mathcal{P}^G(V)$$
$$M \mapsto \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i] \otimes_{H_{[0]}(Z)} M.$$

It holds that $\mathcal{S}$ is an exact functor (between abelian categories) and it is fully faithful. If $e_i + e_j$ is even for all $i, j \in I$ then $\mathcal{S}$ identifies $H_{[0]}(Z)$–mod with a semi-simple Serre subcategory of $\mathcal{P}^G(V)$ (i.e. it is an exact subcategory which is also extension closed and closed under direct summands). Furthermore it is invariant under Verdier duality on $\mathcal{P}^G(V)$.

Remark. Assume that the Springer map is semi-small, the image of the Springer map contains only finitely many G-orbits and each G-orbit is relavant and simply connected, then the Springer functor from above is an equivalence of categories. (The only known example for this is the classical Springer map for $G = \mathbf{Gl}_n$, see later.)

Proof: A similar proof as in the lemma above shows that the Springer functor induces an equivalence on the full subcategory of $\mathcal{P}^G(V)$ generated by finite direct sums of direct summands of $\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i]$. This is a semi-simple category. Assume that $e_i + e_j$ is even for all $i, j \in I$, we have to see that it is extension closed. By composition with the forgetful functor we get a functor

$$H_{[0]}(Z)-\mathrm{mod} \xrightarrow{\mathcal{S}} \mathcal{P}^G(V) \xrightarrow{F} \mathcal{P}^{pt}(V) =: \mathcal{P}(V),$$

by [Cla08] the forgetful functor F is fully faithful. Now, by [Ara01], 4.2.10 the category $\mathcal{P}(V)$ of $D^b(V)$ is closed under extensions and admissible because it is the heart of a t-structure. By the Riemann Hilbert correspondence there exists an abelian category $\mathcal{A}$ (= regular holonomic D-modules on V) and an equivalence of triangulated categories (= the de Rham functor)

$$DR_V\colon D^b(\mathcal{A}) \to D^b(V)$$

such that the standard t-structure on $D^b(\mathcal{A})$ is mapped to the perverse t-structure and it restricts to an equivalence of categories $\mathcal{A} \to \mathcal{P}(V)$. This implies that for $X \cong DR_V(X'), Y \cong DR_V(Y')$ in $\mathcal{P}(V)$ and $n \in \mathbb{N}_0$

$$\mathrm{Ext}^n_{\mathcal{P}(V)}(X,Y) = \mathrm{Ext}^n_{\mathcal{A}}(X',Y') = \mathrm{Hom}_{D^b(\mathcal{A})}(X',Y'[n]) = \mathrm{Hom}_{D^b(V)}(X,Y[n])$$

where the first and the third equality follows from the de Rham functor and the second equality holds because it is the standard t-structure, cp. for example [GM03], p.286.

Now, since we know

$$\mathrm{Hom}_{D^b(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i], (\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i])[1]) = H_{<1>}(Z) = \bigoplus_{i,j\in I} H_{e_i+e_j-1}(Z) = 0$$

because $H_{odd}(Z) = 0$ by lemma 1.3.0.1 and the assumption that $e_i + e_j$ is even for every

$i, j \in I$. We obtain that

$$\mathrm{Ext}^1_{\mathcal{P}(V)}(\bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i], \bigoplus_{i\in I}(\pi_i)_*\underline{\mathbb{C}}_{E_i}[e_i]) = 0,$$

i.e. the semi-simple category generated by the direct image of the Springer map is extension closed. □

The next section consists of concepts of classical Springer theory in the context of a more general collapsing of a homogeneous bundle.

1.6 Orbital varieties, Springer fibres and strata in the Steinberg variety

In this section we work over an arbitrary algebraically closed field K. In the example of classical Springer theory (see later) orbital varieties have been introduced by Spaltenstein in [Spa77]. He proved them to be in bijection to irreducible components of Springer fibres. This idea has been further applied by several authors (for example Reineke [Rei03], Vargas [Var79], Melnikov and Pagnon [MP06]). We give the analogue here to an arbitrary collapsing of a homogeneous bundle under the (reasonable) assumptions on the orbits $\mathcal{O}_v \subset V$ to be isomorphic to the quotients $G/\mathrm{Stab}(v)$, where $\mathrm{Stab}(v) = \{g \in G \mid gv = v\}$ is the stabilizer of v. This property can be characterized as follows.

We think this lemma is well-known (but we do not have a source for it).

Lemma 8. *Let G be an algebraic group over an algebraically closed field K. Let V be a G-scheme of finite type over K. Let $v \in V(K)$ and denote by $\mathcal{O}_v \subset V$ the orbit endowed with the reduced subscheme structure. Let $m\colon G \to \mathcal{O}_v, g \mapsto gv$ be the multiplication map. Then, the following are equivalent.*

1) *m induces an isomorphism $\mathcal{O}_v \cong G/\mathrm{Stab}(v)$.*

2) *m is separable.*

3) *$T_e m\colon T_eG \to T_v\mathcal{O}_v$ is surjective where $e \in G(K)$ is the neutral element.*

Furthermore, if one of the conditions is fulfilled the map m is open and closed.

Example. In characteristic zero, the map m is always separated. Also in the example of quiver-graded Springer theory (see a later section), for $\mathbf{Gl}_{\underline{d}}$-orbits in $\mathrm{R}_{\mathrm{Q}}(\underline{\mathrm{d}})$ the property 3) is true over any algebraically closed field because of Voigt's lemma [Gab75], Prop. 1.1

Definition 2. Let (G, P, V, F) be the construction data for a Springer theory (i.e. we assume the finite set I consists of a single element). Then the irreducible components of $\mathcal{O}_v \cap F$ are called **orbital varieties** (for v).

Lemma 9 (Reineke, [Rei03], Lemma 3.1)**.** *There is an isomorphism*

$$G \times^{\mathrm{Stab}(v)} \pi^{-1}(v) \cong \pi^{-1}(\mathcal{O}_v) \cong G \times^P (\mathcal{O}_v \cap F).$$

Proof: The first isomophism follows from lemma 77. Looking at π

$$\pi\colon G \times^P F \to V, (g, f) \mapsto gf$$

gives

$$\begin{aligned}\pi^{-1}(\mathcal{O}_v) &= \{(g, f) \in G \times^P F \mid gf \in \mathcal{O}_v\} \\ &= \{(g, f) \in G \times^P F \mid f \in \mathcal{O}_v\} \\ &= G \times^P (\mathcal{O}_v \cap F)\end{aligned}$$

□

We get the immediate corollary.

Corollary 1.6.0.1. *There is an isomorphism of equivariant Chow groups tensored with* $\mathbb{Q}$

$$A^{\mathrm{Stab}(v)}_{j+\dim \mathrm{Stab}(v)}(G \times \pi^{-1}(v)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong A^P_{j+\dim P}(G \times (\mathcal{O}_v \cap F)) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where equivariant Chow groups are meant in the sense of Edidin and Graham (see [EG98a]).

Secondly, there is a more intimate relation between the topology of associated fibre bundles and their fibres, we cite from Bongartz the following

Lemma 10. *([Bon98], Lemma 5.16) Let G be a connected algebraic group with a closed subgroup P. Let F be a quasi-projective P-variety. Then, the map*

$$U \mapsto G \times^P U$$

induces a bijection between P-invariant subvarieties of F and G-invariant subvarieties of $G \times^P F$. The bijection respects inclusions, closures and geometric properties like irreducibility, smoothness and normality.

This induces the bijection between the $\mathrm{Stab}(v)$-invariant subvarieties of $\pi^{-1}(v)$ and P-invariant subvarieties of $\mathcal{O}_v \cap F$.

Definition 3. Let (G, P, F, V) be the construction data for a Springer theory. A dense P-orbit in F will be called a **Richardson orbit**.

Corollary 1.6.0.2. *Let (G, P, F) be as in the previous lemma. Then the following are equivalent.*

(1) $G \times^P F$ has a dense G-orbit.

(2) F has a dense P-orbit.

If furthermore, V is a G-representation with $F \subset V$ a P-subrepresentation and there exists $v \in V$ such that $GF = \overline{\mathcal{O}_v} \subset V$, then (1) and (2) are also equivalent to

(3) $\pi^{-1}(v)$ has a dense $\mathrm{Stab}(v)$*-orbit.*

We will see later that condition (1) in quiver-graded Springer theory can be restated in terms of representation theory to an existence statement for a tilting module of a given dimension vector over a certain tensor algebra ($KQ \otimes K\mathbb{A}_\nu$, see chapter 6). There is one important case where we can always find a Richardson orbit.

Corollary 1.6.0.3. *Let (G, P, V, F) be the construction data for a Springer theory with G connected. Assume that there is $v \in V$ such $GF = \overline{\mathcal{O}_v}$ and $\pi^{-1}(v)$ consists of a single point. Then $F \cap \mathcal{O}_v$ is a Richardson orbit.*

Proof: Obviously (3) in the previous corollary is fulfilled, therefore also (1) and (2). As $\pi^{-1}(v)$ is a single Stab(v)-orbit, it follows that $F \cap \mathcal{O}_v$ is a single P-orbit. □

A special case of the previous corollary is the Richardson orbit theorem, cp. Carter [Car85a], 5.2, p.132.

Coming back to the lemma 10, we can realize this bijection between the Stab(v)-invariant subvarieties of $\pi^{-1}(v)$ and P-invariant subvarieties of $\mathcal{O}_v \cap F$ alternatively via the following construction given by Melnikov and Pagnon in [MP06], section 2.

Corollary 1.6.0.4. *Let (G, P, V, F) the construction data for a Springer theory. Let $v \in V$ be an element such that* Stab(v) *is connected. Let $G_v := \{g \in G \mid gv \in F\}$ and*

$$\mathcal{O}_v \cap F \xleftarrow{\;f_1\;} G_v \xrightarrow{\;f_2\;} \pi^{-1}(v) \qquad\qquad gv \;\leftarrow\!\!\shortmid\; g \;\mapsto\; gP$$

Then, the map $U \mapsto \Phi(U) := f_1(f_2^{-1}(U))$ realizes the bijection between Stab(v)*-invariant subvarieties of $\pi^{-1}(v)$ and P-invariant subvarieties of $\mathcal{O}_v \cap F$ respecting inclusion. It restricts to bijections*

1) between the irreducible components of $\pi^{-1}(v)$ and the irreducible components of $\mathcal{O}_v \cap F$. Furthermore, for any irreducible components C of $\pi^{-1}(v)$ it holds

$$\dim \Phi(C) = \dim C + \dim P - \dim \mathrm{Stab}(v).$$

2) Given two irreducible components C, C' of $\pi^{-1}(v)$ and $r \in \mathbb{N}_0$, there is an induced bijection between the irreducible components of $C \cap C'$ of fixed codimension r in C and irreducible components in $\Phi(C) \cap \Phi(C')$ of codimension r in $\Phi(C)$.

3) between Stab(v)*-orbits in $\pi^{-1}(v)$ and P-orbits in $\mathcal{O}_v \cap F$ respecting the degeneration order.*

Remark. In the later example of quiver-graded Springer theory, the stabilizers coincide with the groups $\mathrm{Aut}_{KQ}(M)$. The algebraic group $\mathrm{Aut}_{KQ}(M)$ is connected since it is open in the affine space $\mathrm{End}_{KQ}(M)$.

Relation between the Steinberg variety and orbital varieties

We fix (G, P, P', V, F, F') construction data for a Springer theory. We consider $(Z := (G \times^P F) \times_V (G \times^{P'} F'), m, p)$ with $m\colon Z \to G/P \times G/P', p\colon Z \to V$ defined as before. Assume furthermore that G operates on the image of p with finitely many orbits $\mathcal{O}_{v_1}, \ldots, \mathcal{O}_{v_r}$ and the stabilizers of $v_1, \ldots, v_r$ are irreducible and reduced. The image of p is the closure of the unique maximal dimensional orbit. Furthermore, the point $\{0\} \subset V$ is always the unique minimal dimensional orbit. We write

$$Z_{\mathcal{O}} := p^{-1}(\mathcal{O}), \quad \mathcal{O} \in \{\mathcal{O}_{v_1}, \ldots, \mathcal{O}_{v_r}\}$$

Then, for $\mathcal{O} = \mathcal{O}_v$, we have $Z_{\mathcal{O}} = G \times^{\mathrm{Stab}(v)} (\pi^{-1}(v) \times (\pi')^{-1}(v))$ with $\pi\colon G \times^P F \to V, \pi'\colon G \times^{P'} F' \to V$ the collapsing maps, together with the bijection between orbital varieties and irreducible components of the Springer fibre we have

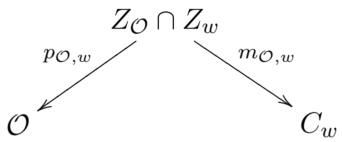
Remark. There are bijections between the following three sets

(1) Pairs Y_1, Y_2 of orbital varieties in $\mathcal{O} \cap F$ and $\mathcal{O} \cap F'$ respectively.

(2) $C_1 \times C_2$ irreducible components of $\pi^{-1}(v) \times (\pi')^{-1}(v)$

(3) Irreducible components of $Z_{\mathcal{O}}$.

We keep the notation of the previous subsections. Observe that we have a G-equivariant roof diagram



where $p_{\mathcal{O},w} := p|_{Z_{\mathcal{O}} \cap Z_w}, m_{\mathcal{O},w} := m|_{Z_{\mathcal{O}} \cap Z_w}$. Then by lemma 77, we directly get for $\mathcal{O} = \mathcal{O}_v$

Lemma 11.

$$G \times^{\mathrm{Stab}(v)} [(\pi^{-1}(v) \times (\pi')^{-1}(v)) \cap C_w] \cong Z_{\mathcal{O}} \cap Z_w \cong G \times^{P \cap w(P')} [F \cap w(F') \cap \mathcal{O}]$$

We get the immediate corollary.

Corollary 1.6.0.5. *There is an isomorphism of equivariant Chow groups tensored with* $\mathbb{Q}$

$$\begin{aligned} &A^{\mathrm{Stab}(v)}_{j+\dim \mathrm{Stab}(v)}(G \times [(\pi^{-1}(v) \times (\pi')^{-1}(v)) \cap C_w]) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &\cong A^{P\cap w(P')}_{j+\dim[P\cap w(P')]}(G \times [F \cap w(F') \cap \mathcal{O}]) \otimes_{\mathbb{Z}} \mathbb{Q}, \end{aligned}$$

where equivariant Chow groups are meant in the sense of Edidin and Graham (see [EG98a]).

As in the previous subsection, we get an inclusion preserving bijection between the $\mathrm{Stab}(v)$-invariant subvarieties of $(\pi^{-1}(v) \times (\pi')^{-1}(v)) \cap C_w$, G-invariant subvarieties of $Z_{\mathcal{O}} \cap Z_w$ and $P \cap w(P')$-invariant subvarieties of $F \cap w(F') \cap \mathcal{O}$.

1.7 What is Springer theory ?

One possible definition:

> Springer theory (for $(G, P_i, V, F_i)_{i \in I}$ and a choice of H) is to understand the Steinberg algebra together with its graded modules.

But I think today it is sensible to say Springer theory is the study of all categories and algebras (and modules over it) which have a construction originating in some Springer theory data $(G, P_i, V, F_i)_{i \in I}$. Then, this includes

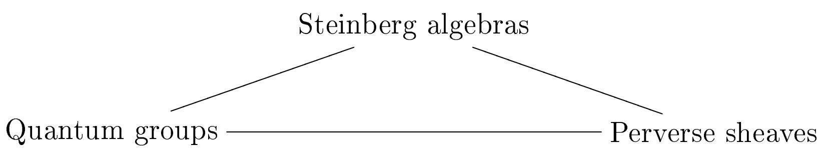
(1) Monoidal categories coming from multiplicative families of Steinberg algebras and their Grothendieck rings. In particular, this includes Lusztig's categories of perverse sheaves (see [Lus91] and the example quiver-graded Springer theory later).

(2) Noncommutative resolutions[10] corresponding to the Springer map. In particular, this includes Bezrukavnikov's noncommutative counterparts of the Springer map in [Bez06] and Buchweitz, Leuschke and van den Bergh's articles [BLB10] and [BLB11].

(3) Categories of flags of (KQ-)submodules for given quivers because their isomorphism classes parametrize orbits of (quiver-graded) Springer fibres. This includes for example Ringel's and Zhang's work on submodule categories and preprojective algebras [RZ12]. Also certain Δ-filtered modules studied in [BHRR99], [BH00b].

An (of course) incomplete overview can be found in the flowchart at the end of this article.

We would also like you to observe that in the two examples we explore connections between objects roughly related to the following triangle



1.8 Classical Springer theory

This is the case of the following initial data

- $(*)$ G an arbitrary reductive group,
- $(*)$ $P = B$ a Borel subgroup of G, denote its Levi decomposition by $B = TU$ with T maximal torus, U unipotent.
- $(*)$ $V = \mathfrak{g}$ the adjoint representation,
- $(*)$ $F = \mathfrak{n} := \mathrm{Lie}(U)$.

[10] *here*: This means just a tilting vector bundle on E, because this gives t-structures in the category of coherent sheaves on E

We set $\mathcal{N} := G\mathfrak{n}$, i.e. the image of the Springer map, and call it the nilpotent cone. We consider the Springer map as $\pi\colon E = G \times^B \mathfrak{n} \to \mathcal{N}$. Explicitly, we can write the Springer triple as

$$\mathcal{N} \xleftarrow{\pi=pr_1} E = \{(n, gB) \in \mathcal{N} \times G/B \mid n \in {}^g\mathfrak{b} := \mathrm{Lie}(gBg^{-1})\} \xrightarrow{\mu=pr_2} G/B$$

For $G = \mathbf{Gl}_n$ we identify $\mathbf{Gl}_n/B$ with the variety Fl_n of complete flags in $\mathbb{C}^n$ and

$$E = \{(A, U^\bullet) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \times Fl_n \mid A^n = 0, A(U^k) \subset U^k,\ 1 \leq k \leq n\}.$$

It turns out, π can be identified with the moment map of G, in particular, $E \cong T^*(G/B)$ is the cotangent bundle over G/B and π is a resolution of singularities for $\mathcal{N}$. But most importantly, this makes the Springer map a symplectic resolution of singularities and one can use symplectic geometry to study it (see for example [CG97]).
The Steinberg variety is given by

$$\mathcal{N} \xleftarrow{p=pr_1} Z = \{(n, gB, hB) \in \mathcal{N} \times G/B \times G/B \mid n \in {}^g\mathfrak{b} \cap {}^h\mathfrak{b}\} \xrightarrow{m=pr_{2,3}} G/B \times G/B$$

For $G = \mathbf{Gl}_n$ we can write it as

$$\begin{aligned} Z = \{(A, U^\bullet, V^\bullet) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \times Fl_n \times Fl_n \mid \\ A^n = 0, A(U^k) \subset U^k, A(V^k) \subset V^k,\ 1 \leq k \leq n\}. \end{aligned}$$

Recall, that we had the stratification by relative position $Z^w := m^{-1}(G\cdot(eB, wB)), w \in W$ where W is the Weyl group of G with respect to a maximal torus $T \subset B$. Since $Z^w \to G \cdot (eB, wB)$ is a vector bundle, we can easily calculate its dimension

$$\begin{aligned} \dim Z^w &= \dim G \cdot (eB, wB) + \dim \mathfrak{n} \cap {}^w\mathfrak{n} \\ &= \dim G - \dim B \cap {}^wB + \dim \mathfrak{n} \cap {}^w\mathfrak{n} = \dim G - \dim T \\ &= \dim E. \end{aligned}$$

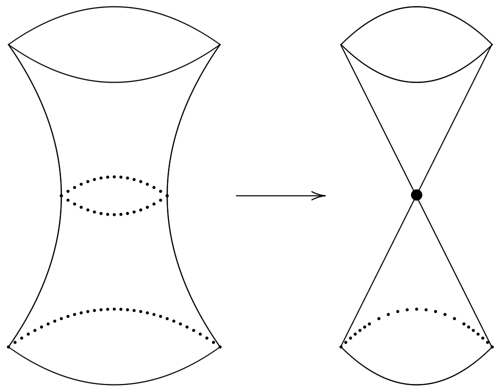
We conclude that Z is equidimensional of dimension $e := \dim E$, in particular the Springer map is semi-small. Also we see that the irreducible components of Z are given by $\overline{Z^w}, w \in W$, that implies that the top-dimensional Borel-Moore homology group $H_{top}(Z)$ has a $\mathbb{C}$-vector space basis given by the cycles $[\overline{Z^w}]$. In the semi-small case we know $H_{[0]}(Z) = H_{<0>}(Z) = H_{top}(Z)$ is a sub- and quotient algebra of $H_*(Z)$.

Example. $G = \mathbf{Sl}_2$, $B = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid b \in \mathbb{C}, a \in \mathbb{C} \setminus \{0\}\}$. Then $\mathcal{N} = \{(x, y, z) \in \mathbb{C}^3 \mid$

$x^2 + zy = 0\}$ and

$$\begin{aligned} E &= \{(A, L) \in \mathcal{N} \times \mathbb{P}^1 \mid L \subset \ker A\} \\ &= \{(\left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right), [a\colon b]) \in M_2(\mathbb{C}) \times \mathbb{P}^1 \mid\ x^2 + zy = 0, xa + yb = 0, za - xb = 0\}, \end{aligned}$$

the Springer map can be seen as the following picture



This is well-known to be the crepant resolution of the A_2-singularity from the MacKay correspondence. In general, if G is semisimple of type ADE, then there exists a slice of the nilpotent cone such that the restricted map is the crepant resolution of the corresponing type singularity, see [Slo80b] for more details.

Theorem 1.8.1. *(roughly Springer [Spr76]) There is an isomorphism of* $\mathbb{C}$*-algebras*

$$\begin{aligned} H_{top}(Z) &\cong \mathbb{C}[W] \\ [\overline{Z^s}] &\mapsto s - 1 \end{aligned}$$

The Springer functor (due to Clausen, [Cla08]) takes the form

$$\begin{aligned} \mathbb{C}W-\text{mod} &\to \mathcal{P}^G(\mathcal{N}) \\ M &\mapsto \pi_* \underline{\mathbb{C}}[e] \otimes_{\mathbb{C}W} M \end{aligned}$$

and identifies $\mathbb{C}W-$mod with a semi-simple Serre subcategory of $\mathcal{P}^G(\mathcal{N})$. This implies an injection on simple objects, which are in $\mathcal{P}^G(\mathcal{N})$ the intersection cohomology complexes associated to $(\mathcal{O}, \mathcal{L})$ with $\mathcal{L}$ a simple G-equivariant local system on an G-orbit $\mathcal{O} \subset \mathcal{N}$. As a consequence we get the bijection called *Springer correspondence* from thm 1.4.2

$$\begin{aligned} \text{Simp}(W) &\leftrightarrow \{t = (\mathcal{O}, \mathcal{L}) \mid \quad L_t \neq 0\} \\ &= \{(s, \chi) \mid s \in \mathcal{N} \text{ rep of } G\text{-orbits }, \chi \in \text{Simp}(C(s)), (H_{top}(\pi^{-1}(s)))_\chi \neq 0\} \end{aligned}$$

where $L_t = \bigoplus_d L_{t,d}$ is the multiplicity vector space in the BBD-decomposition of $\pi_* \underline{\mathbb{C}}$ and $\text{Simp}(W)$ is the set of isomorphism classes of simple objects in $\mathbb{C}W-$mod. The inverse of the map is given by $(s, \chi) \mapsto (H_{top}(\pi^{-1}(s)))_\chi$. For this Springer map all orbits in $\mathcal{N}$ are relevant, i.e. we also have an injection

$$\{G\text{-orbits in } \mathcal{N}\} \to \operatorname{Simp}(W)$$
$$Gs \mapsto H_{top}(\pi^{-1}(s))^{C(s)}$$

Remark. We remark that there are several alternative constructions of the group operation of W on the Borel-Moore homology/ singular cohomology of the Springer fibres. In [Ara01], section 5.5 you find an understandable treatment of Lusztig's approach to this operation using intermediate extensions for perverse sheaves and Arabia provides a list of other authors and approaches to this (first Springer [Spr76],[Spr78], then Kazhdan-Lusztig [KL80], Slodowy [Slo80a], Lusztig [Lus81], Rossmann [Ros91]) and these operations differ between each other by at most multiplication with a sign character (see [Hot81]).
Also, Springer proves with taking (co)homology of Springer fibres with rational coefficients that the simple W-representations are all even defined over $\mathbb{Q}$, a result which our approach does not give because the simple $C(s)$-modules are not necessarily all defined over $\mathbb{Q}$ (cp. [CG97], section 3.5, p.170). In Carter's book [Car85a], p. 388, you find for simple adjoint groups the component groups $C(s), s \in \mathcal{N}$ are one of the following list $(\mathbb{Z}/2\mathbb{Z})^r, \quad S_3, S_4, S_5, \quad r \in \mathbb{N}_0$ as a consequence he gets that the simple modules over the group ring are already defined over $\mathbb{Q}$.

In the introduction of the book [BBM89] you find for a semisimple group G a triangle

$$\begin{array}{ccc} & \text{simple } \mathbb{C}W\text{-modules} & \\ \nearrow & & \nwarrow \\ \text{primitive ideals in } U(\mathfrak{g}) & \longrightarrow & G\text{-orbits in the nilpotent cone} \end{array}$$

They explain it as follows (i.e. this is a summary of a their summary).

* There is an injection of G-orbits in $\mathcal{N}$ into simple $\mathbb{C}W$-modules by the Springer correspondence.

* A primitive ideal in $U(\mathfrak{g})$ is a kernel of some simple $U(\mathfrak{g})$-representation. The classification of primitive ideals is archieved as a result of the proof of the Kazhdan-Lusztig conjectures (see Beilinson-Bernstein [BB81], Brylinski-Kashiwara [BK81]). Any ideal in $U(\mathfrak{g})$ has an associated subvariety of $\mathfrak{g}$. The associated variety of a primitive ideal is the closure of an orbit in $\mathcal{N}$, this was first conjectured by Borho and Jantzen.

* Joseph associated to a primitive ideal a W-harmonic polynomial in $\mathbb{C}[\mathfrak{t}]$ (=Goldie rank polynomial) which is a basis element of one of the simple $\mathbb{C}W$-modules.

We also have to mention the following important results which use K-theory instead of Borel-Moore homology.

1.8.1 Parametrizing simple modules over Hecke algebras.

This field goes back to the work of Kazhdan and Lusztig on the proof of the Deligne-Langlands conjecture for Hecke algebras, see [KL87]. They realize simple modules over Iwahori Hecke algebra as Grothendieck groups of equivariant (with respect to certain groups) coherent sheaves on the Springer fibres. This is now known as Deligne-Langlands correspondence and we call similar results which come later for different Hecke algebras still DL-correspondence.
Let G be an algebraic group and X a G-variety, let $K_0^G(X) := K_0(coh^G(X))$ be the Grothendieck group of the category of G-equivariant coherent sheaves on X. The group $\mathbb{C}^*$ operates on the (classical) Steinberg variety Z via $(n, gB, hB) \cdot t := (t^{-1}n, gB, hB)$, the convolution product construction gives a ring structure on $K_0^{G\times\mathbb{C}^*}(Z)$.
Recall, for a reductive group we fix a maximal torus and a Borel subgroup $T \subset B \subset G$ and call (W, S) the associated Weyl group with set of simple roots. We write $X(T) = \mathrm{Hom}(T, \mathbb{C})$ as an additive group and have for $Y(T) = \mathrm{Hom}(\mathbb{C}^*, T)$ the natural perfect pairing $\langle -, - \rangle \colon X(T) \times Y(T) \to \mathbb{Z}, (\lambda, \sigma) \mapsto m$ with $\lambda \circ \sigma(z) = z^m, z \in \mathbb{C}^*$. For the definition of the dual roots $\alpha_s^* \in Y(T), s \in S$ see [CG97], chapter 7.1, p.361.

Theorem 1.8.2. *([CG97], thm 7.2.5, thm 8.1.16 - DL-corresp. for affine Hecke algebras) Let G be a connected, simply connected semi-simple group over $\mathbb{C}$.*

(a) It holds $K_0^{G\times\mathbb{C}^*}(Z) \cong \mathcal{H}$ where $\mathcal{H}$ is the affine Hecke algebra associated to (W, S), i.e., the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\{e^\lambda T_w \mid w \in W, \lambda \in X(T), e^0 = 1\}$ with relations

(i) $(T_s + 1)(T_s - q) = 0, s \in S$, and $T_x T_y = T_{xy}$ for $x, y \in W$ with $\ell(xy) = \ell(x)\ell(y)$.

(ii) The $\mathbb{Z}[q, q^{-1}]$-subalgebra spanned by e^λ is isomorphis to $(\mathbb{Z}[q^\pm])[X_1^\pm, \ldots, X_n^\pm]$, $n = rk(T)$.

(iii) For $\langle \lambda, \alpha_s^* \rangle = 0$ it holds $T_s e^\lambda = e^\lambda T_s$.
For $\langle \lambda, \alpha_s^* \rangle = 0$ it holds $T_s e^\lambda T_s = q e^\lambda$.

(b) The operation of $\mathcal{H}$ on a simple module factors over $H_*(Z^a)$, with $a = (g, t) \in G \times \mathbb{C}^*$ a semisimple element, in particular is $H_*((\pi^{-1}(s))^g)$ via the convolution construction a $\mathcal{H}$- module. The operation of the component group
$C(a) = \mathrm{Stab}_{G\times\mathbb{C}^*}(a)/(\mathrm{Stab}_{G\times\mathbb{C}^*}(a))^o$ on $H_*(\pi^{-1}(s)^g)$ commutes with the $H_*(Z^a)$-operation and gives

$$H_*(\pi^{-1}(s)^g) = \bigoplus_{\chi \in \mathrm{Simp}(C(a))} K_{a,x,\chi} \otimes \chi$$

for some $H_*(Z^a)$-modules $K_{a,x,\chi}$ which are called ***standard modules****.*
If $t \in \mathbb{C}$ is not a root of unity, then there is a(n explicit) bijection between

(1) $\{G - conj.\ cl.\ of\ (g, x, \chi) \mid g \in G^{ss},\ gxg^{-1} = tx, \chi \in \mathrm{Simp}(C(g, t)), K_{(s,g),x,\chi} \neq 0\}$,
where $G^{ss} \subset G$ denote the semisimple elements.

(2) Simple $\mathcal{H}$-modules where q acts by multiplication with t.

All simples are constructed from the standard modules, in general it is difficult to determine when the candidates are nonzero. For t a root of unity there is an injection of the set (2) in (1).

1.9 Quiver-graded Springer theory

Let Q be a finite quiver with set of vertices Q_0 and set of arrows Q_1. Let us fix a dimension vector $\underline{d} \in \mathbb{N}_0^{Q_0}$ and a sequence of dimension vectors $\underline{\mathbf{d}} := (0 = \underline{d}^0, \dots, \underline{d}^\nu =: \underline{d})$, $\underline{d}_i^k \leq \underline{d}_i^{k+1}$ for all $i \in Q_0$. Quiver-graded Springer theory arises from the following initial data

$$\left[\begin{array}{l}
(*)\ G = \mathbf{Gl}_{\underline{d}} := \prod_{i \in Q_0} \mathbf{Gl}_{d_i}, \\
(*)\ P = P(\underline{\mathbf{d}}) := \prod_{i \in Q_0} P(d_i^\bullet) \text{ where } P(d_i^\bullet) \text{ is the parabolic in } \mathbf{Gl}_{d_i} \text{ fixing a} \\
\qquad \text{(standard) flag } V_i^\bullet \text{ in } \mathbb{C}^{d_i} \text{ with dimensions given by } d_i^\bullet, \\
(*)\ V = \mathrm{R_Q}(\underline{\mathrm{d}}) := \prod_{(\mathrm{i} \to \mathrm{j}) \in \mathrm{Q}_1} \mathrm{Hom}(\mathbb{C}^{\mathrm{d_i}}, \mathbb{C}^{\mathrm{d_j}}) \text{ with the operation } (\mathrm{g_i})(\mathrm{M_{i \to j}}) = (\mathrm{g_j M_{i \to j} g_i^{-1}}) \\
\qquad \text{is called representation space.} \\
(*)\ F = F(\underline{\mathbf{d}}) := \{(M_{i \to j}) \in \mathrm{R_Q}(\underline{\mathrm{d}}) \mid \mathrm{M_{i \to j}}(\mathrm{V_i^k}) \subset \mathrm{V_j^k},\ \ 0 \leq \mathrm{k} \leq \nu\}
\end{array}\right.$$

Given $\underline{d}$ and an (arbitrary) finite set $I := \{\underline{\mathbf{d}} = (0 = \underline{d}^0, \dots, \underline{d}^\nu) \mid \nu \in \mathbb{N}, \underline{d}^\nu = \underline{d}\}$, we can describe the quiver-graded Springer correspondence explicitly for $\underline{\mathbf{d}} \in I$ via

$$E_{\underline{\mathbf{d}}} = \{(M, U^\bullet) \in \mathrm{R_Q}(\underline{\mathrm{d}}) \times \mathrm{Fl}_{\underline{\mathbf{d}}} \mid \ \mathrm{i} \xrightarrow{\forall \alpha} \mathrm{j} \in \mathrm{Q}_1 \colon \mathrm{M}_\alpha(\mathrm{U_i^k}) \subset \mathrm{U_j^k},\ 1 \leq \mathrm{k} \leq \nu\}$$

$$\mathrm{R_Q}(\underline{\mathrm{d}}) \xleftarrow{pr_1} E_{\underline{\mathbf{d}}} \xrightarrow{pr_2} \mathrm{Fl}_{\underline{\mathbf{d}}}$$

where $\mathrm{Fl}_{\underline{\mathbf{d}}} = \prod_{\mathrm{i} \in \mathrm{Q}_0} \mathrm{Fl}_{\underline{\mathbf{d}}_\mathrm{i}}$ and $\mathrm{Fl}_{\underline{\mathbf{d}}_\mathrm{i}}$ is the variety of flags of dimension $(0, d_i^1, d_i^2, \dots, d_i^\nu = d_i)$ inside $\mathbb{C}^{d_i}$ and we set $E := \bigsqcup_{\underline{\mathbf{d}} \in I} E_{\underline{\mathbf{d}}}$,

$$\begin{aligned}
Z_{\underline{\mathbf{d}}, \underline{\mathbf{d}}'} &:= E_{\underline{\mathbf{d}}} \times_{\mathrm{R_Q}(\underline{\mathrm{d}})} E_{\underline{\mathbf{d}}'} \\
&\{(M, U^\bullet, V^\bullet) \in \mathrm{R_Q}(\underline{\mathrm{d}}) \times \mathrm{Fl}_{\underline{\mathbf{d}}} \times \mathrm{Fl}_{\underline{\mathbf{d}}'} \mid \mathrm{i} \xrightarrow{\forall \alpha} \mathrm{j} \in \mathrm{Q}_1 \colon \mathrm{M}_\alpha(\mathrm{U_i^k}) \subset \mathrm{U_j^k}, \mathrm{M}_\alpha(\mathrm{V_i^k}) \subset \mathrm{V_j^k}\}
\end{aligned}$$

$$\mathrm{R_Q}(\underline{\mathrm{d}}) \xleftarrow{pr_1} Z_{\underline{\mathbf{d}}, \underline{\mathbf{d}}'} \xrightarrow{pr_{2,3}} \mathrm{Fl}_{\underline{\mathbf{d}}} \times \mathrm{Fl}_{\underline{\mathbf{d}}'}$$

and the Steinberg variety is $Z := \bigsqcup_{\underline{\mathbf{d}}, \underline{\mathbf{d}}' \in I} Z_{\underline{\mathbf{d}}, \underline{\mathbf{d}}'}$. This description goes back to Lusztig (cp.

for example [Lus91]). It holds

$$\dim E_{\underline{\mathbf{d}}} = \dim \mathrm{Fl}_{\underline{\mathbf{d}}} + \dim \mathrm{F}(\underline{\mathbf{d}}) = \sum_{\mathrm{i}\in \mathrm{Q}_0} \sum_{\mathrm{k}=1}^{\nu-1} \mathrm{d}_\mathrm{i}^\mathrm{k}(\mathrm{d}_\mathrm{i}^{\mathrm{k}+1} - \mathrm{d}_\mathrm{i}^\mathrm{k}) + \sum_{(\mathrm{i}\to \mathrm{j})\in \mathrm{Q}_1} \sum_{\mathrm{k}=1}^{\nu} (\mathrm{d}_\mathrm{i}^\mathrm{k} - \mathrm{d}_\mathrm{i}^{\mathrm{k}-1})\mathrm{d}_\mathrm{j}^\mathrm{k},$$

We define $\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle := \dim \mathbf{Gl}_{\underline{d}} - \dim E_{\underline{\mathbf{d}}}$ and when Q is without oriented cycles this is the Tits form for the algebra $\mathbb{C}Q \otimes \mathbb{C}\mathbb{A}_{\nu+1}$ (cp. [Wol09], Appendix)

$$\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle = \sum_{k=0}^{\nu} \langle \underline{d}^k, \underline{d}^k \rangle_{\mathbb{C}Q} - \sum_{k=0}^{\nu-1} \langle \underline{d}^k, \underline{d}^{k+1} \rangle_{\mathbb{C}Q}.$$

Let $\{\underline{\mathbf{d}}_i \mid i \in I\}$ be the set of complete dimension filtrations of a given dimension vector $\underline{d}$. The ($\mathbf{Gl}_{\underline{d}}$-equivariant) Steinberg algebra is the quiver Hecke algebra (for $Q, \underline{d}$). If the quiver Q has no loops, the image of the injective map from lemma 2 has been calculated by Varagnolo and Vasserot in [Var09]. With generators and relations of the algebra they check that this is the same algebra as has been introduced by Khovanov and Lauda in [KL09] (which was previously conjectured by Khovanov and Lauda). Independently, this has been proven by Rouquier in [Rou11].

Theorem 1.9.1. *(quiver Hecke algebra, [Var09], [Rou11]) Let Q be a quiver without loops and $\underline{d} \in \mathbb{N}_0^{Q_0}$ be a fixed dimension vector. The ($\mathbf{Gl}_{\underline{d}}$-equivariant) quiver-graded Steinberg algebra for complete dimension filtrations $R_{\underline{d}}^G := H_*^G(Z)$ for $(Q, \underline{d})$ is as graded $\mathbb{C}$-algebra generated by*

$$1_i, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d, \quad \sigma_i(s), i \in I, s \in \{(1,2),(2,3),\ldots,(d-1,d)\} =: \mathbb{S},$$

where $d := \sum_{a\in Q_0} d_a$, $I := I_{\underline{d}} := \{(i_1, \ldots, i_d) \mid \ i_k \in Q_0, \sum_{k=1}^{d} i_k = \underline{d}\}$ and we see $\mathbb{S} \subset S_d$ as permutations of $\{1, \ldots, d\}$, we also define

$$h_i((\ell, \ell+1)) = h_{i_{\ell+1}, i_\ell} = \#\{\alpha \in Q_1 \mid \alpha\colon i_{\ell+1} \to i_\ell\}$$

and let

$$\deg 1_i = 0, \ \ \deg z_i(k) = 2, \ \ \deg \sigma_i((\ell,\ell+1)) = \begin{cases} 2h_i((\ell,\ell+1)) - 2 & , \textit{ if } i_\ell = i_{\ell+1} \\ 2h_i((\ell,\ell+1)) & , \textit{ if } i_\ell \neq i_{\ell+1} \end{cases}$$

For $s = (k, k+1)$, $i = (i_1, \ldots, i_d)$ we write $is := (i_1, \ldots, i_{k+1}, i_k, \ldots, i_d)$.
The following relations hold.

*(1) (*orthogonal idempotents*)*

$$1_i 1_j = \delta_{i,j} 1_i,$$
$$1_i \sigma_i(s) 1_{is} = \sigma_i(s)$$
$$1_i z_i(k) 1_i = z_i(k)$$

(2) *(*polynomial subalgebras*)*

$$z_i(k)z_i(k') = z_i(k')z_i(k)$$

(3) For $s = (k, k+1)$, $i = (i_1, \ldots, i_d)$ we set

$$\alpha_s := \alpha_{i,s} := z_i(k) - z_i(k+1)$$

if it is clear from the context which i is meant.

$$\sigma_i(s)\sigma_{is}(s) = \begin{cases} 0, & \text{if } is = i \\ (-1)^{h_{is}(s)}\alpha_s^{h_i(s)+h_{is}(s)}, & \text{if } is \neq i. \end{cases}$$

(4) *(*straightening rule*)*
For $s = (\ell, \ell+1)$ we set

$$s(z_i(k)) = \begin{cases} z_i(k+1), & \text{if } k = \ell \\ z_i(k-1), & \text{if } k = \ell + 1 \\ z_i(k), & \text{else.} \end{cases}$$

$$\sigma_i(s)z_{is}(k) - s(z_{is}(k))\sigma_i(s) = \begin{cases} -1_i, & \text{if } is = i, s = (k, k+1) \\ 1_i, & \text{if } is = i, s = (k-1, k) \\ 0, & , \text{ if } is \neq i. \end{cases}$$

(5) *(*braid relation*)*
Let $s, t \in \mathbb{S}, st = ts$, then

$$\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s).$$

Let $i \in I$, $s = (k, k+1), t = (k+1, k+2)$. We set $s(\alpha_t) := (z_i(k) - z_i(k+2)) =: t(\alpha_s)$

$$\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t) = \begin{cases} P_{s,t} & \text{if } ists = i, is \neq i, it \neq i \\ 0, & \text{else.} \end{cases}$$

where

$$P_{s,t} := \alpha_s^{h_i(s)} \frac{\alpha_t^{h_{is}(s)} - (-1)^{h_{is}(s)}\alpha_s^{h_{is}(s)}}{\alpha_s + \alpha_t} - \alpha_t^{h_{is}(s)} \frac{\alpha_s^{h_i(s)} - (-1)^{h_i(s)}\alpha_t^{h_i(s)}}{\alpha_s + \alpha_t}$$

is a polynomial in $z_i(k), z_i(k+1), z_i(k+2)$.

We call this the ***quiver Hecke algebra*** *for $Q, \underline{d}$.*

Using the degeneration of the spectral sequence argument from lemma 1.3.0.1 we get

Corollary 1.9.1.1. *Let Q be a quiver without loops and $\underline{d} \in \mathbb{N}_0^{Q_0}$. The not-equivariant*

Steinberg algebra $R_{\underline{d}} := H_{[*]}(Z)$ *is the graded* $\mathbb{C}$*-algebra generated by*

$$1_i, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d \quad \sigma_i(s), i \in I, s \in \{(1,2),(2,3),\dots,(d-1,d)\}$$

with the same degrees and relations as $R^G_{\underline{d}}$ *and the additional relations*

$$P(z_i(1), \dots, z_i(n)) = 0, \quad i \in I, \text{ for all } P \in \mathbb{C}[x_1, \dots x_d]^{S_{\underline{d}}}.$$

where $S_{\underline{d}} = S_{d_{i_1}} \times \cdots \times S_{d_{i_n}}$ *with* $Q_0 = \{i_1, \dots, i_n\}$ *is the Weyl group of* $G = \mathbf{Gl}_{\underline{d}}$

What about Springer fibre modules and the decomposition theorem?

This is not investigated yet. We make some remarks on it.

Remark. (1) If Q is a Dynkin quiver[11], the images of all quiver-graded Springer maps have finitely many orbits. For all quiver Q and dimension vector $\underline{d} \in \mathbb{N}_0^{Q_0}$ all $\mathbf{Gl}_{\underline{d}}$-orbits in $\mathrm{R_Q}(\underline{\mathrm{d}})$ are connected, i.e. $C(s) = \{e\}$ for all $x \in \mathrm{R_Q}(\underline{\mathrm{d}})$.

(2) In the case of finitely many orbits in the image of the Springer map, semi-smallness of the Springer map (associated to a dimension filtration $\underline{\mathbf{d}}$ of a dimension vector $\underline{d}$) is equivalent to for every $s \in \mathrm{R_Q}(\underline{\mathrm{d}})$ it holds

$$2 \dim \pi_{\underline{\mathbf{d}}}^{-1}(s) \leq \dim \mathrm{Ext}^1_{\mathbb{C}Q}(s,s) = \mathrm{codim}_{\mathrm{R_Q}(\underline{\mathrm{d}})}\, Gs.$$

It is very rarely fulfilled.

(3) If Q is a Dynkin quiver and $\underline{d} \in \mathbb{N}_0^{Q_0}$ a complete set of the isomorphism classes of simple modules for the quiver Coxeter algebra $R_{\underline{d}}$ is parametrized by the $G := \mathbf{Gl}_{\underline{d}}$-orbits in $\mathrm{R_Q}(\underline{\mathrm{d}})$. For $s \in \mathrm{R_Q}(\underline{\mathrm{d}})$ we have a simple module of the form

$$L_{Gs} := \bigoplus_{\underline{\mathbf{d}}} L^{(\underline{\mathbf{d}})}_{Gs}$$

where $\underline{\mathbf{d}}$ runs over all complete dimension filtrations of $\underline{d}$ and $L^{(\underline{\mathbf{d}})}_{Gs}$ is the multiplicity vector space occurring in the decomposition of $(\pi_{\underline{\mathbf{d}}})_* \underline{\mathbb{C}}[e_{\underline{\mathbf{d}}}]$. By the work of Reineke (see [Rei03]) there exists for every $s \in \mathrm{R_Q}(\underline{\mathrm{d}})$ a complete dimension filtration $\underline{\mathbf{d}}$ such that Gs is dense in the image of $\pi_{\underline{\mathbf{d}}}$. This implies by the considerations from subsection 1.4.3 that

$$L^{(\underline{\mathbf{d}})}_{Gs,-*}(-d_{Gs}) = H_{[*]}(\pi_{\underline{\mathbf{d}}}^{-1}(s)) \quad (\neq 0),$$

as graded vector spaces, where $d_{Gs} = \dim Gs$. In fact, Reineke even shows that there exists a $\underline{\mathbf{d}}$ for every x such that the Springer map is a bijection over Gs, in which case $\dim L^{(\underline{\mathbf{d}})}_{Gs} = 1$.

[11] i.e. the underlying graph is a Dynkin diagram of type $A_n, D_n, E_{6/7/8}$.

For Q Dynkin, there are parametrizations of indecomposable graded projective modules in terms of Lyndon words, see [HMM12], which are not yet understood in the context of the decomposition theorem.

1.9.1 Monoidal categorifications of the negative half of the quantum group

Again let Q be a finite quiver without loops. First Lusztig found the monoidal categorification of the negative half of the quantum group via perverse sheaves, then Khovanov and Lauda did the same with (f.g. graded) projective modules over quiver Hecke algebras. In the following theorem's we are explaining the following diagrams of isomorphisms of twisted Hopf algebras over $\mathbb{Q}(q)$.

$$\begin{array}{ccc} & K_0(proj^{\mathbb{Z}} \bigoplus H_*^G(Z)) \otimes \mathbb{Q}(q) & \\ \swarrow & & \searrow \\ \mathcal{U}^- := U_q^-(Q) & \longleftarrow & K_0(\mathcal{P}) \otimes \mathbb{Q}(q) \end{array}$$

In all three algebras there exists a notion of canonical basis which is mapped to each other under the isomorphisms. Also, there is a triangle diagram with isomorphisms defined over $\mathbb{Z}[q, q^{-1}]$ which gives the above situation after applying $- \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$.

The negative half of the quantum group.

The negative half $\mathcal{U}^- := U_q^-(Q)$ of the quantized enveloping algebra (defined by Drinfeld and Jimbo) associated to the quiver Q is defined via: Let $a_{i,j} := \#\{\alpha \in Q_1 \mid \alpha\colon i \to j, \text{ or } \alpha\colon j \to i\}, i \neq j \in Q_0$. It is the $\mathbb{Q}(q)$-algebra generated by $F_i, i \in Q_0$ with respect to the (quantum Serre relations)

$$\sum_{p=0}^{N+1} [p, N+1-p] F_j^p F_i F_j^{N+1-p} = 0, \quad N = a_{ij}, i \neq j$$

where

$$[n]_! := \prod_{k=1}^{n} \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [n, m] := \frac{[n+m]_!}{[n]_! [m]_!}.$$

Lusztig calls this $'\mathbf{f}$

A Hopf algebra is a bialgebra (i.e. an algebra which also has the structure of a coalgebra such that the comultiplication and counit are algebra homomorphisms) which also has an antipode, i.e. an anti-automorphism which is uniquely determined by the bialgebra through commuting diagrams. A twisted Hopf algebra differs from the Hopf algebra by: The comultiplication and the antipode are only homomorphisms if you *twist* the algebra structure by a bilinear form (see the example below). For more details on the definition see [LZ00]. The twisted $\mathbb{Q}(q)$-Hopf algebra structure is given by the following, it is by definition a $\mathbb{Q}(q)$-algebra which is $\mathbb{N}^{Q_0}$-graded and it has

(1) (comultiplication)
If we give $\mathcal{U}^- \otimes_{\mathbb{Q}(q)} \mathcal{U}^-$ the algebra structure

$$(x_1 \otimes x_2)(x_1' \otimes x_2') := q^{|x_2|\cdot|x_1'|} x_1x_1' \otimes x_2x_2'$$

where for $x \in \mathcal{U}^-$ we write $|x| \in \mathbb{N}_0^{Q_0}$ for its degree and the symmetric bilinear form

$$\cdot\colon \mathbb{Z}_0^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}, \;\; i \cdot i := 2, \; i \cdot j := -a_{i,j} \text{ for } i \neq j$$

Then the comultiplication is the $\mathbb{Q}(q)$ algebra homomorphism

$$\mathcal{U}^- \to \mathcal{U}^- \otimes \mathcal{U}^-, \;\; F_i \mapsto F_i \otimes 1 + 1 \otimes F_i$$

(2) (counit) $\epsilon\colon \mathcal{U}^- \to \mathbb{Q}(q), \;\; F_i \mapsto 0$

(3) (antipode)
Let $\mathcal{U}^-_{tw}$ be the algebra with the multiplication $x * y := q^{|y|\cdot|x|}xy$
The antipode is the algebra anti-homomorphism

$$\mathcal{U}^- \to \mathcal{U}^-_{tw}, \;\; F_i \mapsto -F_i$$

Lusztig's category of perverse sheaves.

Lusztig writes complete dimension filtrations as words in the vertices $i = (i_1, \ldots, i_d)$, $i_t \in Q_0$. We set $\underline{d} := \sum_{t=1}^d i_t$ and define

$$L_i := (\pi_i)_* \underline{\mathbb{C}}[e_i]$$

where $\pi_i\colon E_i := \mathbf{Gl}_{\underline{d}} \times^{P_i} F_i \to R_Q(\underline{d})$ is the quiver-graded Springer map and $e_i = \dim_{\mathbb{C}} E_i$. Let us call $\mathcal{P}_{Q_0}$ the additive category generated by shifts of the L_i, $i = (i_1, \ldots, i_d), i_t \in Q_0$. The set $\mathrm{Hom}(L_i, L_j[n])$ in this category is zero unless $\underline{d} = \sum i_t = \sum j_k$ and then it is given by $1_j * H^{\mathbf{Gl}_{\underline{d}}}_{[n]}(Z) * 1_i$. The category can be endowed with the structure of a monoidal category via

$$L_i * L_j := L_{ij}$$

where ij is the concatenation of the sequence i and then j.

Lemma 12. *(Lusztig, [Lus91], Prop. 7.3) Let $\mathcal{P}$ be the idempotent completion of $\mathcal{P}_{Q_0}$ (i.e. we take the smallest additive category generated by direct summands of the L_i in $D^b_{\mathbf{Gl}_{\underline{d}}}(R_Q(\underline{d}))$ and their shifts). It carries a monoidal structure and the inclusion induces*

$$K_0(\mathcal{P}_{Q_0}) = K_0(\mathcal{P})$$

where the Grothendieck group has the ring structure from the monoidal categories and a $\mathbb{Z}[q, q^{-1}]$-module structure via the shift, i.e. $q \cdot [M] := [M[1]]$, M an object in $\mathcal{P}$.

Remark. We call the monoidal category $\mathcal{P}$ **Lusztig's category of perverse sheaves**. Even though these are not perverse sheaves since we allow shifts of them and Lusztig originally defined them inside $\bigsqcup D^b(R_Q(\underline{d}))$ which of course gives a different category (for example in his category $\mathrm{Hom}(L_i, L_j[n]) = 1_j * H_{[n]}(Z) * 1_i$ if $\sum i_t = \sum j_k$ and zero else). Nevertheless the two categories have the same Grothendieck group. In the view of the context here we think it is more apropriate to define it in the equivariant derived categories.

Remark. The previous lemma is no longer true if you allow your quiver to have loops. For example if Q is the quiver with one loop. Then, let Z_n be the Steinberg algebra associated to $(G = \mathbf{Gl}_n, B_n, \mathfrak{gl}_n, \mathfrak{n}_n)$ with $B_n \subset \mathbf{Gl}_n$ the upper triangular matrices, $\mathfrak{n}_n$ the Lie algebra of the unipotent radical of B_n. We claim

$$
\begin{aligned}
K_0(\mathcal{P}) &= \bigoplus_{n\in\mathbb{N}_0} K_0(\text{f.d. proj. graded } H_{[*]}(Z_n) - \text{modules}) \\
&= \bigoplus_{n\in\mathbb{N}_0} K_0(\text{f.d. simple graded } H_{[0]}(Z_n) - \text{modules}) \\
&= \bigoplus_{n\in\mathbb{N}_0} K_0(\text{f.d. graded } \mathbb{C}S_n - \text{modules}) \\
&= \left(\bigoplus_{n\in\mathbb{N}_0} K_0(\text{f.d.}\mathbb{C}S_n - \text{modules})\right) \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}] \\
&= (\text{Symmetric functions}) \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}]
\end{aligned}
$$

The first isomorphism is implied by the Corollary 1.4.0.3. The second equality is implied by semi-smallness of the classical Springer maps. For the third result see the section on classical Springer theory. The last equality is well-known, it maps the simple module S_λ (=Specht module) corresponding to a partition λ to the Schur function corresponding to λ.
But the category $\mathcal{P}_{Q_0}$ corresponds to the submonoidal category given by finite direct sums of shifts of finite-dimensional free modules. This is a monoidal category generated by direct sums of shifts of one object $E = S_1$ and an arrow $s\colon E^2 := E \otimes E \to E^2$ of degree 0 with the relation $(sE) \circ (Es) \circ (sE) = (Es) \circ (sE) \circ (Es)$ (see also [Rou11]). In this case $K_0(\mathcal{P}_{Q_0}) = \mathbb{Z}[q, q^{-1}, T],\ [E] \mapsto T$ which is much smaller than $K_0(\mathcal{P})$.

Now, let again be Q without loops. $K_0(\mathcal{P})$ has the structure of a twisted $\mathbb{Z}[q, q^{-1}]$-Hopf algebra. The algebra structure is given by the monoidal structure on $\mathcal{P}$ which is defined by induction functors. A restriction functor for the category $\mathcal{P}$ defines the structure of a coalgebra. For the geometric construction of these functors see [Lus91].

Theorem 1.9.2. *(Lusztig, [Lus91], thm 10.17) Consider the map*

$$
\begin{aligned}
\lambda_Q\colon \mathcal{U}^- &\to K_0(\mathcal{P}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \\
F_i &\mapsto [L_i] \otimes 1, \quad i \in Q_0
\end{aligned}
$$

where we see $i \in Q_0$ as a sequence in the vertices of length 1. This defines an isomorphism of twisted $\mathbb{Q}(q)$-Hopf algebras.

Definition 4. We call $\mathbf{B} := \{[L_i] \otimes 1 \mid i = (i_1, \ldots, i_d), i_t \in Q_0\}$ **canonical basis** for $K_0(\mathcal{P}) \otimes \mathbb{Q}(q)$.
We also call $\lambda_Q^{-1}(\mathbf{B})$ **canonical basis** in $\mathcal{U}^-$.
Also the image in $K_0(proj^{\mathbb{Z}} \bigoplus R_{\underline{d}}^{\mathbf{Gl}_{\underline{d}}}) \otimes \mathbb{Q}(q)$ is called **canonical basis**.

There are two intrinsic alternative definitions of the canonical basis for $\mathcal{U}^-$ given by again Lusztig in [Lus90] for the finite type case and in general by Kashiwara's crystal basis, see [Kas91].

Generators and relations for $\mathcal{P}_{Q_0}$.

This is due to Rouquier (cp. [Rou11]), it is the observation that the generators and relations of the quiver Hecke algebra rather easily give generators and relations for the monoidal category $\mathcal{P}_{Q_0}$. In the category, we use the convention: Instead of $E \to E'(n)$ we write $E \to E'$ is a morphism of degree n. A composition $g \circ f$ of a morphism $f \colon E \to E'$ of degree n and $g \colon E' \to E''$ of degree m is the homomorphism $E \to E''$ of degree $n+m$ given by $E \xrightarrow{f} E'(n) \xrightarrow{g(n)} E''(n+m)$.

Let Q be a quiver without loops. Let $\mathcal{B}$ be the monoidal category generated by finite direct sums of shifts of objects $E_a =: E_a(0), a \in Q_0$ and arrows

$$z_a \colon E_a \to E_a, \quad \sigma_{a,b} \colon E_a E_b \to E_b E_a, \quad a, b \in Q_0$$

of degrees

$$\deg z_a = 2, \quad \deg \sigma_{a,b} = \begin{cases} -2 & , \text{ if } a = b \\ 2h_{b,a} & , \text{ if } a \neq b \end{cases}$$

where as before $h_{a,b} := \#\{\alpha \in Q_1 \mid \alpha \colon a \to b\}, \quad a, b \in Q_0$, and assume relations

(1) $(s^2 = 1)$

$$\sigma_{ab} \circ \sigma_{ba} = \begin{cases} (-1)^{h_{b,a}} (E_b z_a - z_b E_a)^{h_{a,b} + h_{b,a}} & , \text{ if } a \neq b \\ 0 & , \text{ if } a = b \end{cases}$$

(2) (straightening rule)

$$\sigma_{ab} \circ z_a E_b - E_b z_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ E_a E_a & , \text{ if } a = b, \end{cases}$$

$$\sigma_{ab} \circ E_a z_b - z_b E_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ -E_a E_a & , \text{ if } a = b, \end{cases}$$

(3) (braid relations) for $a, b, c \in Q_0$ we have the following inclusion of $\mathbb{C}$-algebras. Let

$\mathbb{C}[\alpha_s, \alpha_t]$ be the set of polynomials in α_s, α_t.

$$\begin{aligned} J_{a,b,c}\colon \mathbb{C}[\alpha_s, \alpha_t] &\to \mathrm{End}_{\mathcal{B}}(E_a E_b E_c) \\ \alpha_s &\mapsto E_a z_b E_c - z_a E_b E_c \\ \alpha_t &\mapsto E_a E_b z_c - E_a z_b E_c, \end{aligned}$$

we set $t(\alpha_s^h) := (\alpha_s + \alpha_t)^h =: s(\alpha_t^h) \in \mathbb{C}[\alpha_s, \alpha_t], h \in \mathbb{N}_0$. Then, the relation is

$$\begin{aligned} &\sigma_{ab} E_c \circ E_a \sigma_{cb} \circ \sigma_{ca} E_b - E_b \sigma_{ca} \circ \sigma_{cb} E_a \circ E_c \sigma_{ab} \\ &= \begin{cases} J_{bab}(\alpha_s^{h_{a,b}} \frac{\alpha_t^{h_{b,a}} - (-1)^{h_{b,a}} \alpha_s^{h_{b,a}}}{\alpha_s + \alpha_t} - \alpha_t^{h_{b,a}} \frac{\alpha_s^{h_{a,b}} - (-1)^{h_{a,b}} \alpha_t^{h_{a,b}}}{\alpha_s + \alpha_t}) & , \text{ if } a = c, a \neq b, \\ 0 & , \text{ else.} \end{cases} \end{aligned}$$

for $i = (i_1, \ldots, i_n), i_t \in Q_0$ we set $E_i := E_{i_1} E_{i_2} \cdots E_{i_n}$. Let $I_{\underline{d}} := \{i = (i_1, \ldots, i_n) \mid \sum_t i_t = \underline{d}\}$. Then, by construction there is an isomorphism of algebras

$$\begin{aligned} R_{\underline{d}}^{\mathbf{Gl}_{\underline{d}}} &\to \bigoplus_{i,j \in I_{\underline{d}}} \mathrm{Hom}_{\mathcal{B}}(E_i, E_j) \\ 1_i &\mapsto \mathrm{id}_{E_i} \\ z_i(t) &\mapsto E_{i_1} E_{i_2} \cdots E_{i_{t-1}} z_{i_t} E_{i_{t+1}} \cdots E_{i_n} \\ \sigma_i(s) &\mapsto E_{i_1} \cdots E_{i_{\ell-1}} \sigma_{i_{\ell+1}, i_\ell} E_{i_{\ell+2}} \cdots E_{i_n}, \quad , \text{if } s = (\ell, \ell+1) \in S_n \end{aligned}$$

Theorem 1.9.3. *([Rou11]) There is an equivalence of monoidal categories*

$$\begin{aligned} \mathcal{P}_{Q_0} &\to \mathcal{B} \\ L_i &\mapsto E_i \end{aligned}$$

which is on morphisms the isomorphism of algebras from above.

Since we have not more knowledge on the decomposition theorem for quiver-graded Springer maps, we can not expect to find a similar easy description for the category $\mathcal{P}$.

Khovanov and Lauda's catgorification of the negative half of the quantum group.

Many years later Khovanov and Lauda have a different approach to the same *monoidal categorification* as Lusztig. Instead the category $\mathcal{P}$ they consider the category of projective graded (f.g.) modules over quiver Hecke algebras $R_{\underline{d}}^G := R_{\underline{d}}^{\mathbf{Gl}_{\underline{d}}}$, $\underline{d} \in \mathbb{N}_0^{Q_0}$

$$proj^{\mathbb{Z}} \bigoplus_{\underline{d} \in \mathbb{N}_0^{Q_0}} R_{\underline{d}}^G.$$

It is easy to see that we have natural injective maps $\mu\colon R_{\underline{d}}^G \otimes R_{\underline{e}}^G \to R_{\underline{d}+\underline{e}}^G$ compatible with the algebra multiplication. We write $1_{\underline{d},\underline{e}} := \mu(1 \otimes 1)$. From this there are (well-defined see [KL09],section 2.6) induction and restriction functors

$$\mathrm{Ind}_{\underline{d},\underline{e}}^{\underline{d}+\underline{e}} \colon proj^{\mathbb{Z}}(R_{\underline{d}}^G \otimes R_{\underline{e}}^G) \to proj^{\mathbb{Z}}(R_{\underline{d}+\underline{e}}^G), \quad X \mapsto R_{\underline{d}+\underline{e}}^G 1_{\underline{d},\underline{e}} \otimes_{R_{\underline{d}}^G \otimes R_{\underline{e}}^G} X$$
$$\mathrm{Res}_{\underline{d},\underline{e}}^{\underline{d}+\underline{e}} \colon proj^{\mathbb{Z}}(R_{\underline{d}+\underline{e}}^G) \to proj^{\mathbb{Z}}(R_{\underline{d}}^G \otimes R_{\underline{e}}^G), \quad Y \mapsto 1_{\underline{d},\underline{e}} Y$$

The induction functor gives $proj^{\mathbb{Z}} \bigoplus_{\underline{d} \in \mathbb{N}_0^{Q_0}} R_{\underline{d}}^G$ the structure of a monoidal category via $X \circ X' := \mathrm{Ind}_{\underline{d},\underline{e}}^{\underline{d}+\underline{e}} X \boxtimes X'$ where $X \boxtimes X'$ is the natural graded $R_{\underline{d}}^G \otimes R_{\underline{e}}^G$-module structure. Obviously, it is a $\mathbb{Z}[q, q^{-1}]$-algebra with q operating as the shift (1) on the graded modules, i.e. $q \cdot [M] := [M(1)]$. The comultiplication is given by $[\mathrm{Res}][P] := \sum_{\underline{d},\underline{e}\colon \underline{d}+\underline{e}=\underline{f}} [\mathrm{Res}_{\underline{d},\underline{e}}^{\underline{f}}(P)]$. It even defines a twisted $\mathbb{Z}[q, q^{-1}]$-Hopf algebra structure on $K_0(proj^{\mathbb{Z}} \bigoplus_{\underline{d} \in \mathbb{N}_0^{Q_0}} R_{\underline{d}}^{\mathbf{Gl}_{\underline{d}}})$.

Theorem 1.9.4. *(Khovanov, Lauda, [KL09]) The map*

$$\kappa_Q \colon \mathcal{U}^- \to K_0(proj^{\mathbb{Z}} \bigoplus_{\underline{d} \in \mathbb{N}_0^{Q_0}} R_{\underline{d}}^{\mathbf{Gl}_{\underline{d}}}) \otimes \mathbb{Q}(q)$$
$$F_i \mapsto [R_i^{\mathbf{Gl}_1}] \otimes 1, \quad i \in Q_0$$

where we consider $i \in Q_0$ as an element in $\mathbb{N}_0^{Q_0}$, is an isomorphism of twisted $\mathbb{Q}(q)$-Hopf algebras.

Khovanov and Lauda invented the quiver Hecke algebra, which later had been proven in [Var09] to be the same as the Steinberg algebra of quiver-graded Springer theory. The explicit description (generators and relations for the algebra) and diagram calculus (which we leave out in this survey) are a major step forward from Lusztig's description. Their work sparked a big interest in this subject.

Remark. Let Q be a Dynkin quiver. Then, the objects of the category $\mathcal{P}$ are direct sums of shifts of $IC_{\mathcal{O}}$ where $\mathcal{O} \subset R_Q(\underline{d})$ is a $\mathbf{Gl}_{\underline{d}}$-orbit (we do not write a local system if the trivial local system is meant). These are in bijection with isomorphism classes of $\mathbb{C}Q$-modules. The monoidal structure on $\mathcal{P}$ is constructed such that $K_0(\mathcal{P}) \otimes \mathbb{Q}(q)$ is the twisted Ringel-Hall algebra (over $\mathbb{Q}(q)$). The isomorphism between the twisted Ringel-Hall algebra and the negative half of the quantum group associated to the underlying graph of the quiver is a theorem of Ringel, see for example [Rin93].

Table 1.1: List of known Steinberg algebras.

	$H_{top}(Z,\mathbb{C})$	$H_*(Z,\mathbb{C})$	$H_*^G(Z,\mathbb{C})$	$K_0^{G\times\mathbb{C}^*}(Z)\otimes_{\mathbb{Z}}\mathbb{C}$
$(G,B,\mathfrak{g},\mathfrak{u})$ classical ST	$\mathbb{C}W$	$\mathbb{C}[\mathfrak{t}]/I_W\#\mathbb{C}[W]$	$\mathbb{C}[\mathfrak{t}]\#\mathbb{C}[W]$ degenerate affine Hecke algebra	affine Hecke algebra
$(G,B,\{0\},\{0\})$ nil ST i.e. $Z=G/B\times G/B$	$\mathbb{C}$	$\mathrm{End}_{\mathbb{C}-lin}(H^*(G/B))$	$\mathrm{End}_{H_G^*(pt)}(H_G^*(G/B))$ (affine) nil Hecke algebra	?
quiver-graded ST (complete dim filtrations)	?	$R_{\underline{d}}$	quiver Hecke algebra (= KLR-algebra)	?

Further known examples are:

(1) There is an exotic Springer theory (by Kato [Kat09], [Kat11], Achar and Henderson [AH08]). The Steinberg algebra $K_0^{G\times(\mathbb{C}^*)^3}(Z)\otimes_{\mathbb{Z}}\mathbb{C}$ is isomorphic to the Hecke algebra with unequal parameters of type $C_n^{(1)}$. Also Kato gave an exotic Deligne-Langlands correspondence.

(2) Quiver-graded Springer theory for the oriented cycle quiver (allowing only nilpotent representations) gives that $H_*^G(Z)$ is isomorphic to the quiver Schur algebra (compare the work of Stroppel and Webster, [SW11].)

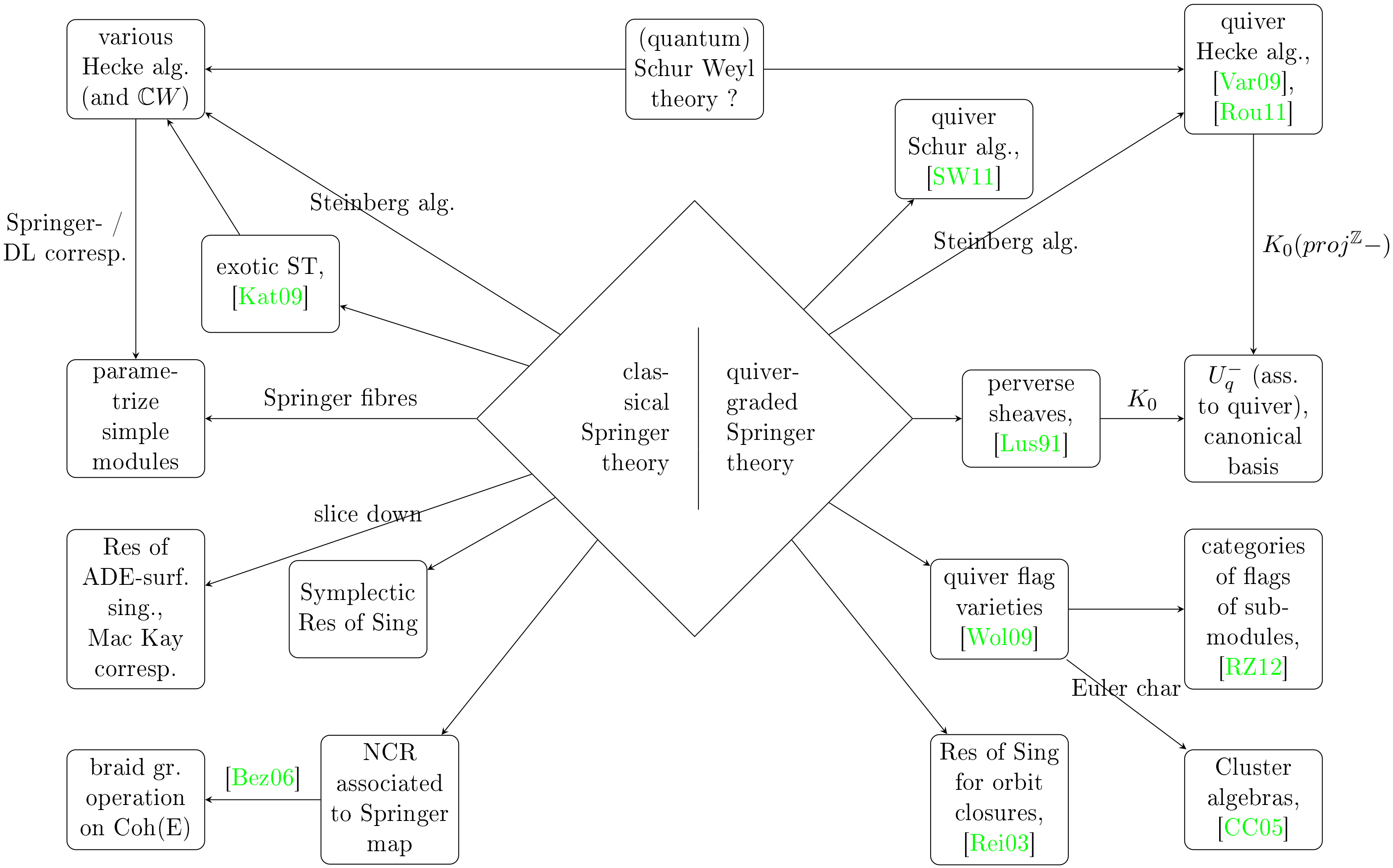


Figure 1.1: Springer theory and related fields

1.9.2 Literature review

Collapsings of homogeneous vector bundles are quite ubiquitous (for example see [Kem76]).

(1) Classical Springer theory (cp remark 1.8):
Classical Springer theory is usually defined for semi-simple algebraic groups and goes back to first Springer [Spr76],[Spr78], then Kazhdan-Lusztig [KL80], Slodowy [Slo80a], Lusztig [Lus81], Rossmann [Ros91]) and the defined convolution operations differ between each other by at most multiplication with a sign character (see [Hot81]).
Also relevant is the earlier work on the topology of Springer fibres of Spaltenstein (see [Spa76], [Spa77]) and Vargas (see [Var79]) and the Springer map already occurrs in Steinberg's work (for example [Ste74]). A book on classical Springer theory is written by Borho, Brylinski and Mac Pherson [BBM89]. A comprehensive treatment can be found in chapter 3 of [CG97] and a short one using perverse sheaves in [Ara01] if you speak French. I apologize to the many other authors who I do not mention.

(2) Quiver-graded Springer theory:
First considered by Lusztig, see [Lus91]. Later, Reineke started to look at it as an analogue of the classical Springer theory, see [Rei03], also see [Wol09].
The quiver Hecke algebras as Steinberg algebras first occurred in the work of Varagnolo and Vasserot, cp. [Var09], and independently also in Rouquier's article [Rou11].

Open problems/ wild speculations:

(O1) Are Springer fibre modules always semi-simple modules over the Steinberg algebra?

(O2) Which Steinberg algebras are affine cellular algebras?
Which have finite global dimension?
Partial answers: Brundan, Kleshchev and McNamara showed that KLR-algebras for Dynkin quivers are affine cellular (see [BKM12]).
Certain Steinberg algebras (including KLR-algebras for Dynkin quivers) have been shown to have finite global dimension (see [Kat13]). In [BKM12], the authors write that they expect that KLR-algebras have finite global dimension if and only if the quiver is Dynkin.

(O3) Are there Kazhdan-Lusztig polynomials and even a theory of canonical basis for Steinberg algebras?
Do we have Standard modules for Steinberg algebras?
Partial answers: Standard modules have been defined in [Kat13] under some assumptions (finitely many orbits in the image of the Springer map,...).
The original definition of Kazhdan-Lusztig polynomials has been inspired by studying a base change between two bases in the Steinberg variety associated to classical Springer theory.

(O4) Can we describe noncommutative resolutions of singularities corresponding to Springer maps?

Can we adapt the notion of a noncommutative resolution of singularities using constructible instead of coherent sheaves?

Partial answers exists for the coherent sheaf theory: Bezrukavnikov studied it for classical Springer theory (see [Bez06]). For quiver-graded Springer theory with $Q = A_2$ noncommutative resolutions have been studied by Buchweitz, Leuschke and van den Bergh (see [BLB10], [BLB11]).

(O5) Does there exist a Schur-Weyl theory relating classical and quiver-graded Springer theory (for example via Morita equivalences of the associated Steinberg algebras)?

Partial answers only for type A-situations i.e. *the quiver and the reductive group is of type A* (so maybe it only exists in this case): due to Brundan, Kleshchev [BK09], see also for example [Web13].

Chapter 2

Generalized quiver Hecke algebras

Summary. We generalize the methods of Varagnolo and Vasserot, [Var09] and partlially [VV11], to generalized quiver representations introduced by Derksen and Weyman in [DW02]. This means we have a general geometric construction of an interesting class of algebras (the Steinberg algebras for generalized quiver-graded Springer theory) containing skew group rings of Weyl groups with polynomial rings, (affine) nil Hecke algebras and KLR-algebras (=quiver Hecke algebras). Unfortunately this method works only in the *Borel case*, i.e. all parabolic groups in the construction data of a Springer theory are Borel groups. Nevertheless, we try to treat also the parabolic case as far as this is possible here.
This is a short reminder of Derksen and Weyman's generalized quiver representations from [DW02].

Definition 5. A *generalized quiver with dimension vector* is a triple $(\mathbb{G}, G, V)$ where $\mathbb{G}$ is a reductive group, G is a centralizer of a Zariski closed abelian reductive subgroup H of $\mathbb{G}$,i.e.

$$G = C_{\mathbb{G}}(H) = \{g \in \mathbb{G} \mid ghg^{-1} = h \quad \forall h \in H\}$$

(then G is also reductive, see lemma below) and V is a representation of G which decomposes into irreducible representations which also appear in $\mathcal{G} := \text{Lie}(\mathbb{G})$ seen as an G-module.
A *generalized quiver representation* is a quadruple $(\mathbb{G}, G, V, Gv)$ where $(\mathbb{G}, G, V)$ is a generalized quiver with dimension vector, v in V and Gv is the G-orbit.

Remark. Any such reductive abelian group is of the form $H = A \times S$ with A finite abelian and S a torus, this implies that there exists finitely many elements $h_1, \dots, h_m$ such that $C_{\mathbb{G}}(H) = \bigcap_{i=1}^m C_{\mathbb{G}}(h_i)$, see for example Humphreys' book [Hum75], Prop. in 16.4, p.107.

We would like to work with the associated Coxeter systems, therefore it is sensible to assume $\mathbb{G}$ connected and replace G by its identity component G^o. There is the following proposition

Proposition 1. *Let $\mathbb{G}$ be a connected reductive group and $H \subset \mathbb{G}$ an abelian group which lies in a maximal torus. We set $G := C_G(H)^o = (\bigcap_{i=1}^m C_{\mathbb{G}}(h_i))^o$. Then it holds*

(1) For any maximal torus $T \subset \mathbb{G}$, the following three conditions are equivalent:

(i) $T \subset G$.

(ii) $H \subset T$.

(iii) $\{h_1, \ldots, h_m\} \subset T$.

(2) G *is a reductive group.*

(3) *If* $\underline{\Phi}$ *is the set of roots of* $\mathbb{G}$ *with respect to a maximal torus* T *with* $H \subset T$*, then* $\Phi := \{\alpha \in \underline{\Phi} \mid \alpha(h) = 1 \quad \forall h \in H\}$ *is the set of roots for* G *with respect to* T*, its Weyl group is* $\langle s_\alpha \mid \alpha \in \Phi \rangle$ *and for all* $\alpha \in \Phi$ *the weight spaces are equal* $\mathfrak{g}_\alpha = \mathcal{G}_\alpha$ *(and* 1*-dimensional* $\mathbb{C}$*-vector spaces).*

(4) *There is a surjection*

$$\{\mathbb{B} \subset \mathbb{G} \mid \mathbb{B} \text{ Borel subgroup, } H \subset \mathbb{B}\} \to \{B \subset G \mid B \text{ Borel subgroup }\}$$
$$\mathbb{B} \mapsto \mathbb{B} \cap G$$

If $\underline{\Phi}^+$ *is the set of positive roots with respect to* $(\mathbb{G}, \mathbb{B}, T)$ *with* $H \subset T$*, then* $\Phi^+ := \Phi \cap \underline{\Phi}^+$ *is the set of positive roots for* $(G, G \cap \mathbb{B}, T)$ *.*

proof: Ad (1): This is easy to prove directly.
(2)-(4) are proven if $G = C_{\mathbb{G}}(h)^o$ for one semisimple element $h \in \mathbb{G}$ in Carters book [Car85b], section 3.5. p.92-93. In general $G = (\bigcap_{i=1}^m C_{\mathbb{G}}(h_i)^o)^o$ for certain $h_i \in H, 1 \leq i \leq m$. The result follows via induction on m. Set $\mathbb{G}_1 := C_G(h_1)^o$. It holds $G = (\bigcap_{i=2}^m C_{\mathbb{G}_1}(h_i)^o)^o = C_{\mathbb{G}_1}(H)^o \subset \mathbb{G}_1$ and $\mathbb{G}_1$ is a connected reductive group. By induction hypothesis, all statements are true for $(G, \mathbb{G}_1)$, so in particular G is a reductive group. The other statements are then obvious. □

Notational conventions

We fix the ground field for all algebraic varieties and Lie algebras to be $\mathbb{C}$.
For a Lie algebra $\mathfrak{g}$ we define the k-th power inductively by $\mathfrak{g}^1 := \mathfrak{g}, \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$. If we denote an algebraic group by double letters (or indexed double letters) like $\mathbb{G}, \mathbb{B}, \mathbb{U}, ...$ (or $\mathbb{G}'$, $\mathbb{P}_J$, etc.) we take the calligraphic letters for the Lie algebras, i.e. $\mathcal{G}, \mathcal{B}, \mathcal{U}, ...$ (or $\mathcal{G}', \mathcal{P}_J$, etc) respectively. If we denote an algebraic group by roman letters (or indexed roman letters) like $G, B, U, ...$ (or G', P_J, etc.) we take the small frakture letters for the Lie algebras, i.e. $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, ..$ (or $\mathfrak{g}', \mathfrak{p}_J$) respectively.
If we habe a subgroup $P \subset G$ of a group and an element $g \in G$ we write ${}^gP := gPg^{-1}$ for the conjugate subgroup.
We also recall the following.

Remark. Let (W, S) be a Coxeter system, $J \subset S$. Then $(W_J := \langle J \rangle, J)$ is again a Coxeter system with the length function is the restriction of the length function of (W, S) to elements in W_J. Then, the set W^J of minimal length coset representatives $W^J \subset W$ for W/W_J is defined via: An element w lies in W^J if and only if for all $s \in J$ we have $l(ws) >$

$l(w)$. Also there is a factorization $W = W^J W_J$ and if $w = xy$ with $x \in W^J, y \in W_J$, their lengths satisfy $l(w) = l(x) + l(y)$. We will fix the bijection $c_J \colon W^J \to W/W_J, w \mapsto wW_J$. The Bruhat order of (W, S) can be restricted to W^J and transferred via the bijection to W/W_J.
For two subsets $K, J \subset S$ define ${}^K W^J := (W^K)^{-1} \cap W^J$, the projection $W \to W_K \backslash W/W_J$ restricts to a bijection ${}^K W^J \to W_K \backslash W/W_J$.
Let (G, B, T) be a reductive group with Borel subgroup and maximal torus and (W, S) be its associated Coxeter system. We fix for any element in W a lift to the group G and denote it by the same letter.

2.1 Generalized quiver-graded Springer theory

We define a generalized quiver-graded Springer theory for generalized quiver representations in the sense of Derksen and Weymann. Given $(\mathbb{G}, \mathbb{P}_J, \mathcal{U}, H, V)$ (and some not mentioned $H \subset T \subset \mathbb{B} \subset \mathbb{P}_J$) with

* $\mathbb{G}$ is a connected reductive group, $H \subset T$ is a subgroup of a maximal torus in $\mathbb{G}$, we set $G = C_{\mathbb{G}}(H)^o$ (then G is also reductive with $T \subset G$ is a maximal torus in G).

* $T \subset \mathbb{B} \subset \mathbb{G}$ a Borel subgroup, then $B := \mathbb{B} \cap G$ is a Borel subgroup of G,
We write $(\mathbb{W}, \mathbb{S})$ for the Coxeter system associated with $(\mathbb{G}, \mathbb{B}, T)$ and (W, S) for the one associated to (G, B, T). Observe, that $W \subset \mathbb{W}$. For any $J \subset \mathbb{S}$ we set $\mathbb{P}_J := \mathbb{B}\langle J \rangle \mathbb{B}$ and call it a **standard parabolic group**.

* Now fix a subset $J \subset \mathbb{S}$. We call a $\mathbb{P}_J$-subrepresentation $\mathcal{U}' \subset \mathcal{G} = \mathrm{Lie}(\mathbb{G})$ (of the adjoint representation which we denote by $(g, x) \mapsto {}^g x$, $g \in \mathbb{G}, x \in \mathcal{G}$) *suitable* if

  - $(\mathcal{U}')^T = \{0\}$,
  - $\mathcal{U}' \cap {}^s\mathcal{U}'$ is $\mathbb{P}_J$-stable for all $s \in \mathbb{S}$.

  Let $\mathcal{U} = \bigoplus_{k=1}^r \mathcal{U}^{(k)}$ a $\mathbb{P}_J$-representation with each $\mathcal{U}^{(k)}$ is suitable. (Examples of suitable $\mathbb{P}_J$-representations are given by $\mathcal{U}' = \mathcal{U}_{J'}^t$ where $J \subset J' \subset \mathbb{S}$, $\mathcal{U}_{J'} = \mathrm{Lie}(\mathbb{U}_{J'})$ with $\mathbb{U}_{J'} \subset \mathbb{P}_{J'}$ is the unipotent radical and $\mathcal{U}_{J'}^t$ is the t-th power, $t \in \mathbb{N}$). We define $\mathbb{W}_J := \langle J \rangle$ and $\mathbb{W}^J$ be the set of minimal coset representatives in $\mathbb{W}/\mathbb{W}_J$, $I_J := W \backslash \mathbb{W}^J \subset W \backslash \mathbb{W}$ and

$$\bigsqcup_{J \subset \mathbb{S}} I_J$$

  We call $I := I_\emptyset$ the set of **complete dimension filtrations**. Let $\{x_i \in \mathbb{W} \mid i \in I_J\}$ be a complete representing system of the cosets in I_J. Every element of the Weyl groups $\mathbb{W}$ (and W) we lift to elements in $\mathbb{G}$ (and G) and denote the lifts by the same letter. For every $i \in I_J$ we set

$$P_i := {}^{x_i}\mathbb{P}_J \cap G,$$

Observe that $H \subset T = {}^{w}T \subset {}^{w}\mathbb{P}_J$ for all $w \in \mathbb{W}$, therefore ${}^{w}\mathbb{P}_J \cap G$ is a parabolic subgroup in G for any $w \in \mathbb{W}$.

* $V = \bigoplus_{k=1}^{r} V^{(k)}$ with $V^{(k)} \subset \mathcal{G}$ is a G-subrepresentation.
$F_i = \bigoplus_{k=1}^{r} F_i^{(k)}$ with $F_i^{(k)} := V^{(k)} \cap {}^{x_i}\mathcal{U}^{(k)}$ is a P_i-subrepresentation of $V^{(k)}$.

We define

$$V \xleftarrow{\ \pi_i\ } E_i := G \times^{P_i} F_i \xrightarrow{\ \mu_i\ } G/P_i \qquad\qquad gf \longleftarrow \overline{(g,f)} \longrightarrow gP_i.$$

Now, there are closed embeddings $\iota_i \colon G/P_i \to \mathbb{G}/\mathbb{P}_J,\ gP_i \mapsto gx_i\mathbb{P}_J$ with for any $i \neq i'$ in I_J it holds $\operatorname{Im}\iota_i \cap \operatorname{Im}\iota_{i'} = \emptyset$. Therefore, we can see $\bigsqcup_{i\in I_J} G_i/P_i$ as a closed subscheme of $\mathbb{G}/\mathbb{P}_J$. It can be identified with the closed subvariety of the fixpoints under the H-operation $(\mathbb{G}/\mathbb{P}_J)^H = \{g\mathbb{P}_J \in \mathbb{G}/\mathbb{P}_J \mid hg\mathbb{P}_J = g\mathbb{P}_J \text{ for all } h \in H\}$.

$$V \xleftarrow{\ \pi_J\ } E_J := \bigsqcup_{i\in I_J} E_i \xrightarrow{\ \mu_J\ } \mathbb{G}/\mathbb{P}_J$$

We also set

$$V \xleftarrow{\ p_{ij}\ } Z_{ij} := E_i \times_V E_j \xrightarrow{\ m_{ij}\ } G/P_i \times G/P_j$$

$$V \xleftarrow{\ p_J\ } Z_J := \bigsqcup_{i,j\in I_J} Z_{ij} \xrightarrow{\ m_J\ } \mathbb{G}/\mathbb{P}_J \times \mathbb{G}/\mathbb{P}_J.$$

In an obvious way all maps are G-equivariant. We are primarily interested in the following **Steinberg variety**

$$Z := Z_{\emptyset}.$$

The equivariant Borel-Moore homology of a Steinberg variety together with the convolution operation (defined by Ginzburg) defines a finite dimensional graded $\mathbb{C}$-algebra. We set

$$\mathcal{Z}_G := H_*^G(Z)$$

which we call **(G-equivariant) Steinberg algebra**. The aim of this section is to describe $\mathcal{Z}_G$ in terms of generators and relation (for $J = \emptyset$). This means all P_i are Borel subgroups of G.

If we set

$$H^G_{[p]}(Z) := \bigoplus_{i,j \in I} H^G_{e_i+e_j-p}(Z_{i,j}), \quad e_i = \dim_{\mathbb{C}} E_i$$

then $H^G_{[*]}(Z)$ is a graded $H^*_G(pt)$-algebra Then, we denote the right $\mathbb{W}$-operation on $I = W \setminus \mathbb{W}$ by $(i,w) \mapsto iw$, $i \in I, w \in \mathbb{W}$. We prove the following.

Theorem 2.1.1. *Let $J = \emptyset$ and $\mathcal{E}_i = \mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_i(1), \ldots, x_i(n)]$, $i \in I$. Then $\mathcal{Z}_G \subset \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W-mod}(\bigoplus_{i\in I} \mathcal{E}_i)$ is the graded $\mathbb{C}$-subalgebra generated by*

$$1_i, i \in I, \quad z_i(t), 1 \le t \le n = rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

defined as follows for $k \in I$, $f \in \mathcal{E}_k$.

$$1_i(f) := \begin{cases} f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$z_i(t)(f) := \begin{cases} x_i(t)f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$\sigma_i(s)(f) := \begin{cases} q_i(s)\frac{s(f)-f}{\alpha_s}, & (\in \mathcal{E}_i) \text{ if } i = is = k, \\ q_i(s)s(f) & (\in \mathcal{E}_i) \text{ if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

where

$$q_i(s) := \prod_{\alpha \in \Phi_{\mathcal{U}}, s(\alpha) \notin \Phi_{\mathcal{U}}, x_i(\alpha) \in \Phi_V} \alpha \qquad \in \mathcal{E}_i.$$

and $\Phi_{\mathcal{U}} = \bigsqcup_k \Phi_{\mathcal{U}^{(k)}}$, $\Phi_{\mathcal{U}^{(k)}} \subset \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}]$ is the set of T-weights for $\mathcal{U}^{(k)}$ and $\Phi_V = \bigsqcup_k \Phi_{V^{(k)}}$, $\Phi_{V^{(k)}} \subset \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ is the set of T-weights for $V^{(k)}$.
Furthermore, it holds

$$\deg 1_i = 0, \ \ \deg z_i(k) = 2, \ \ \deg \sigma_i(s) = \begin{cases} 2(\deg q_i(s)) - 2, & \text{if } is = i \\ 2\deg q_i(s), & \text{if } is \neq i \end{cases}$$

where $\deg q_i(s)$ refers to the degree as homogeneous polynomial in $\mathbb{C}[\mathfrak{t}]$.

The generality of the choice of the $\mathcal{U}$ in the previous theorem is later used to understand the case of an arbitrary J as a an algebra of the form $e_J \mathcal{Z}_G e_J$ for an associated Borel-case Steinberg algebra $\mathcal{Z}_G$ and e_J an idempotent element (this is content of a later article called *parabolic Steinberg algebras*).
For $J = \emptyset, \mathcal{U} = \mathrm{Lie}(\mathbb{U})^{\oplus r}$ for $\mathbb{U} \subset \mathbb{B}$ the unipotent radical we have the following result which generalizes KLR-algebras to arbitrary connected reductive groups and allowing quivers with loops.

Corollary 2.1.1.1. *Let $J = \emptyset, \mathcal{U} = Lie(\mathbb{U})^{\oplus r}$, $\mathbb{U} \subset \mathbb{B}$ the unipotent radical and $\mathcal{E}_i = \mathbb{C}[\mathfrak{t}] =$*

$\mathbb{C}[x_i(1), \ldots, x_i(n)], i \in I$. *Then*

$$\mathcal{Z}_G \subset \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W-mod}(\bigoplus_{i \in I} \mathcal{E}_i)$$

is the graded $\mathbb{C}$*-subalgebra generated by*

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I.$$

Let $f \in \mathcal{E}_k, k \in I,\ \alpha_s \in \Phi^+$ *be the positive root such that* $s(\alpha_s) = -\alpha_s$. *It holds*

$$\sigma_i(s)(f) := \begin{cases} \alpha_s^{h_i(s)} \frac{s(f)-f}{\alpha_s}, & \text{if } i = is = k, \\ \alpha_s^{h_i(s)} s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

where

$$h_i(s) := \#\{k \in \{1, \ldots, r\} \mid x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$$

where $V = \bigoplus_k V^{(k)}$ *and* $\Phi_{V^{(k)}} \subset \underline{\Phi}$ *are the* T*-weights of* $V^{(k)}$.

(1) If $Wx_i \neq Wx_is$ *then*

$$h_i(s) = \#\{k \mid V^{(k)} \subset \mathcal{R}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}.$$

We say that this number counts arrows.

(2) If $Wx_i = Wx_is$*, then*

$$h_i(s) = \#\{k \mid V^{(k)} \subset \mathfrak{g}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}.$$

We say that this number counts loops.

In the case of the previous corollary we call the Steinberg algebra $\mathcal{Z}_G$ **generalized quiver Hecke algebra**. It can be described by the following generators and relations. For a reduced expression $w = s_1 s_2 \cdots s_k$ we set

$$\sigma_i(s_1 s_2 \cdots s_k) := \sigma_i(s_1)\sigma_{is_1}(s_2) \cdots \sigma_{is_1 s_2 \cdots s_{k-1}}(s_k)$$

Sometimes, if it is understood that the definition depends on a particular choice of a reduced expression for w, we write $\sigma_i(w) := \sigma_i(s_1 s_2 \cdots s_k)$. Furthermore, we consider

$$\Phi\colon \bigoplus_{i \in I} \mathbb{C}[x_i(1), \ldots x_i(n)] \cong \bigoplus_{i \in I} \mathbb{C}[z_i(1), \ldots z_i(n)],\ x_i(t) \mapsto z_i(t)$$

as the left $\mathbb{W}$-module $\mathrm{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}]$, we fix the polynomials

$$c_i(s,t) := \Phi(\sigma_i(s)(x_i(t))) \ \in \bigoplus_{i \in I} \mathbb{C}[z_i(1), \ldots z_i(n)], \quad i \in I,\ 1 \leq t \leq n,\ s \in \mathbb{S}.$$

Now, we can describe under some extra conditions the relations of the generalized quiver Hecke algebras.

Proposition 2. *Let $\mathbb{S} \subset \mathbb{W} = Weyl(\mathbb{G}, T)$ be the simple reflections. Under the following assumption for the data $(\mathbb{G}, \mathbb{B}, \mathcal{U} = (\mathrm{Lie}(\mathbb{U}))^{\oplus r}, H, V)$, $J = \emptyset$: We assume for any $s, t \in \mathbb{S}$*

(B2) If the root system spanned by α_s, α_t is of type B_2 (or $stst = tsts$ is the minimal relation), then for every $i \in I$ such that $is = i = it$ it holds $h_i(s), h_i(t) \in \{0, 1, 2\}$.

(G2) If the root system spanned by α_s, α_t is of type G_2 (or $ststst = tststs$ is the minimal relation), then for every $i \in I$ such that $is = i = it$ it holds $h_i(s) = 0 = h_i(t)$.

Then the generalized quiver Hecke algebra for $(\mathbb{G}, \mathbb{B}, \mathcal{U} = (Lie(\mathbb{U}))^{\oplus r}, H, V), J = \emptyset$ is the graded $\mathbb{C}$-algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

in degrees

$$\deg 1_i = 0, \;\; \deg z_i(k) = 2, \;\; \deg \sigma_i(s) = \begin{cases} 2h_i(s) - 2, & \textit{if } is = i \\ 2h_i(s), & \textit{if } is \neq i \end{cases}$$

and relations

*(1) (*orthogonal idempotents*)*

$$1_i 1_j = \delta_{i,j} 1_i,$$
$$1_i z_i(t) 1_i = z_i(t),$$
$$1_i \sigma_i(s) 1_{is} = \sigma_i(s)$$

*(2) (*polynomial subalgebras*)*

$$z_i(t) z_i(t') = z_i(t') z_i(t)$$

(3) (relation implied by $s^2 = 1$*)*

$$\sigma_i(s)\sigma_{is}(s) = \begin{cases} 0 & , \textit{ if } is = i, \; h_i(s) \textit{ is even} \\ -2\alpha_s^{h_i(s)-1}\sigma_i(s) & , \textit{ if } is = i, \; h_i(s) \textit{ is odd} \\ (-1)^{h_{is}(s)}\alpha_s^{h_i(s)+h_{is}(s)} & , \textit{ if } is \neq i \end{cases}$$

*(4) (*straightening rule*)*

$$\sigma_i(s) z_i(t) - s(z_i(t))\sigma_i(s) = \begin{cases} c_i(s,t), & , \textit{ if } is = i \\ 0 & , \textit{ if } is \neq i. \end{cases}$$

*(5) (*braid relations*)*
Let $s, t \in \mathbb{S}, st = ts$, then

$$\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$$

Let $s, t \in \mathbb{S}$ not commuting such that $x := sts\cdots = tst\cdots$ minimally, $i \in I$. There exists explicit polynomials $(Q_w)_{w<x}$ in $\alpha_s, \alpha_t \in \mathbb{C}[\mathfrak{t}]$ such that

$$\sigma_i(sts\cdots) - \sigma_i(tst\cdots) = \sum_{w<x} Q_w \sigma_i(w)$$

(observe that for $w < x$ there exists just one reduced expression).

The proof you find in the end, see Prop. 5.

2.2 Relationship between parabolic groups in G and $\mathbb{G}$

For later on, we need to understand the relationship between parabolic subgroups in $\mathbb{G}$ and in G. Recall that a parabolic subgroup is a subgroup which contains a Borel subgroup, every parabolic subgroups is conjugated to a standard parabolic subgroup. The standard parabolic subgroups wrt (G, B, T) are in bijection with the set of subsets of S, via $J \mapsto B\langle J\rangle B =: P_J$. As a first step, we need to study the relationship of the Coxeter systems (W, S) and $(\mathbb{W}, \mathbb{S})$.

Lemma 13. *It holds $G \cap \mathbb{W} = W$. It holds $W \cap \mathbb{S} \subset S$. Let l_S be the length function with respect ot (W, S) and $l_{\mathbb{S}}$ be the length function with respect to $(\mathbb{W}, \mathbb{S})$. For every $w \in W$ it holds $l_S(w) \leq l_{\mathbb{S}}(w)$.*

proof: $N_{\mathbb{G}}(T) \cap G = N_G(T)$ implies $G \cap \mathbb{W} = W$. The inculsion $\Phi^+ \cap s(-\Phi^+) \subset \underline{\Phi}^+ \cap s(-\underline{\Phi}^+)$ for any $s \in \mathbb{S}$ implies $W \cap \mathbb{S} \subset S$.
Let $w = t_1 \cdots t_r \in W$, $t_i \in S$ reduced expression and assume $l_{\mathbb{S}}(w) < r$. It must be possible in $\mathbb{W}$ to write w as a subword of $t_1 \cdots \hat{t_i} \cdots t_r$ for some $i \in \{1, \dots, r\}$. But then $r = l_S(w) \leq l_S(t_1 \cdots \hat{t_i} \cdots t_r) < r$. □

Definition 6. We call $J \subset \mathbb{S}$. We say that J is S-**adapted** if for all $s \in S$ with $s = s_1 \cdots s_r$ a reduced expression in $(\mathbb{W}, \mathbb{S})$ such that there exists $i \in \{1, \dots, r\}$ with $s_i \in J$ then it also holds $\{s_1, \dots, s_r\} \subset J$.

Lemma 14. *(a) Intersection with G defines a map*

$$\{\mathbb{P}_J \mid J \subset \mathbb{S} \textit{ is } S-\textit{adapted}\} \to \{P_J \mid J \subset S\}$$
$$\mathbb{P}_J \mapsto \mathbb{P}_J \cap G = P_{S \cap \mathbb{W}_J}$$

(b) Let $G \cap {}^x\mathbb{B}$ is a Borel subgroup of G with $\mathbb{B} \subset G$ a Borel subgroup and $x \in \mathbb{W}$. Let $s \in \mathbb{S}$, then it holds

(1) If $Wxs \neq Wx$ then $G \cap {}^{xs}\mathbb{B} = G \cap {}^x\mathbb{B}$.

(2) If $Wxs = Wx$, then ${}^xs \in W$ and $G \cap {}^{xs}\mathbb{B} = {}^{{}^xs}[G \cap {}^x\mathbb{B}]$.

This gives an algorithm to find for any $x \in \mathbb{W}$ a $z \in W$ such that $G \cap {}^x\mathbb{B} = {}^z[G \cap \mathbb{B}]$. Also, for every $J \subset \mathbb{S}$ it then holds $G \cap {}^x\mathbb{P}_J = {}^z[G \cap \mathbb{P}_J]$ and $W \cap {}^x\mathbb{W}_J = {}^z[W \cap \mathbb{W}_J]$

where $x \in \mathbb{W}, z \in W$ *as before and for every* S*-adapted* $J \subset \mathbb{S}$

$$G \cap {}^x\mathbb{P}_J = {}^z\mathbb{P}_{S \cap \mathbb{W}_J}.$$

proof:

(a) It holds by the previous lemma $G \cap \mathbb{W}_J = W \cap \mathbb{W}_J$ and because J is S-adapted it holds $W \cap \mathbb{W}_J = \langle S \cap \mathbb{W}_J \rangle$, to see that:
Let $w = t_1 \cdots t_r \in \mathbb{W}_J$ with $t_i \in S$ an S-reduced expression, we need to see $t_i \in \mathbb{W}_J, 1 \leq i \leq r$. Wlog assume $t_1 \notin \mathbb{W}_J$. As J is S-adapted, there exists a $\mathbb{S}$-reduced expression with elements in J of w which is a subword of $t_2 \cdots t_r$. But this means a word of S-length r is a subword of a word of S-length $r-1$, therefore $t_1 \in \mathbb{W}_J$.
Now, the following inclusion is obvious

$$P_{S \cap \mathbb{W}_J} = B \langle G \cap \mathbb{W}_J \rangle B \subset G \cap \mathbb{P}_J.$$

Because $B \subset \mathbb{P}_J \cap G$ there has to exist $(\mathbb{W}_J \cap S) \subset J' \subset S$ such that $\mathbb{P}_J \cap G = P_{J'}$, we need to see $(S \cap \mathbb{W}_J) = J'$. Let $s \in J'$, then $s \in \mathbb{P}_J = \mathbb{B}\mathbb{W}_J\mathbb{B}$ implies $s \in \mathbb{W}_J$.

(b) Let $s \in \mathbb{S}, {}^x s \notin W$, then $\pm x(\alpha_s) \notin \Phi$ and this implies

$$\Phi \cap xs(\underline{\Phi}) = \Phi \cap [x(\underline{\Phi}) \setminus \{x(\alpha_s)\} \cup \{-x(\alpha_s)\}] = \Phi \cap x(\underline{\Phi}).$$

Therefore, the Lie algebras of the Borel groups $G \cap {}^x\mathbb{B}$ and $G \cap {}^{xs}\mathbb{B}$ have the same weights for T, this proves they are equal.
The point (2) is obvious.

□

Remark. In the setup of the beginning, we can always find unique representatives $x_i \in \mathbb{W}, i \in I$ for the elements in $W \setminus \mathbb{W}$ which fulfill

$$B_i = G \cap {}^{x_i}\mathbb{B} = G \cap \mathbb{B} = B.$$

This follows because for every $i \in I$ there is a bijection

$$\begin{aligned} Wx_i &\to \{\text{ Borel subgroups of } G \text{ containing } T\} \\ vx_i &\mapsto {}^v[G \cap {}^{x_i}\mathbb{B}] \end{aligned}$$

Then, there exists a unique $v \in W$ such that ${}^v[G \cap {}^{x_i}\mathbb{B}] = G \cap \mathbb{B}$, replace x_i by vx_i as a representative for Wx_i.
We will call these representatives **minimal coset representatives**[1]. Observe for $is \neq i$ it holds $x_{is} = x_i s$ by lemma 14, (b), (2).

[1] if G is a Levi-group in $\mathbb{G}$ they are the minimal coset representatives, in this more general situation the notion is not defined.

But since the images of $G/B_i, i \in I$ inside $\mathbb{G}/\mathbb{B}$ are disjoint, we prefer not to identify all $B_i, i \in I$.
In general, in the parabolic setup, it holds $P_i \neq P_j$ for $i \neq j$.

Lemma 15. *(factorization lemma) Let $J, K \subset \mathbb{S}$ be S-adapted and set $L := S \cap \mathbb{W}_J, M := S \cap \mathbb{W}_K$.*

(1) It holds $W^L = W \cap \mathbb{W}^J$ and for every element in $w \in W$ the unique decomposition as $w = w^J w_J$, $w^J \in \mathbb{W}^J, w_J \in \mathbb{W}_J$ fulfills $w^J \in W^L = W \cap \mathbb{W}_J$, $w_J \in W_L = W \cap \mathbb{W}_J$.

(2) It holds ${}^J\mathbb{W}^K \cap W = {}^LW^M$. In particular, every double coset $\mathbb{W}_J w \mathbb{W}_K$ with $w \in W$ contains a unique element of ${}^LW^M$.

proof:

(1) It holds $W_L(W \cap \mathbb{W}_J) = W = W \cap \mathbb{W}^J\mathbb{W}_J \supset (W \cap \mathbb{W}^J)(W \cap \mathbb{W}^J)$, the uniqueness of the factorization in W implies $(W \cap \mathbb{W}^J) \subset W^L$.
Now take $a \in W^L$, we can factorize it in $\mathbb{W}$ as $a = a^J a_J$ with $a^J \in \mathbb{W}^J, a_J \in \mathbb{W}_J$. We show that $a_J \in W$. Write $a = t_1 \cdots t_r$ S-reduced expression, assume $a_J \neq e$, then there exists a unique $i \in \{1, \ldots, r\}$ such that a_J is a subword of $t_i \cdots t_r$ but no subword of $t_{i+1} \cdots t_r$. Then, t_i must have a subword contained in $\mathbb{W}_J$, as J is S-adapted we get $t_i \in \mathbb{W}_J$. Continue with $t_i^{-1} a_J$ being a subword of $t_{i+1} \cdots t_r$. By iteration you find $a_J = t_{i_1} \cdots t_{i_k} \in W$ for certain $i = i_1 < \cdots < i_k$, $i_j \in \{1, \ldots, r\}$. This implies $a_J = e$ and $a = a^J \in W \cap \mathbb{W}^J$.

(2) By definition ${}^J\mathbb{W}^K \cap W = (\mathbb{W}^J)^{-1} \cap \mathbb{W}^K \cap W = (W^L)^{-1} \cap W^M = {}^LW^M$.

□

2.3 The equivariant cohomology of flag varieties

Lemma 16. ***(The (co)-homology rings of a point)***
Let G be reductive group, $T \subset P \subset G$ with P a parabolic subgroup and T a maximal torus, we write W for the Weyl group associated to (G, T) and $X(T) = \mathrm{Hom}_{Gr}(T, \mathbb{C}^*)$ for the group of characters. Let ET be a contractible topological space with a free T-operation from the right.

(1) For every character $\lambda \in X(T)$ denote by

$$S_\lambda := ET \times^T \mathbb{C}_\lambda$$

the associated T-equivariant line bundle over $BT := ET/T$ to the T-representation $\mathbb{C}_\lambda$ which is $\mathbb{C}$ with the operation $t \cdot c := \lambda(t)c$. The first chern class defines a homomorphism of abelian groups

$$c \colon X(T) \to H^2(BT), \quad \lambda \mapsto c_1(S_\lambda).$$

Let $\mathrm{Sym}_{\mathbb{C}}(X(T))$ *be the symmetric algebra with complex coefficients generated by* $X(T)$*, it can be identified with the ring of regular function* $\mathbb{C}[\mathfrak{t}]$ *on* $\mathfrak{t} = Lie(T)$ *(with doubled degrees), where* $X(T) \otimes_{\mathbb{Z}} \mathbb{C}$ *is mapped via taking the differential (of elements in* $X(T)$*) to* $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}-lin}(\mathfrak{t}, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}]$ *(both are the degree 2 elements).*

The previous map extends to an isomorphism of graded $\mathbb{C}$*-algebras*

$$\mathbb{C}[\mathfrak{t}] \to H_T^*(pt) = H^*(BT)$$

In fact this is a W*-linear isomorphism where the* W*-operation on* $\mathbb{C}[\mathfrak{t}]$ *is given by,* $(w, f) \mapsto w(f), w \in W, f \in \mathbb{C}[\mathfrak{t}]$ *with*

$$w(f)\colon \mathfrak{t} \to \mathbb{C}, t \mapsto f(w^{-1}tw).$$

We can choose ET *such that it also has a free* G*-operation from the right (i.e.* $ET := EG$*), then* $BT = ET/T$ *has an induced Weyl group action from the right given by* $xT \cdot w := xwT$*,* $w \in W, x \in ET$*. The pullbacks of this group operation induce a left* W*-operation on* $H_T^*(pt)$*.*

(2) $H_T^*(pt) = H_{-*}^T(pt)$, $H_G^*(pt) = (H_T^*(pt))^W = (H_{-*}^T(pt))^W = H_{-*}^G(pt)$.

proof:

(1) For the isomorphism see for example and the explanation of the W-operation see (L. Tu; Characteristic numbers of a homogeneous space, axiv, [Tu03])

(2) Use the definition and Poincare duality for the first isomorphism, for the second also use the splitting principle.

□

Lemma 17. ***(The cohomology rings of homogeneous vector bundles over*** G/P***)*** *Let* G *be reductive group,* $T \subset B \subset P \subset G$ *with* B *a Borel subgroup,* P *parabolic and* T *a maximal torus.*

(1) *For* $\lambda \in X(T)$ *we denote be* $L_\lambda := G \times^B \mathbb{C}_\lambda$ *the associated line bundle to the* B*-representation* $\mathbb{C}_\lambda$ *given by the trivial representation when restricted to the unipotent radical and* λ *when restricted to* T*. Let* $\mu\colon E \to G/B$ *be a* G*-equivariant vector bundle. Then,* $\mu^*(L_\lambda)$ *is a line bundle on* E *and*

$$K_\lambda := EG \times^G \mu^*(L_\lambda) \to EG \times^G E$$

is a line bundle over $EG \times^G E$*. There is an isomorphism of graded* $\mathbb{C}$*-algebras*

$$\mathbb{C}[\mathfrak{t}] \to H_G^*(E) = H^*(EG \times^G E)$$
$$X(T) \ni \quad \lambda \mapsto c_1(K_\lambda).$$

with $deg\lambda = 2$ for $\lambda \in X(T)$.
(By definition, equivariant chern classes are defined as $c_1^G(\mu^* L_\lambda) := c_1(K_\lambda)$).

(2) *Let $\mu \colon E \to G/P$ be a G-equivariant vector bundle, then there is an isomorphism of graded $\mathbb{C}$-algebras*

$$H_G^*(E) \to (H_T^*(pt))^{W_L}.$$

proof:

(1) Arabia proved that $H_G^*(G/B) \cong H_T^*(pt)$ as graded $\mathbb{C}$-algebras (cp. [Ara85]), the composition with the isomorphism from the previous lemma gives an isomorphism

$$c \colon \mathbb{C}[\mathfrak{t}] \to H_G^*(G/B), : \lambda \mapsto c_1(EG \times^G L_\lambda) =: c_1^G(L_\lambda)$$

Now, we show that for a vector bundle $\mu \colon E \to G/P$ with $P \subset G$ parabolic, the induced pullback map

$$\mu^* \colon H_G^*(G/P) \to H_G^*(E), \quad c_1^G(L_\lambda) \mapsto c_1^G(\mu^* L_\lambda)$$

is an isomorphism of graded $H_G^*(pt)$-algebras. We already know that it is a morphism of graded $H_G^*(pt)$-algebras, to see it is an isomorphism, apply the definition and Poincare duality to get a commutative diagram

$$\begin{array}{ccc} H_G^k(G/P) & \xrightarrow{\mu^*} & H_G^k(E) \\ \cong \downarrow & & \cong \downarrow \\ H^G_{2\dim G/P-k}(G/P) & \xrightarrow{\mu^*} & H^G_{2\dim E-k}(E) \end{array}$$

the lower morphism μ^* is the pullback morphism which gives the Thom isomorphism, therefore the upper μ^* is also an isomorphism.

(2) By the last proof, we already know $H_G^*(E) \cong H_G^*(G/P)$. Then apply the isomorphism of Arabia see [Ara85], this gives $H_G^*(G/P) \cong H_P^*(pt)$. Now, P homotopy-retracts on its Levy subgroup L, this implies $H_P^*(pt) = H_L^*(pt)$, together with the (2) in the previous lemma we are done.

□

Lemma 18. ***(The cohomology ring of the flag variety as subalgebra of the Steinberg algebra)***
Let G be reductive group, $T \subset P \subset G$ with P parabolic and T a maximal torus. Let V be a G-representation and $F \subset V$ be a P-subrepresentation, let $E := G \times^P F$ and $Z := E \times_V E$ be the associated Steinberg variety. The diagonal morphism $E \to E \times E$ factorizes over Z and induces an isomorphism $E \to Z_e$ which induces an isomorphism of algebras

$$H_G^*(G/P) \to H^G_{2\dim E-*}(Z_e),$$

recall that the convolution product on $H_*^G(Z_e)$ maps degrees $(i,j) \mapsto i+j-2\dim E$.

proof: Obviously you have an isomorphism

$$H_G^*(G/B) \xrightarrow{\mu^*} H_G^*(E) \cong H_G^*(Z_e) \to H^G_{2\dim E-*}(Z_e)$$

where the last isomorphism is Poincare duality. But we need to see that this is a morphism of algebras where $H_*^G(Z_e)$ is the convolution algebra with respect to the embedding $Z_e \cong E \xrightarrow{diag} E \times E$. This follows from [CG97], Example 2.7.10 and section 2.6.15.

□

We observe that the algebra $\mathbb{C}[\mathfrak{t}]$ with generators $t \in \mathfrak{t}^*$ in degree 2 plays three different roles in the last lemmata. It is the T-equivariant cohomology of a point, it is the G-equivariant cohomology of a complete flag variety G/B, it can be found as the subalgebra $H_*^G(Z_e) \subset H_*^G(Z)$.

2.4 Computation of fixed points

Recall the following result, for example see [Här99], satz 2.12, page 13.

Lemma 19. *Let $T \subset P \subset G$ be reductive group with a parabolic subgroup P and a maximal torus T. Let W be the Weyl group associated to (G,T) and $Stab(P) := \{w \in W \mid wPw^{-1} = P\}$. For $w = xStab(P) \in W/Stab(P)$ we set $wP := xP \in G/P$. Then, it holds*

$$(G/P)^T = \{wP \in G/P \mid w \in W/Stab(P)\}$$

Lemma 20. *Let $P_1, P_2 \subset G$ be a reductive group with two parabolic subgroup, $F_1, F_2 \subset V$ a G-representation with a P_1 and P_2-subrepresentation. Assume $(GF_i)^T = \{0\}$. We write $(E_i = G \times^{P_i} F_i, \mu_i\colon E_i \to G/P_i, \pi\colon E_i \to V)$ for the associated Springer triple and $Z := E_1 \times_V E_2, m\colon Z \to (G/P_1) \times (G/P_2)$ for the Steinberg variety.*
Then, there are induced a bijections $\mu_i^T\colon E_i^T \to (G/P_i)^T, m^T\colon Z^T \to (G/P_1)^T \times (G/P_2)^T$. More explicit we have

$$\begin{aligned} E_i^T &= \{\phi_w := (0, wP_i) \in V \times G/P_i \mid w \in W/Stab(P_i)\} \subset E_i \\ Z^T &= \{\phi_{x,y} := (0, xP_1, yP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W/Stab(P_1), y \in W/Stab(P_2)\} \\ &\subset Z. \end{aligned}$$

Furthermore, for any $w \in W/Stab(P_2)$ let $Z^w := m^{-1}(G \cdot (P_1, wP_2))$ and

$$m_w := m|_{Z^w}\colon Z^w \to G \cdot (P_1, wP_2)$$

be the induced map. There is an induced Bruhat order $\leq$ on $W/Stab(P_2)$ by taking the

Bruhat order of minimal length representatives.

$$\begin{aligned}(Z^w)^T &= \{\phi_{x,xw} = (0, xP_1, xwP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W\}\\ \overline{Z^w}^T &= \{\phi_{x,xv} \mid x \in W, v \leq w\} = \bigcup_{v \leq w} (Z^v)^T\end{aligned}$$

There is a bijection $W/(Stab(P_1) \cap {}^w Stab(P_2)) \to (Z^w)^T,\ x \mapsto \phi_{x,xw}$.

proof Obviously, it holds $E_i^T \subset V^T \times (G/P_i)^T = \{0\} \times (G/P_i)^T$. But we also have a zero section s of the vector bundle $\pi\colon E_i \to G/P_i$ which gives the closed embedding $G/P_i \to E_i \subset V \times (G/P_i), gP_i \mapsto (0, gP_i)$.
It holds $Z^T \subset V^T \times (G/P_1)^T \times (G/P_2)^T = \{0\} \times (G/P_1)^T \times (G/P_2)^T$. But using the description of $Z = \{(v, gP_1, hP_2) \in V \times G/P_1 \times G/P_2 \mid (v, gP_1) \in E_1, (v, hP_2) \in E_2\}$, we see that $\{0\} \times (G/P_1)^T \times (G/P_2)^T \subset Z$ and these are obviously T-fixed points.
We have $(Z^w)^T \subset Z^w \cap Z^T = \{\phi_{x,xw} \mid x \in W\}$ and one can see the other inclusion, too. Also, we have $\overline{Z^w}^T \subset (\bigcup_{v \leq w} Z^v)^T = \bigcup_{v \leq w} (Z^v)^T$. Consider the closed embedding

$$s\colon G/P_1 \times G/P_2 \to Z, \quad (gP_1, hP_2) \mapsto (0, gP_1, hP_2).$$

Clearly $s(G(P_1, wP_2)) \subset Z^w \subset \overline{Z^w}$, but since s is a closed embedding we have

$$\bigcup_{v \leq w} (Z^v)^T \subset s(\overline{G(P_1, wP_2)}) = \overline{s(G(P_1, wP_2))} \subset \overline{Z^w}$$

which yields the other inclusion. □

2.4.1 Notation for the fixed points

Now, in the set-up of the beginning this gives the following:
Observe, that $(\bigsqcup_{i \in I_J} G_i/P_i)^T = ((\mathbb{G}/\mathbb{P}_J)^H)^T = (\mathbb{G}/\mathbb{P}_J)^T$, and $(\mathbb{G}/\mathbb{P}_J)^T = \{w\mathbb{P}_J \mid w \in \mathbb{W}^J\}$. For any $w \in \mathbb{W}^J$ there exists a unique $i \in I_J$ such that $x^i := wx_i^{-1} \in W$, this implies $w\mathbb{P}_J = x^i(x_i\mathbb{P}_J) = \iota_i(x^iP_i) \in (G/P_i)^T$. Therefore, we write

$$\begin{aligned}(\bigsqcup_{i \in I_J} G_i/P_i)^T = (\mathbb{G}/\mathbb{P}_J)^T &= \bigsqcup_{i \in I_J} \{wx_i\mathbb{P}_J \mid w \in W/{}^{x_i}\mathbb{W}_J \subset \mathbb{W}/{}^{x_i}\mathbb{W}_J\}\\ E_J^T &= \{\phi_{wx_i} = (0, wx_i\mathbb{P}_J) \mid i \in I_J, w \in W/{}^{x_i}\mathbb{W}_J\}\\ Z_J^T &= \bigsqcup_{i,j \in I_J} \{\phi_{wx_i,vx_j} = (0, wx_i\mathbb{P}_J, vx_j\mathbb{P}_J) \mid w \in W/{}^{x_i}\mathbb{W}_J, v \in W/{}^{x_j}\mathbb{W}_J\}.\end{aligned}$$

Let $w, v \in \mathbb{W}^J$ and $i, j \in I_J$ such that $w^i := wx_i^{-1} \in W, v^j := vx_j^{-1} \in W$. We then set $\phi_w := \phi_{w^ix_i}$, $\phi_{w,v} = \phi_{w^ix_i,v^j,x_j}$.
As we have bijections $(0, wx_i\mathbb{P}_J) \mapsto wx_i\mathbb{P}_J, (0, wx_i\mathbb{P}_J, vx_j\mathbb{P}_J) \mapsto (wx_i\mathbb{P}_J, vx_j\mathbb{P}_J)$ between E^T and $(\mathbb{G}/\mathbb{P}_J)^T$, Z^T and $(\mathbb{G}/\mathbb{P}_J \times \mathbb{G}/\mathbb{P}_J)^T$, we denote the T-fixed by the same symbols.

2.4.2 The fibres over the fixpoints

Remember, by definition we have $F_i = \mu_J^{-1}(\phi_{x_i})$. For any $w = w^i x_i \in \mathbb{W}^J, w^i \in W$ We set

$$F_w := \mu_J^{-1}(\phi_w) = \mu_i^{-1}(\phi_{w^i x_i}) = {}^{w^i}F_i = \bigoplus_{k=1}^{r} V^{(k)} \cap {}^{w}\mathcal{U}^{(k)}$$

and if also $x \in \mathbb{W}/(\mathbb{W}_J \cap {}^{w}\mathbb{W}_J)$ (i.e. the definition does not depend on the choice of a representative in the coset)

$$\begin{aligned} F_{x,xw} &:= m_J^{-1}(\phi_{x,xw}) = F_x \cap F_{xw} \\ &= \bigoplus_{k=1}^{r} V^{(k)} \cap {}^{x}[\mathcal{U}^{(k)} \cap {}^{w}\mathcal{U}^{(k)}] \end{aligned}$$

For $J = \emptyset, \mathcal{U} = \mathrm{Lie}(\mathbb{U})^{\oplus r}$**:** We choose $V = \bigoplus_{k=1}^{t} V^{(k)} \oplus \bigoplus_{k=t+1}^{r} V^{(k)}$ with $V^{(k)} \subset \mathcal{R}, 1 \leq k \leq t$, $V^{(k)} = \mathfrak{g}^{(k)}$ with $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ is a direct summand, $t+1 \leq k \leq r$. The fibres look like

$$F_w = \bigoplus_{k=1}^{t} V^{(k)} \cap {}^{w}\,\mathrm{Lie}(\mathbb{U}) \oplus \bigoplus_{k=t+1}^{r} V^{(k)} \cap {}^{w}\mathfrak{u}^{(k)}$$

where $\mathfrak{u}^{(k)}$ is the Lie subalgebra spanned by the weights > 0 in $\mathfrak{g}^{(k)}$.

$$F_{x,xw} = \bigoplus_{k=1}^{t} V_k \cap {}^{x}[(\mathrm{Lie}(\mathbb{U})) \cap {}^{w}(\mathrm{Lie}(\mathbb{U}))] \oplus \bigoplus_{k=t+1}^{r} V_k \cap {}^{x}[\mathfrak{u}^{(k)} \cap {}^{w}\mathfrak{u}^{(k)}]$$

Lemma 21. *Assume $J = \emptyset, \mathcal{U} = \mathrm{Lie}(\mathbb{U})^{\oplus r}$. Let $x \in \mathbb{W}, s \in \mathbb{S}$ we set*

$$h_{\overline{x}}(s) := \#\{k \in \{1, \ldots r\} \mid x(\alpha_s) \in \Phi_{V^{(k)}}\}$$

where $V = \bigoplus_{k=1}^{r} V^{(k)}$ and $\Phi_{V^{(k)}} \subset \underline{\Phi}$ are the T-weights of $V^{(k)}$. If $x = x^i x_i$ with $x^i \in W$, then $h_{\overline{x}}(s) = h_{\overline{x_i}}(s) =: h_i(s)$. It holds

$$F_{x_i}/F_{x_i,x_i s} = (\mathcal{G}_{x_i(\alpha_s)})^{\oplus h_i(s)}.$$

(1) If ${}^{x}s \notin W$ then

$$h_i(s) = \#\{k \mid V^{(k)} \subset \mathcal{R},\ x_i(\alpha_s) \in \Phi_{V^{(k)}}\}.$$

(2) If ${}^{x}s \in W$, then

$$h_i(s) = \#\{k \mid V^{(k)} \subset \mathfrak{g},\ x_i(\alpha_s) \in \Phi_{V^{(k)}}\}.$$

proof: Without loss of generality $V \subset \mathcal{G}, \mathcal{U} = \mathrm{Lie}(\mathbb{U})$, set $x := x_i$, we have a short exact sequence

$$0 \to V \cap {}^{x}[\mathcal{U} \cap {}^{s}\mathcal{U}] \to V \cap {}^{x}\mathcal{U} \to V \cap \mathcal{G}_{x(\alpha_s)} \to 0$$

Now, $V \cap \mathcal{G}_{x(\alpha_s)} = 0$ if and only if $x(\alpha_s) \notin \Phi_V$.

(1) If ${}^{x}s \notin W$ then $x(\alpha_s) \notin \Phi$ where Φ are the T-weights of $\mathfrak{g}$. That means, if $V \subset \mathfrak{g}$ we get $h_i(s) = 0$.

(2) If ${}^{x}s \in W$, then $x(\alpha_s) \in \Phi$. This means, if $V \subset \mathcal{R}$ we get $h_i(s) = 0$.

□

2.5 Relative position stratification

2.5.1 In the flag varieties

Let $J \subset \mathbb{S}$, $w \in {}^{J}\mathbb{W}^{J}$, $i, j \in I_J$. We define

$$\begin{aligned}
C^{w} &:= \mathbb{G}\phi_{e,w} \cap \left(\bigsqcup_{i \in I_J} G_i/P_i \times \bigsqcup_{i \in I_J} G_i/P_i \right) \\
C^{\leq w} &:= \overline{\mathbb{G}\phi_{e,w}} \cap \left(\bigsqcup_{i \in I_J} G_i/P_i \times \bigsqcup_{i \in I_J} G_i/P_i \right) \\
C^{w}_{i,j} &:= C^{w} \cap (G/P_i \times G/P_j) \\
C^{\leq w}_{i,j} &:= C^{\leq w} \cap (G/P_i \times G/P_j)
\end{aligned}$$

For an arbitrary $w \in \mathbb{W}$ there exists a unique $v \in {}^{J}\mathbb{W}^{J}$ such that $\mathbb{W}_J w \mathbb{W}_J = \mathbb{W}_J v \mathbb{W}_J$, we set $C^{w} := C^{v}, C^{w}_{i,j} := C^{v}_{i,j}, C^{\leq w} := C^{\leq v}, C^{\leq w}_{i,j} := C^{\leq v}_{i,j}$. We remark that $C^{\leq w}, C^{\leq w}_{i,j}$ are closed (but not necessary the closure of $C^{w}, C^{w}_{i,j}$, because it can happen that $C^{w}_{i,j} = \emptyset, C^{\leq w}_{i,j} \neq \emptyset$, see next lemma (3)).
Let $i, j \in I_J, \mathcal{C}_{i,j} := \{C^{w}_{i,j} \mid w \in {}^{J}\mathbb{W}^{J}, C^{w}_{i,j} \neq \emptyset\}, Orb_{i,j} := \{G\text{-orbits in } G/P_i \times G/P_j\}$, we have the following commutative diagram

$$\begin{array}{ccc}
Orb_{i,j} & \xrightarrow{\quad rp \quad} & W \cap {}^{x_i}\mathbb{W}_J \setminus W / W \cap {}^{x_j}\mathbb{W}_J \\
\Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Psi} \\
\mathcal{C}_{i,j} & \xrightarrow{\quad rp_{\mathbb{W}} \quad} & \{({}^{x_i}\mathbb{W}_J) w ({}^{x_j}\mathbb{W}_J) \mid w \in W\}
\end{array}$$

defined as follows

$$\begin{aligned}
rp(G\phi_{x_i, w x_j}) &:= (W \cap {}^{x_i}\mathbb{W}_J) w (W \cap {}^{x_j}\mathbb{W}_J), \\
rp_{\mathbb{W}}(C^{w}_{i,j}) &:= {}^{x_i}\mathbb{W}_J (x_i w x_j^{-1}) {}^{x_j}\mathbb{W}_J \\
\Phi(G\phi_{x_i, w x_j}) &:= C^{x_i^{-1} w x_j}_{i,j} \quad (\supset G\phi_{x_i, w x_j}) \\
\Psi((W \cap {}^{x_i}\mathbb{W}_J) w (W \cap {}^{x_j}\mathbb{W}_J)) &:= ({}^{x_i}\mathbb{W}_J) w ({}^{x_j}\mathbb{W}_J),
\end{aligned}$$

$rp, rp_{\mathbb{W}}$ are bijections and Φ, Ψ are surjections. We will from now on assume that Φ, Ψ are bijections as well, i.e. for every nonempty $C^{w}_{i,j}$ there is a $w_0 \in W$ such that $\mathbb{W}_J x_i^{-1} w_0 x_j \mathbb{W}_J = \mathbb{W}_J w \mathbb{W}_J$ and $C^{w}_{i,j} = G\phi_{x_i, w_0 x_j} \subset G/P_i \times G/P_j$, this implies

$$C^{w}_{i,j} \cong G/(P_i \cap {}^{w_0}P_j \cap G).$$

Lemma 22. *Let $J \subset \mathbb{S}, s \in \mathbb{S} \setminus J, i, j \in I_J$.*

(1) $C^{\leq s}$ is smooth, it equals $C^s \cup C^e$.

(2) $C^{\leq s}_{ij} = \emptyset$ unless $Wx_j\mathbb{W}_J \in \{Wx_i\mathbb{W}_J, Wx_is\mathbb{W}_J\}$.

(3) Assume that $Wx_i\mathbb{W}_J \neq Wx_is\mathbb{W}_J$ and let $j \in I_J$ such that $x_isx_j^{-1} \in W$, then it holds

$$\iota_i(G/P_i) \neq \iota_j(G/P_j), \quad C^{\leq s}_{i,j} = C^s_{i,j}, \quad C^{\leq s}_{i,i} = C^e_{i,i}$$

and $G \cap {}^{x_i}[\mathbb{P}_J \cap {}^s\mathbb{P}_J] = G \cap {}^{x_i}\mathbb{P}_{J\cap {}^sJ}$, $C^s_{ij} = G/(G \cap {}^{x_i}\mathbb{P}_{J\cap {}^sJ})$.

(4) Assume that $Wx_i\mathbb{W}_J = Wx_is\mathbb{W}_J = Wx_j\mathbb{W}_J$, then it holds $i = j$, in particular

$$\iota_i(G/P_i) = \iota_j(G/P_j), \quad C^w_{i,j} = C^w_{i,i}, \text{ for all } w$$

and the first equality implies $({}^{x_i}\mathbb{P}_J) \cap G \neq ({}^{x_is}\mathbb{P}_J) \cap G$, there is an isomorphism of G-varieties

$$G \times^{P_i} \left(({}^{x_i}\mathbb{P}_{J\cup\{s\}} \cap G)/P_i\right) \to C^{\leq s}_{i,i}, \quad (g, hP_i) \mapsto (gP_i, ghP_i).$$

proof:

(1) The variety $\bigsqcup_{i\in I_J} G_i/P_i$ is a smooth subvariety of $\mathbb{G}/\mathbb{P}_J$ because each $G/P_i, i \in I_J$ is smooth. It is known that $\overline{\mathbb{G}\phi_{e,s}} = \mathbb{G}\phi_{e,s} \cup \mathbb{G}\phi_{e,e}$ is smooth in $\mathbb{G}/\mathbb{P}_J$, therefore its intersection (i.e. pullback) is smooth in $(\mathbb{G}/\mathbb{P}_J)^H$.

(2) Now, $C^{\leq s}_{i,j} = C^s_{i,j} \cup C^e_{i,j}$ and $C^s_{i,j} \neq \emptyset$ iff it contains a T-fixed point ϕ_{x_i,vx_j} for a $v \in W$, that implies $x_i^{-1}vx_j\mathbb{P}_J = s\mathbb{P}_J$, i.e. there is an $f \in \mathbb{P}_J$ such that $vx_jf = x_is$, therefore $f \in \mathbb{P}_J \cap \mathbb{W} = \mathbb{W}_J$ and $Wx_j\mathbb{W}_J = Wx_is\mathbb{W}_J$. Similar $C^e_{i,j} \neq \emptyset$ iff $Wx_j\mathbb{W}_J = Wx_i\mathbb{W}_J$.

(3) The intersection $(G/P_i) \cap (G/P_j)$ is a G-equivariant subset of $\mathbb{G}/\mathbb{P}_J$, therefore it is nonempty iff it contains all T-fixed points $vx_i\mathbb{P}_J = wx_j\mathbb{P}_J$ with $v, w \in W$. But this is equivalent to $Wx_i\mathbb{W}_J = Wx_j\mathbb{W}_J$.
As we have seen before $Wx_is\mathbb{W}_J = Wx_j\mathbb{W}_J$ implies $C^e_{i,j} = \emptyset, C^s_{i,i} = \emptyset$ and therefore $C^{\leq s}_{i,j} = C^s_{i,j}, C^{\leq s}_{i,i} = C^e_{i,i}$.
Let $Wx_i\mathbb{W}_J \neq Wx_is\mathbb{W}_J$, we need to show $G \cap {}^{x_i}[\mathbb{P}_J \cap {}^s\mathbb{P}_J] = G \cap {}^{x_i}\mathbb{P}_{J\cap {}^sJ}$ Let $\underline{\Phi} = \underline{\Phi}_+ \cup \underline{\Phi}^-$ be the set of roots for $(\mathbb{G}, \mathbb{B}, T)$ decomposing as positive and negative roots, let $\underline{\Delta}_J \subset \underline{\Phi}_+$ be the simple roots corresponding to $J \subset \mathbb{S}$ and let Φ be the roots for (G, T). It is enough to prove that the T-weights on $\mathrm{Lie}(G \cap {}^{x_i}[\mathbb{P}_J \cap {}^s\mathbb{P}_J])$ equal the T-weights on $\mathrm{Lie}(G \cap {}^{x_i}\mathbb{P}_{J\cap {}^sJ})$.

Now, $Wx_i\mathbb{W}_J \neq Wx_is\mathbb{W}_J$ implies ${}^{x_i}s \notin W$ or equivalently $x_i(\alpha_s) \notin \Phi$ where $\alpha_s \in \underline{\Phi}_+$ is the simple root negated by s. We have the T-weights of $\mathrm{Lie}({}^{x_i}\mathbb{P}_J)$ are $\{x_i(\alpha) \mid \alpha \in \underline{\Phi}_+ \cup -\underline{\Delta}_J\}$,
the T-weights of $\mathrm{Lie}({}^{x_is}\mathbb{P}_J)$ are $\{x_i(\alpha) \mid \alpha \in \underline{\Phi}_+ \setminus \{\alpha_s\} \cup -s(\underline{\Delta}_J) \cup \{-\alpha_s\}\}$.

It follows that the T-weights of $\mathrm{Lie}(G \cap {}^{x_i}\mathbb{P}_J \cap {}^{x_i s}\mathbb{P}_J)$ are

$$\begin{aligned} &\{x_i(\alpha) \mid \alpha \in (\underline{\Phi}_+ \cup [-\underline{\Delta}_J \cap -s(\underline{\Delta}_J)]) \cap \Phi\} \\ =& \{x_i(\alpha) \mid \alpha \in (\underline{\Phi}_+ \cup -\underline{\Delta}_{J \cap {}^s J}) \cap \Phi\} \end{aligned}$$

and these are the T-weights of $\mathrm{Lie}(G \cap {}^{x_i}\mathbb{P}_{J \cap {}^s J})$.

(4) The first part is by definition. Assume $Wx_i\mathbb{W}_J = Wx_i s\mathbb{W}_J$ implies $x_i s x_i^{-1} = ab$ with $a \in W, b \in {}^{x_i}\mathbb{W}_J$. Now ${}^{x_i}\mathbb{P}_J \cap G$ is a parabolic subgroup of G conjugated to $P_{J\cap S}$, therefore

$$^{x_i s}\mathbb{P}_J \cap G = ({}^{x_i s x_i^{-1}}({}^{x_i}\mathbb{P}_J)) \cap G = {}^a({}^{x_i}\mathbb{P}_J) \cap G = {}^a(({}^{x_i}\mathbb{P}_J) \cap G)$$

and assume that this is equal ${}^{x_i}\mathbb{P}_J \cap G$ that implies $a \in x_i\langle J \cap S\rangle x_i^{-1}$, then $x_i s x_i^{-1} = ab \in {}^{x_i}\mathbb{W}_J$ that implies $s \in J$ contradicting our assumption $s \notin J$.
Finally, consider the closed embedding $G \times^{P_i} \left(({}^{x_i}\mathbb{P}_{J\cup\{s\}} \cap G)/P_i\right) \to G \times^{P_i} G/P_i$ and compose it with the G-equivariant isomorphism

$$G \times^{P_i} G/P_i \to G/P_i \times G/P_i, \overline{(g, hP_i)} \mapsto (gP_i, ghP_i).$$

The image is precisely $C^s_{i,i} \cup C^e_{i,i}$.

□

2.5.2 In the Steinberg variety

Let $w \in \mathbb{W}^J$, $i, j \in I_J$, recall that we have a map $m_J \colon Z_J \to \mathbb{G}/\mathbb{P}_J$.

$$\begin{aligned} Z^w_{i,j} &:= m^{-1}_{i,j}(C^w_{i,j}) \\ Z^w = Z^w_J &:= \bigsqcup_{i,j\in I_J} Z^w_{i,j} \\ Z^{\leq w} = Z^{\leq w}_J &= \bigcup_{v \leq w, v\in\mathbb{W}^J} Z^v_J \\ Z^{\leq w}_{i,j} &:= \bigcup_{v \leq w, v\in\mathbb{W}^J} Z^v_{i,j} \end{aligned}$$

Lemma 23. *(a) If $C^w_{i,j} \neq \emptyset$, the restriction $m_{i,j} \colon Z^w_{i,j} \to C^w_{i,j}$ is a vector bundle with fibres isomorphic to $F_i \cap {}^{x_i w x_j^{-1}} F_j$, it induces a bijection on T-fixed points. In particular, all nonempty $Z^w_{i,j}$ are smooth.*

(b) For any $s \in \mathbb{S}$ the restriction $m \colon \overline{Z^s} \to C^{\leq s}$ is a vector bundle over its image, in particular $\overline{Z^s}$ is smooth. More precisely, it is a disjoint union $\overline{Z^s_{i,j}} \to C^{\leq s}_{i,j}$ with

(1) $\overline{Z^s_{i,j}} \neq \emptyset$ implies $Wx_j\mathbb{W}_J = Wx_i s\mathbb{W}_J$.

(2) Assume that $Wx_i\mathbb{W}_J \neq Wx_i s\mathbb{W}_J$, then $\overline{Z^s_{i,j}} = Z^s_{i,j}$ and $\overline{Z^s_{i,i}} = \emptyset$.

(3) Assume that $Wx_i\mathbb{W}_J = Wx_i s\mathbb{W}_J$, then it holds $\overline{Z^s_{i,i}} \to C^{\leq s}_{i,i}$ is a vector bundle.

proof:

(a) As $C^w_{i,j}$ is assumed to be a diagonal G-orbit in $G/P_i \times G/P_j$, it is a homogeneous space and the statement easily follows from a wellknown lemma, cp. [Slo80b], p.26, lemma 4.

(b) (1) If $\overline{Z^s_{i,j}} \neq \emptyset$, then $C^s_{i,j} \neq \emptyset$ and by the proof of the previous lemma 22, (2), the claim follows.

(2) If $Wx_i\mathbb{W}_J \neq Wx_is\mathbb{W}_J$, then by lemma 22, (3), $C^{\leq s}_{i,j} = C^s_{i,j}$ is already closed, therefore $Z^s_{i,j}$ is closed as well. Also, $C^{\leq s}_{i,i} = C^e_{i,i}$ is already closed, therefore $Z^e_{i,i}$ is closed as well.

(3) If $Wx_i\mathbb{W}_J = Wx_is\mathbb{W}_J$, then $C^{\leq s}_{i,i}$ is the closure of the G-orbit $C^s_{i,i}$ and by lemma 22, (4) we have $G \times^{P_i} \big(({}^{x_i}\mathbb{P}_{J\cup\{s\}} \cap G)/P_i\big) \to C^{\leq s}_{i,i}, \quad (g, hP_i) \mapsto (gP_i, ghP_i)$ is an isomorphism. We set $X :=$

$$X := \{(gf, gP_i, ghP_i) \in G(F_i \cap {}^{x_is}F_i) \times G/P_i \times G/P_i \mid \\ g \in G, f \in F_i \cap {}^{x_isx_i^{-1}}F_i, h \in {}^{x_i}\mathbb{P}_{J\cup\{s\}} \cap G\}$$

and we claim $\overline{Z^s_{i,i}} = X$. First, observe that $X \subset Z_{i,i}$ because $gf = gh(h^{-1}f)$ with $h^{-1}f \in F_i \cap {}^{x_isx_i^{-1}}F_i$. One can easily check the following steps.

(*) $X \to C^{\leq s}_{i,i}$ is a vector bundle with fibre over $F_i \cap {}^{x_isx_i^{-1}}F_i$. In particular, we get that X is smooth irreducible and $\dim X = \dim Z^s_{i,i}$.

(*) $Z^s_{i,i} \subset X$.

(*) X is closed in $Z_{i,i}$ because we can write it as $X = p^{-1}(G(F_i \cap {}^{x_isx_i^{-1}}F_i)) \cap m^{-1}(C^{\leq s}_{i,i})$. Since $F_i \cap {}^{x_isx_i^{-1}}F_i$ is (by definition) $B_i = {}^{x_i}B$-stable, we get $G(F_i \cap {}^{x_isx_i^{-1}}F_i)$ is closed in V. This implies X is closed.

2.6 A short lamentation on the parabolic case

From the next section on we assume that all $P_i = B_i$ are Borel subgroups. What goes wrong with the more general assumption (which we call the *parabolic case*)?

(1) We do not know whether $C^w_{i,j}$ (see previous section) is always a G-orbit. That is relevant for Euler class computation in Lemma 14.

(2) The cellular fibration property has to be generalized because $C^w := \{gP, gwP' \mid g \in G\} \subset G/P \times G/P' \xrightarrow{pr_1} G/P$ is not a vector bundle (its fibres are unions of Schubert cells). This complicates Lemma 12.

(3) We do not know what is the analogue of lemma 13, i.e. what can we say about $\mathcal{Z}^{\leq x} * \mathcal{Z}^{\leq y}$?

(4) The cycles $[\overline{Z^s_{i,j}}]$ are not in general multiplicative generators. If we try to understand more generally $[\overline{Z^w_{i,j}}]$, the multiplicity formular does not give us as much information

as for $[\overline{Z^s_{i,j}}]$ because $\overline{Z^s_{i,j}}$ is even smooth. Also understanding the $[Z^w_{i,j}]$ is not enough, since they do not give a basis as a free $\mathcal{E}$-module because the rank is wrong (cp. failing of cellular fibration lemma).

The point (4) is the biggest problem. Even for $H^G_*(G/P \times G/P)$ we do not know a set of generators and relations (see next chapter).
So, from now on we assume $J = \emptyset$.

2.7 Convolution operation on the equivariant Borel-Moore homology of the Steinberg variety

Definition 7. Let $H \in \{pt, T, G\}$ with $T \subset G$ where T is a maximal torus.
We define the *H-equivariant algebra of a point* to be $H^*_H(pt)$ with product equals the cup-product, we will always identify it with $H^H_*(pt) := H^{-*}_H(pt)$. It is a graded $\mathbb{C}$-algebra concentrated in negative even degrees.
We define the *H-equivariant Steinberg algebra* to be the H-equivariant Borel-Moore homology algebra of the Steinberg variety, the product is the convolution product, see [CG97], [Var09].
We say (H-equivariant)**company algbra** to the H-equivariant cohomology algebra of E, the product is the cup-product.

$$\mathcal{Z}_H := H^H_*(Z) \text{ for the } H\text{-equivariant Steinberg algebra,}$$
$$\mathcal{E}_H := H^*_H(E) \text{ for the } H\text{-equivariant company algebra.}$$

For $H = pt$ we leave out the adjective H-equivariant and leave out the index H.

Recall, that $\mathcal{Z}_H$ and $\mathcal{E}_H$ are left graded modules over Λ_H. Furthermore, $\mathcal{E}_H$ is a left module over $\mathcal{Z}_H$. This follows from considering $M_1 = M_2 = M_3 = E$ smooth manifolds and $Z \subset M_1 \times M_2, E = E \times \{\overline{(e,0)}\} \subset M_2 \times M_3$. Then the set-theoretic convolution gives $Z \circ E = E$, which implies the operation.

Also, $\mathcal{Z}_H$ is a left module over $\mathcal{E}_H$. This follows from considering $M_1 = M_2 = M_3 = E$ smooth manifolds ($\dim_{\mathbb{C}} E =: e$) and $E \hookrightarrow M_1 \times M_2$ diagonally, $Z \subset M_2 \times M_3$, then the set-theoretic convoltion gives $E \circ Z = Z$, that implies that we have a map

$$H^H_{2e_i-p}(E_i) \times H^H_{e_i+e_j-q}(Z_{i,j}) \to H^H_{e_i+e_j-(p+q)}(Z_{i,j})$$

Using Poincare duality we get $H^H_{2e_i-p}(E) \cong H^p_H(E)$ and the grading
$H^H_{[q]}(Z) := \bigoplus_{i,j} H^H_{e_i+e_j-q}(Z)$ the previous map gives an operation of the $H^*_H(E)$ on $H^H_{[*]}(Z)$ which is $H^H_*(pt)$-linear. We denote the operations by

$$\begin{aligned} *\colon \mathcal{Z}_H \times \mathcal{E}_H &\to \mathcal{E}_H \\ \diamond\colon \mathcal{E}_H \times \mathcal{Z}_H &\to \mathcal{Z}_H \end{aligned}$$

Furthermore, there are forgetful algebra homomorphisms

$$\begin{aligned} H^*_G(pt) \to H^*_T(pt) &\to H*(pt) = \mathbb{C}, \\ \mathcal{Z}_G \to \mathcal{Z}_T &\to \mathcal{Z}, \\ \mathcal{E}_G \to \mathcal{E}_T &\to \mathcal{E}. \end{aligned}$$

Let us investigate some elementary properties of the convolution operations. From [VV11], section 5, p.606, we know that the operation of $\mathcal{Z}_G$ on $\mathcal{E}_G$ is faithful, i.e. we get an injective $\mathbb{C}$-algebra homomorphism

$$\mathcal{Z}_G \hookrightarrow \operatorname{End}(\mathcal{E}_G).$$

We have the following cellular fibration property. We choose a total order $\leq$ refining Bruhat order on $\mathbb{W}$. For each $i, j \in I$ we get a filtration into closed G-stable subsets of $Z_{i,j}$ by setting $Z^{\leq w}_{i,j} := \bigcup_{v \leq w} Z^v_{i,j},\ w \in \mathbb{W}$. Via the first projection $pr_1 \colon C^v_{i,j} \to G/B_i$ is a G-equivariant vector bundle with fibre $B_i v B_j/B_j$, we call its (complex) dimension $d^v_{i,j}$, also $Z^v_{i,j} \to C^v_{i,j}$ is a G-equivariant vector bundle, we define the complex fibre dimension $f^v_{i,j}$. By the G-equivariant Thom isomorphism (applied twice) we get

$$H^G_m(Z^v_{i,j}) = H^G_{m-2d^v_{i,j}-2f^v_{i,j}}(G/B_i).$$

In particular, it is zero when m is odd and $H^G_*(Z^v_{i,j})$ is a free $H^G_*(pt)$-module with basis $b_x,\ x \in W, \deg b_x = 2\dim(B_i x B_i)/B_i + 2d^v_{i,j} + 2f^v_{i,j}$.
Using the long exact localization sequence in G-equivariant Borel-Moore homology for every $v \in \mathbb{W}$, we see that $Z^v_{i,j}$ is open in $Z^{\leq v}_{i,j}$ with an closed complement $Z^{<v}_{i,j}$. We conclude inductively using the Thom isomorphism that $H^G_{odd}(Z^{\leq v}_{i,j}) = 0$ and that $H^G_*(Z^{\leq w}_{i,j}) = \bigoplus_{v \leq w} H^G_*(Z^v_{i,j})$. We observe, that $\#\{w \in \mathbb{W} \mid Z^w_{i,j} \neq \emptyset\} = \#W$ for every $i, j \in I$. It follows that $H^G_*(Z_{i,j})$ is a free $H^G_*(pt)$-module of rank $\#(W \times W)$, and that every $H^G_*(Z^{\leq v}_{i,j}) \xrightarrow{i_*} H^G_*(Z_{i,j})$ is injective.
We can strengthen this result to the following lemma.

Lemma 24. *Let $\leq$ be a total order refining Bruhat order on $\mathbb{W}$. For any $w \in \mathbb{W}$ set $Z^{\leq w} := m^{-1}(\bigcup_{v \leq w} C^v) = \bigcup_{v \leq w} Z^v$. The closed embedding $i \colon Z^{\leq v} \to Z$ gives rise to an injective morphism of $H^*_G(E)$-modules $i_* \colon \mathcal{Z}^{\leq v}_G := H^G_*(Z^{\leq v}) \to \mathcal{Z}_G$. We identify in the following $\mathcal{Z}^{\leq v}_G$ with its image in $\mathcal{Z}_G$. For all $v \in W$ we have*

$$\mathcal{Z}^{\leq w}_G = \bigoplus_{v \leq w} \mathcal{E}_G \diamond [\overline{Z^v}] \qquad \text{as } \mathcal{E}_G\text{-module}$$

$$1_i * \mathcal{Z}^{\leq w}_G * 1_j = \bigoplus_{v \leq w} \mathcal{E}_i \diamond [\overline{Z^v_{i,j}}] \qquad \text{as } \mathcal{E}_i\text{-module}$$

where $\mathcal{E}_i = H^*_G(E_i)$. Each $[\overline{Z^v}]$ is nonzero (and not necessarily a homogeneous element). In particular, $\mathcal{Z}_G$ (as ungraded module) is a free left $\mathcal{E}_G$-module of rank $\#\mathbb{W}$.

proof: Now first observe that set-theoretically we have $E \circ Z^v = Z^v$ (where we use the diagonal embedding for E again). This implies that the direct sum decomposition $H_*^G(Z) = \bigoplus_{v\in\mathbb{W}} H_*^G(Z^v)$ is already a decomposition of $H_G^*(E)$-modules.

Now we know that we have by the Thom-isomorphism algebra isomorphims

$$H_G^*(E) \cong H_G^*(\bigsqcup_{i\in I} G/B_i) \cong H_G^*(Z^v),$$

using that $\#\{(i,j) \mid Z_{i,j}^v \neq \emptyset\} = \#I$. Now, Poincare duality is given by $H_G^q(Z_{i,j}^v) \to H^G_{2\dim Z_{i,j}^v - q}(Z_{i,j}^v)$, $\alpha \mapsto \alpha \cdot [Z_{i,j}^v]$ the composition gives

$$H_G^p(E_i) \to H^G_{2\dim Z_{i,iv}^v - q}(Z_{i,iv}^v), \quad c \mapsto c \cdot [Z_{i,iv}^v].$$

□

Lemma 25. *For each $x, y \in \mathbb{W}$ with $l(x) + l(y) = l(xy)$ we have*

$$\mathcal{Z}_G^{\leq x} * \mathcal{Z}_G^{\leq y} \subset \mathcal{Z}_G^{\leq xy}$$

proof: By definition of the convolution product, it is enough to check that for all $w \leq x, v \leq y$ it holds for the set theoretic convolution product

$$Z_{i,j}^w \circ Z_{j',k}^v \subset \begin{cases} \emptyset, & j \neq j' \\ Z_{i,k}^{\leq xy}, & j = j' \end{cases}$$

for $i, j, j', k \in I$, because by definition $Z^{\leq x} \circ Z^{\leq y} = \bigcup_{w\leq x, v\leq y} Z^w \circ Z^v$. Now, the case $j \neq j'$ follows directly from the definition. Let $j = j'$. Let $\mathbb{C}^w := \mathbb{G}(\mathbb{B}, w\mathbb{B}) \subset \mathbb{G}/\mathbb{B} \times \mathbb{G}/\mathbb{B}$. According to Hinrich, Joseph [HJ05], 4.3 it holds $\mathbb{C}^w \circ \mathbb{C}^v \subset \mathbb{C}^{wv}$ for all $v, w \in \mathbb{W}$. Now, we can adapt this argument to prove that $C_{i,j}^w \circ C_{j,k}^v \subset C_{j,k}^{wv}$ as follows:
Since $C_{i,j}^w \neq \emptyset, C_{j,k}^v \neq \emptyset$ we have that $w_0 = x_i w x_j^{-1} \in W, v_0 = x_j v x_k^{-1} \in W$ and $C_{i,j}^w = G(B_i, w_0 B_j), C_{j,k}^v = G(B_j, v_0 B_k)$. We pick $M_1 = G/B_i, M_2 = G/B_j, M_3 = G/B_k$ for the convolution and get

$$p_{13}(p_{12}^{-1} C_{i,j}^w \cap p_{23}^{-1} C_{j,k}^v) = \{g(B_i, w_0 b v_0 B_k) \mid g \in G, b \in B_j\}.$$

Now since the length are adding one finds $B_i w_0 B_j v_0 B_k = B_i(w_0 v_0) B_k$, as follows

$$\begin{aligned} w_0 B_j v_0 B_k &= x_i[w({}^{x_j^{-1}}G \cap \mathbb{B}) v({}^{x_k^{-1}}G \cap \mathbb{B})] x_k^{-1} \\ &\subset x_i[w\mathbb{B}v\mathbb{B}]x_k \cap G \subset x_i[\mathbb{B}wv\mathbb{B}]x_k^{-1} \cap G \\ &= [{}^{x_i}\mathbb{B}(x_i wv x_k^{-1})^{x_k}\mathbb{B}] \cap G \qquad = B_i w_0 v_0 B_k \end{aligned}$$

For the last equality, clearly $B_i w_0 v_0 B_k \subset [{}^{x_i}\mathbb{B}(x_i wv x_k^{-1})^{x_k}\mathbb{B}] \cap G$. Assume

$$[{}^{x_i}\mathbb{B}(x_i wv x_k^{-1})^{x_k}\mathbb{B}] \cap G = \bigcup B_i t B_k$$

for certain $t \in W$, then clearly $B_i t B_k \subset [^{x_i}\mathbb{B}(x_i w v x_k^{-1})^{x_k}\mathbb{B}] \cap G \cap [^{x_i}\mathbb{B} t^{x_k}\mathbb{B}] \cap G$ as this intersection is empty if $t \neq (x_i w v x_k^{-1})$, the last equality follows.
Then using $Z_{i,j}^w = \{g(f_i = w_0 f_j, B_i, w_0 B_j) \in V \times G/B_i \times G/B_j \mid g \in G, f_i \in F_i, f_j \in F_j\}$ one concludes by definition that $Z_{i,j}^w \circ Z_{j,k}^v \subset Z_{j,k}^{wv}$ □

We have the following corollary whose proof we have to delay until we have introduced the localization to the T-fixed point.

Corollary 2.7.0.2. *For $s \in \mathbb{S}, w \in \mathbb{W}$ with $l(sw) = l(w) + 1$,*

$$[\overline{Z^s}] * [\overline{Z^w}] = [\overline{Z^{sw}}] \textit{ in } \mathcal{Z}_G^{\leq sw}/\mathcal{Z}_G^{<sw}.$$

Since $[\overline{Z^v}] = \sum_{s,t \in I}[\overline{Z^v_{s,t}}]$ for all $v \in \mathbb{W}$, this is equivalent to $i, j, l, k \in I$ we have

$$[\overline{Z^s_{i,j}}] * [\overline{Z^w_{l,k}}] = \delta_{l,j}[\overline{Z^{sw}_{i,k}}] \textit{ in } \mathcal{Z}_G^{\leq sw}/\mathcal{Z}_G^{<sw}.$$

2.8 Computation of some Euler classes

Definition 8. (Euler class) Let T be a torus and $\mathfrak{t} := \mathrm{Lie}(T)$. Let M be a finite dimensional complex $\mathfrak{t}$-representation. Then, we have a weight space decomposition

$$M = \bigoplus_{\alpha \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})} M_\alpha, \quad M_\alpha = \{m \in M \mid tm = \alpha(t)m\}.$$

We define

$$\mathrm{eu}(M) := \prod_{\alpha \in \mathrm{Hom}(\mathfrak{t},\mathbb{C})} \alpha^{\dim M_\alpha} \quad \in \mathbb{C}[\mathfrak{t}] = H_T^*(pt)$$

For a T-variety X and a T-fixed point $x \in X$, we define the **Euler class** of $x \in X$ to be

$$\mathrm{eu}(X, x) := \mathrm{eu}(T_x X),$$

where the $\mathfrak{t}$-operation on the tangent space $T_x X$ is the differential of the natural T-action. Observe, that $\mathrm{eu}(T_x^* X) = (-1)^{\dim T_x X}\,\mathrm{eu}(T_x X)$.

Recall from an earlier section the notation $Z^w := m^{-1}(C^w)$. We are particularly interested in the following Euler classes, let $w = w^k x_k, x = x^i x_i, y = y^j x_j \in \mathbb{W}, w^k, x^i, y^j \in W$

$$\begin{aligned} \Lambda_w &:= \mathrm{eu}(E, \phi_w) = \mathrm{eu}(T_{\phi_{w^k x_k}} E_k), && \in H_T^*(pt) \\ eu(\overline{Z^w}, \phi_{x,y}) &= (\mathrm{eu}(T_{\phi_{x^i x_i, y^j x_j}} \overline{Z^w_{ij}}))^{-1}, && \in K := Quot(H_T^*(pt)) \end{aligned}$$

Remember $F_w := \mu^{-1}(\phi_w) = \mu_k^{-1}(\phi_{w^k x_k}) = {}^{w^k}F_k$, $F_{x,y} := m^{-1}(\phi_{x,y}) = {}^{x^i}F_i \cap {}^{y^j}F_j = F_x \cap F_y$. In particular, we can see them as $\mathfrak{t}$-representations. We also consider the following $\mathfrak{t}$-representations

$$\mathfrak{n}_w := T_{w^k P_k} G/P_k = \mathfrak{g} \cap {}^w\mathcal{U}^- = {}^{w^k}[\mathfrak{g} \cap {}^{x_k}\mathcal{U}^-]$$

$$\mathfrak{m}_{x,y} := \frac{\mathfrak{n}_x}{\mathfrak{n}_x \cap \mathfrak{n}_y} = \mathfrak{g} \cap \frac{{}^x\mathcal{U}^-}{{}^x\mathcal{U}^- \cap {}^y\mathcal{U}^-}$$

where $\mathcal{U}^- := \mathrm{Lie}(\mathbb{U}^-)$ with $\mathbb{U}^- \subset \mathbb{B}^- := {}^{w_0}\mathbb{B}$ is the unipotent radical where $w_0 \in \mathbb{W}$ is the longest element. Some properties can easily be seen.

(1) $\mathfrak{n}_x = \prod_{\alpha\in\Phi\cap x^{-1}\underline{\Phi}^-} \alpha$.

(2) If $s \in \mathbb{S}$, $x \in \mathbb{W}$ such that ${}^x s \in W$, then

$$\mathrm{eu}(\mathfrak{n}_x) = -\,\mathrm{eu}(\mathfrak{n}_{xs}), \quad \mathrm{eu}(\mathfrak{m}_{x,xs}) = -\,\mathrm{eu}(\mathfrak{m}_{xs,x}) = x(\alpha_s)$$

(3) If $s \in \mathbb{S}$, $x \in \mathbb{W}$ such that ${}^x s \notin W$, then

$$\mathfrak{n}_x = \mathfrak{n}_{xs}, \quad \mathrm{eu}(\mathfrak{m}_{x,xs}) = \mathrm{eu}(\mathfrak{m}_{xs,x}) = 0$$

Furthermore, for $s \in S, x \in \mathbb{W}, i \in I$ we write set as a shortage

$$\begin{aligned} Q_x(s) &:= \mathrm{eu}(F_x/F_{x,xs}), \\ Q_i(s) &:= Q_{x_i}(s), \\ q_i(s) &:= \prod_{\alpha\in\Phi_{\mathcal{U}}, s(\alpha)\notin\Phi_{\mathcal{U}}, x_i(\alpha)\in\Phi_V} \alpha. \end{aligned}$$

for $x = x^i x_i$ with $x^i \in W$ it holds $Q_x(s) = x^i(Q_i(s)), Q_i(s) = x_i(q_i(s))$, i.e.

$$Q_x(s) = x(q_i(s))$$

Lemma 26. *Let $J = \emptyset$, it holds*

(1) for $w \in \mathbb{W}$

$$\Lambda_w = \mathrm{eu}(F_w \oplus \mathfrak{n}_w)$$

(2) If $s \in \mathbb{S}, x \in \mathbb{W}, \alpha_s \in \underline{\Phi}^+$ with $s(\alpha_s) = -\alpha_s$ and ${}^x s \in W$

$$\begin{aligned} \mathrm{eu}(\overline{Z^s}, \phi_{x,xs}) &= \mathrm{eu}(F_{x,xs} \oplus \mathfrak{n}_x \oplus \mathfrak{m}_{x,xs}) = \ x(\alpha_s)\ Q_x(s)^{-1}\ \Lambda_x \\ \mathrm{eu}(\overline{Z^s}, \phi_{x,x}) &= \mathrm{eu}(F_{x,xs} \oplus \mathfrak{n}_x \oplus \mathfrak{m}_{xs,x}) = \ -\,\mathrm{eu}(\overline{Z^s}, \phi_{x,xs}). \end{aligned}$$

(3) If $s \in \mathbb{S}, x \in \mathbb{W}$ and ${}^x s \notin W$

$$\mathrm{eu}(\overline{Z^s}, \phi_{x,xs}) = \mathrm{eu}(F_{x,xs} \oplus \mathfrak{n}_x) = Q_x(s)^{-1}\Lambda_x$$

(4) Let $x, w \in \mathbb{W}$. Then

$$\mathrm{eu}(\overline{Z^w}, \phi_{x,xw}) = \mathrm{eu}(F_{x,xw} \oplus \mathfrak{n}_x \oplus \mathfrak{m}_{x,xw})$$

proof:

(1) We know $\mu_k \colon E_k \to G/B_k,\ B_k = G \cap {}^{x_k}\mathbb{B}$ is a vector bundle, therefore we have a

short exact sequence of tangent spaces

$$0 \to T_{\phi_w}\mu_k^{-1}(w^k B_k) \to T_{\phi_w} E_k \to T_{w^k B_k} G/B_k \to 0$$

which is a split sequence of T-representations implying the first statement.

ad (3,2) Let $i, j \in I_J$ such that $x^i := xx_i^{-1}, y^j := xsx_j^{-1} \in W$.

(2) If ${}^x s \in W$ we have that $i = j$ and $\overline{Z^s_{i,i}} \to C^{\leq s}_{i,i} \cong G \times^{B_i} (G \cap {}^{x_i}\mathbb{P}_{\{s\}})/B_i$ is a vector bundle. For $x' \in \{x, xs\}$ we have a short exact sequence on tangent spaces

$$0 \to F_{x,xs} \to T_{\phi_{x,x'}} \overline{Z^s_{i,i}} \to T_{\phi_{x,x'}} C^{\leq s}_{i,i} \to 0$$

Using the isomorphism $G \times^{B_i} [({}^{x_i}\mathbb{P}_{\{s\}} \cap G)/B_i] \to C^{\leq s}_{i,i}$, $(g, hB_i) \mapsto (gB_i, ghB_i)$ we get

$$\operatorname{eu}(T_{\phi_{x,x'}} C^{\leq s}_{i,i}) = \begin{cases} \operatorname{eu}(T_{\overline{(x^i, B_i)}} G \times^{B_i} [({}^{x_i}\mathbb{P}_{\{s\}} \cap G)/B_i]) = \operatorname{eu}(\mathfrak{n}_x) \cdot \operatorname{eu}(\mathfrak{m}_{xs,x}), & x' = x \\ \operatorname{eu}(T_{\overline{(x^i, {}^{x_i}sB_i)}} G \times^{B_i} [({}^{x_i}\mathbb{P}_{\{s\}} \cap G)/B_i]) = \operatorname{eu}(\mathfrak{n}_x) \cdot \operatorname{eu}(\mathfrak{m}_{xs,x}), & x' = xs \end{cases}$$

It follows $\operatorname{eu}(\overline{Z^s}, \phi_{x,x}) = \operatorname{eu}(F_{x,xs}) \cdot \operatorname{eu}(\mathfrak{n}_x) \cdot \operatorname{eu}(\mathfrak{m}_{xs,x})$ and $\operatorname{eu}(\overline{Z^s}, \phi_{x,xs}) = \operatorname{eu}(F_{x,xs} \oplus \mathfrak{n}_x \oplus \mathfrak{m}_{x,xs})$.

(3) If ${}^x s \notin W$ we get $i \neq j$ and $Z^s_{i,j}$ is closed and a vector bundle over $C^s_{i,j} = G/(G \cap {}^x\mathbb{B})$, we get a short exact sequence on tangent spaces

$$0 \to F_{x,xs} \to T_{\phi_{x,xs}} Z^s_{i,j} \to T_{\phi_{x,xs}} C^s_{i,j} \to 0.$$

We obtain $\operatorname{eu}(\overline{Z^s}, \phi_{x,xs}) = \operatorname{eu}(F_{x,xs}) \operatorname{eu}(\mathfrak{n}_x)$.

(4) Pick $i, j \in I$ such that $x \in Wx_i, xw \in Wx_j$. We have the short exact sequence

$$0 \to F_{x,xw} \to T_{\phi_{x,xw}} \overline{Z^w_{i,j}} \to T_{\phi_{x,xw}} C^w_{i,j} \to 0$$

Then, recall the isomorphism

$$\begin{aligned} C^w_{i,j} = G\phi_{x,xw} &\to G/(G \cap {}^x\mathbb{B} \cap {}^{xw}\mathbb{B}) \\ \phi_{x,xw} &\mapsto \overline{e} := e(G \cap {}^x\mathbb{B} \cap {}^{xw}\mathbb{B}) \end{aligned}$$

Again we have a short exact sequence

$$0 \to T_{\overline{e}}(G \cap {}^x\mathbb{B})/(G \cap {}^x\mathbb{B} \cap {}^{xw}\mathbb{B}) \to T_{\overline{e}} G/(G \cap {}^x\mathbb{B} \cap {}^{xw}\mathbb{B}) \to T_{\overline{e}} G/(G \cap {}^x\mathbb{B}) \to 0$$

Together it implies $\operatorname{eu}(\overline{Z^w_{i,j}}, \phi_{x,xw}) = \operatorname{eu}(F_{x,xw}) \operatorname{eu}(\mathfrak{n}_x/(\mathfrak{n}_x \cap \mathfrak{n}_{xw})) \operatorname{eu}(\mathfrak{n}_x)$.

□

Corollary 2.8.0.3. *Let* $J = \emptyset, \mathcal{U} = \mathrm{Lie}(\mathbb{U})^{\oplus r}$*, it holds*

(1) If $s \in \mathbb{S}, x \in \mathbb{W}$ *and* ${}^x s \in W$*, then* $h_{\overline{x}}(s) = h_{\overline{xs}}(s)$ *and*

$$\Lambda_x = (-1)^{1+h_{\overline{xs}}(s)}\Lambda_{xs}$$
$$\mathrm{eu}(\overline{Z^s}, \phi_{x,xs}) = (x(\alpha_s))^{1-h_{\overline{x}}(s)}\Lambda_x$$

(2) If $s \in \mathbb{S}, x \in \mathbb{W}$ *and* ${}^x s \notin W$

$$\mathrm{eu}(\overline{Z^s}, \phi_{x,xs}) = x(\alpha_s))^{-h_{\overline{x}}(s)}\Lambda_x$$

proof: This follows from $q_x(s) = x(\alpha_s)^{h_{\overline{x}}(s)}$ and
if ${}^x s \in W$ we have that $i = j$ and $h_{\overline{x}}(s) = h_{\overline{xs}}(s)$. Therefore we get

$$\begin{aligned}\mathrm{eu}(F_x) &= x(\alpha_s)^{h_{\overline{x}}(s)}\,\mathrm{eu}(F_{x,xs}) \\ &= (-1)^{h_{\overline{xs}}(s)}(xs(\alpha_s))^{h_{\overline{xs}}(s)}\,\mathrm{eu}(F_{xs,x}) \\ &= (-1)^{h_{\overline{xs}}(s)}\,\mathrm{eu}(F_{xs})\end{aligned}$$

Using that $\mathrm{eu}(\mathfrak{n}_x) = -\,\mathrm{eu}(\mathfrak{n}_{xs})$ we obtain $\Lambda_x = (-1)^{1+h_{\overline{xs}}(s)}\Lambda_{xs}$ □

2.9 Localization to the torus fixed points

Now, we come to the application of localization to T-fixed points. We remind the reader that Z is a cellular fibration and E is smooth, therefore in both cases the odd ordinary (=singular) cohomology groups vanish for Z and E. This implies in particular that E, Z are *equivariantly formal*, which is (in the case of finitely T-fixed points) equivalent to $\mathcal{Z}_G$ and $\mathcal{E}_G$ are free modules over $H^*_G(pt)$.

If we denote by K the quotient field of $H^*_G(pt)$ and for any T-variety X

$$H^T_*(X) \to \mathcal{H}_*(X) := H^T_*(X) \otimes_{H^*_T(pt)} K, \quad \alpha \mapsto \alpha \otimes 1.$$

Lemma 27. *(1)*
$$\mathcal{H}_*(E) = \bigoplus_{w \in \mathbb{W}} K\psi_w, \quad \mathcal{H}_*(Z) = \bigoplus_{x,y \in \mathbb{W}} K\psi_{x,y}$$

where $\psi_w = [\{\phi_w\}] \otimes 1, \psi_{x,y} = [\{\phi_{x,y}\}] \otimes 1$.

(2) For every $i \in I$*,* $w \in Wx_i$ *we have a map* $w\cdot \colon \mathcal{E}_i := H^*_G(E_i) \to \mathbb{C}[\mathfrak{t}]$*, via taking the forgetful map composed with the pullback map under the closed embedding* $i_w \colon \{\phi_w\} \to E_i$

$$\mathcal{E}_i = H^*_G(E_i) \to H^*_T(E_i) \xrightarrow{i_w^*} H^*_T(pt) = \mathbb{C}[\mathfrak{t}],$$

we denote the map by $f \mapsto w(f)$*,* $f \in \mathcal{E}_i, w \in \mathbb{W}$*. Furthermore, composing the* *forgetful map with the map from before we get an injective algebra homomorphism*

$$\begin{aligned}\Theta_i\colon \mathcal{E}_i \to H_T^*(E_i) \hookrightarrow H_T^*(E_i)\otimes K &\cong \bigoplus_{w\in Wx_i} K\psi_w \\ c \quad \longmapsto \quad & \sum_{w\in Wx_i} w(c)\Lambda_w^{-1}\psi_w.\end{aligned}$$

We set $\Theta = \bigoplus_{i\in I}\Theta_i\colon \mathcal{E}_G \to \bigoplus_{w\in\mathbb{W}} K\psi_w$.

proof:

(1) This is GKM-localization theorem for T-equivariant cohomology, for a source also mentioning the GKM-theorem for T-equivariant Borel-Moore homology see for example [Bri00], Lemma 1.

(2) This is [EG98b], Thm 2, using the equivariant cycle class map to identifiy T-equivariant Borel-Moore homology of E with the T-equivariant Chow ring.

□

2.9.1 The $\mathbb{W}$-operation on $\mathcal{E}_G$:

Recall that the ring of regular functions $\mathbb{C}[\mathfrak{t}]$ on $\mathfrak{t} = Lie(T)$ is a left W-module and a left $\mathbb{W}$-module with respect to $w \cdot f(t) = f(w^{-1}tw)$, $w \in \mathbb{W}(\supset W)$. The from W to $\mathbb{W}$ induced representation is given by

$$\operatorname{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}] = \bigoplus_{i\in I} x_i^{-1}\mathbb{C}[\mathfrak{t}],$$

for $w \in \mathbb{W}, i \in I$ the operation of w on $x_i^{-1}\mathbb{C}[\mathfrak{t}]$ is given by

$$\begin{aligned} x_i^{-1}\mathbb{C}[\mathfrak{t}] &\to x_{iw^{-1}}^{-1}\mathbb{C}[\mathfrak{t}] \\ x_i^{-1}f &\mapsto wx_i^{-1}f \end{aligned}$$

where we use that $wx_i^{-1}W = x_{iw^{-1}}^{-1}W$.
Now, we identify $\mathcal{E}_G = \bigoplus_{i\in I}\mathcal{E}_i$ with the left $\mathbb{W}$-module $\operatorname{Ind}_W^{\mathbb{W}}\mathbb{C}[\mathfrak{t}]$ via $\mathcal{E}_i = x_i^{-1}\mathbb{C}[\mathfrak{t}]$.

Furthermore, we have the (left) $\mathbb{W}$-representation on $\bigoplus_{x\in\mathbb{W}} K(\Lambda_x^{-1}\psi_x)$ defined via

$$w(k(\Lambda_x^{-1}\psi_x)) := k(\Lambda_{xw^{-1}}\psi_{xw^{-1}}), \quad k\in K, w\in\mathbb{W}.$$

Lemma 28. *The map* $\Theta\colon \mathcal{E}_G \to \bigoplus_{x\in\mathbb{W}} K(\Lambda_x^{-1}\psi_x)$ *is* $\mathbb{W}$*-invariant.*

proof: Let $w \in \mathbb{W}$, we claim that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x\in\mathbb{W}} K(\Lambda_x^{-1}\psi_x) \\ \downarrow{\scriptstyle w\cdot} & & \downarrow{\scriptstyle w} \\ \mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x\in\mathbb{W}} K(\Lambda_x^{-1}\psi_x) \end{array} \qquad \begin{array}{ccc} c & \longrightarrow & \sum_{x\in\mathbb{W}} i_x^*(c)\Lambda_x^{-1}\psi_x \\ \downarrow & & \downarrow \\ w\cdot c & \longrightarrow & \sum_{x\in\mathbb{W}} i_{xw}^*(c)\Lambda_x^{-1}\psi_x \end{array}$$

We need to see $i_x^*(w \cdot c) = i_{xw}^*(c)$. Let $xw \in Wx_i$, $x \in Wx_{iw^{-1}}$ This means that the diagram

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{w\cdot} & \mathcal{E}_{iw^{-1}} \\ & {}_{i_{xw}^*}\searrow \quad \swarrow_{i_x^*} & \\ & H_T^*(pt) & \end{array}$$

is commutative. But it identifies with

$$\begin{array}{ccc} x_i^{-1}\mathbb{C}[\mathfrak{t}] & \xrightarrow{w\cdot} & x_{iw^{-1}}^{-1}\mathbb{C}[\mathfrak{t}] \\ & {}_{xw\cdot}\searrow \quad \swarrow_{x\cdot} & \\ & \mathbb{C}[\mathfrak{t}] & \end{array} \qquad \begin{array}{ccc} x_i^{-1}f & \longrightarrow & wx_i^{-1}f \\ & \searrow \quad \swarrow & \\ & xwx_i^{-1}f. & \end{array}$$

The diagram is commutative. □

Remark. From now on, we use the following description of the $\mathbb{W}$-operation on $\mathcal{E}_G$. We set $\mathcal{E}_i = \mathbb{C}[\mathfrak{t}]$, $i \in I$. Let $w \in \mathbb{W}$

$$w(\mathcal{E}_i) = \mathcal{E}_{iw^{-1}}, \quad \mathcal{E}_i = \mathbb{C}[\mathfrak{t}] \ni f \mapsto w \cdot f \in \mathbb{C}[\mathfrak{t}] = \mathcal{E}_{iw^{-1}}.$$

The isomorphism $p := \bigoplus_{i \in I} p_i$ defined by

$$\begin{aligned} p_i \colon \mathbb{C}[\mathfrak{t}] &\to x_i^{-1}\mathbb{C}[\mathfrak{t}] \\ f &\mapsto x_i^{-1}(x_i f) \end{aligned}$$

gives the identification with the induced representation $\operatorname{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}]$ which we described before.

2.9.2 Calculations of some equivariant multiplicities

In some situation one can actually say something on the images of algebraic cycle under the GKM-localization map, recall the

Theorem 2.9.1. *(multiplicity formular, [Bri00], section 3) Let X equivariantly formal T-variety with a finite set of T-fixpoints X^T, by the localization theorem,*

$$[X] = \sum_{x \in X^T} \Lambda_x^X [\{x\}] \quad \in H_*^T(X) \otimes K$$

where $\Lambda_x^X \in K$. If X is rationally smooth in x, then $\Lambda_x^X \neq 0$ and $(\Lambda_x^X)^{-1} = eu(X,x) \in H_{2n}^T(X)$, $\quad n = \dim_{\mathbb{C}}(X)$.

Remark. It holds for any $w \in \mathbb{W}$

$$[\overline{Z^w}] = \sum_{i,j \in I} [\overline{Z_{i,j}^w}].$$

Especially $1 = [Z^e] = \sum_{i \in I} [Z_{i,i}^e]$ is the unit and $1_i = [Z_{i,i}^e]$ are idempotent elements,

$1_i * 1_j = 0$ for $i \neq j$, $[Z_{i,j}] = 1_i * [Z] * 1_j$. In particular, for $s \in \mathbb{S}$ by lemma 11, we have

$$[\overline{Z^s}] = \sum_{i \in I:\, is=i} [\overline{Z^s_{i,i}}] + \sum_{i \in I:\, is \neq i} [Z^s_{i,is}].$$

By the multiplicity formula we have

$$[\overline{Z^s_{i,is}}] = \begin{cases} \sum_{x \in W} \Lambda^s_{xx_i,xx_is}\psi_{xx_i,xx_is} + \Lambda^s_{xx_i,xx_i}\psi_{xx_i,xx_i} & \text{, if } i = is \\ \sum_{x \in W} \Lambda^s_{xx_i,xx_is}\psi_{xx_i,xx_is} & \text{, if } is \neq i \end{cases}$$

with $\Lambda^s_{y,z} = (\mathrm{eu}(\overline{Z^s_{i,j}}, \phi_{y,z}))^{-1}$, for all $y, z \in \mathbb{W}$ as above

$$[\overline{Z^w_{i,j}}] = \begin{cases} \sum_{x \in W} \Lambda^w_{xx_i,xx_iw}\psi_{xx_i,xx_iw} + \sum_{v<w} \Lambda^w_{xx_i,xx_iv}\psi_{xx_i,xx_iv} & \text{, if } iw = j \\ 0 & \text{, if } iw \neq j \end{cases}$$

with $\Lambda^w_{xx_i,xx_iw} = (\mathrm{eu}(\overline{Z^w_{i,iw}}, \phi_{xx_i,xx_iw}))^{-1}$ for all $x \in W$,

2.9.3 Convolution on the fixed points

The following key lemma on convolution products of T-fixed points

Lemma 29. *For any $w, x, y \in \mathbb{W}$ it holds*

$$\psi_{x,w} * \psi_w = \Lambda_w \psi_x, \quad \psi_{x,w} * \psi_{w,y} = \Lambda_w \psi_{x,y}$$

proof: We take $M_1 = M_2 = M_3 = E$ and $Z_{1,2} := \{\phi_{x,w} = ((0, xB), (0, wB))\} \subset E \times E$, $Z_{2,3} := \{\phi_{w',y}\} \subset E \times E$, then the set theoretic convolution gives

$$\{\phi_{x,w}\} \circ \{\phi_{w',y}\} = \begin{cases} \{\phi_{x,y}\}, & \text{if } w = w' \\ \emptyset, & \text{if } w \neq w' \end{cases}$$

Similar, take $M_1 = M_2 = E, M_3 = pt$, $Z_{12} := \{\phi_{x,w}\}, Z_{23} = \phi_{w'} \times pt$, then

$$\{\phi_{x,w}\} \circ \{\phi_w\} = \begin{cases} \{\phi_x\} & \text{if } w = w' \\ \emptyset, & \text{else} \end{cases}$$

To see that we have to multiply with Λ_w, we use the following proposition

Proposition 3. *(see [CG97], Prop. 2.6.42, p.109) Let $X_i \subset M, i = 1, 2$ be two closed (complex) submanifolds of a (complex) manifold with $X := X_1 \cap X_2$ is smooth and $T_xX_1 \cap T_xX_2 = T_xX$ for all $x \in X$. Then, we have*

$$[X_1] \cap [X_2] = e(\mathcal{T}) \cdot [X]$$

where $\mathcal{T}$ is the vector bundle $T_*M/(T_*X_1 + T_*X_2)$ on X and $e(\mathcal{T}) \in H^*(X)$ is the (non-equivariant) Euler class of this vector bundle, $\cap\colon H^{BM}_*(X_1) \times H^{BM}_*(X_2) \to H^{BM}_*(X)$ is the intersection pairing (cp. Appendix, or [CG97], 2.6.15) and $\cdot$ on the right hand side stands for the $H^*(X)$-operation on the Borel-Moore homology (introduced in [CG97], 2.6.40)

Set $E_T := E \times^T ET, (\phi_x)_T := \{\phi_x\} \times^T ET (\cong ET/T = BT)$. We apply the proposition for $M = E_T^3$, $X_1 := (\phi_x)_T \times (\phi_w)_T \times E_T, X_2 := E_T \times (\phi_w)_T \times (\phi_y)_T$, $X_1 \cap X_2 \cong \{\phi_{x,y}\}_T (\cong BT)$, then $\mathcal{T} = (T_{\phi_w}E) \times^T ET$ and the (non-equivariant) Euler class is the top chern class of this bundle which is the T-equivariant top chern class of the constant bundle $T_{\phi_w}E$ on the point $\{\phi_{x,y}\}$. Since $T_{\phi_w}E = \bigoplus_\lambda \mathbb{C}_\lambda$ for one-dimensional T-representations $\mathbb{C}_\lambda$ with $t \cdot c := \lambda(t)c, \ \ t \in T, c \in \mathbb{C} = \mathbb{C}_\lambda$. It holds

$$c^T_{top}(T_{\phi_w}E) = \prod_\lambda c_1^T(\mathbb{C}_\lambda) = \prod_\lambda \lambda = \Lambda_w.$$

Secondly, apply the proposition with $M = E_T^2 \times (pt)_T, X_1 = (\phi_x)_T \times (\phi_w)_T \times (pt)_T, X_2 := E_T \times (\phi_w)_T \times (pt)_T$, to see again $e(\mathcal{T}) = \Lambda_w$.

□

Now we can give the missing proof of Corollary 2.7.0.2

proof of Corollary 2.7.0.2: By the lemma 25 we know that there exists a $c \in \mathcal{E}_G$ such that $[\overline{Z^s_{i,j}}] * [\overline{Z^w_{j,k}}] = c \diamond [\overline{Z^{sw}_{i,k}}]$ in $\mathcal{Z}_G^{\leq sw}/\mathcal{Z}_G^{<sw}$. We show that $c = 1$. We pass with the forgetful map to T-equivariant Borel-Moore homology and tensor over $K = \mathrm{Quot}(H^T_*(pt))$ and write $[\overline{Z^x_{s,t}}], x \in \mathbb{W}, s, t \in I$ for the image of the same named elements. Let $i, j, k \in I$ with $x_j w x_k^{-1} \in W$.

$$\begin{aligned}
[\overline{Z^s_{i,j}}] * [\overline{Z^w_{j,k}}] &= (\sum_{x\in W} \Lambda^s_{xx_i,xx_is}\psi_{xx_i,xx_is} + \Lambda^s_{xx_i,xx_i}\psi_{xx_i,xx_i})* \\
&\quad (\sum_{x\in W} \Lambda^w_{xx_j,xx_jw}\psi_{xx_j,xx_jw} + \sum_{v<w}\Lambda^w_{xx_j,xx_jv}\psi_{xx_j,xx_jv}) \\
&= \sum_{x\in W} \Lambda^s_{xx_i,xx_is}\Lambda^w_{xx_is,xx_isw}\Lambda_{xx_is}\psi_{xx_i,xx_isw} + \underbrace{\cdots}_{\text{terms in } \mathcal{Z}_G^{<sw}}
\end{aligned}$$

Now, this has to be equal to $c\sum_{x\in W}\Lambda_{xx_i,xx_isw}\psi_{xx_i,xx_isw}$ in $\mathcal{Z}_G^{\leq sw}/\mathcal{Z}_G^{<sw}$. Comparing coefficients at x gives

$$c = \frac{\mathrm{eu}(E_j, \phi_{xx_is})\,\mathrm{eu}(\overline{Z^{sw}_{i,k}}, \phi_{xx_i,xx_isw})}{\mathrm{eu}(\overline{Z^s_{i,j}}, \phi_{xx_i,xx_is})\,\mathrm{eu}(\overline{Z^w_{j,k}}, \phi_{xx_is,xx_isw})}$$

$$= \frac{\mathrm{eu}(\mathfrak{g}\cap {}^{xx_is}\mathcal{U}^- \oplus \mathfrak{g}\cap {}^{xx_i}\mathcal{U}^- \oplus \mathfrak{g}\cap {}^{xx_i}(\frac{\mathcal{U}^-}{\mathcal{U}^-\cap^{sw}\mathcal{U}^-}))}{\mathrm{eu}(\mathfrak{g}\cap {}^{xx_i}\mathcal{U}^- \oplus \mathfrak{g}\cap {}^{xx_i}(\frac{\mathcal{U}^-}{\mathcal{U}^-\cap^{s}\mathcal{U}^-}) \oplus \mathfrak{g}\cap {}^{xx_is}\mathcal{U}^- \oplus \mathfrak{g}\cap {}^{xx_i}(\frac{{}^s\mathcal{U}^-}{{}^s\mathcal{U}^-\cap^{sw}\mathcal{U}^-}))}$$
$$\cdot \prod_{l=1}^r \frac{\mathrm{eu}(V^{(l)}\cap {}^{xx_i}({}^s\mathcal{U}^{(l)}) \oplus V^{(l)}\cap {}^{xx_i}(\mathcal{U}^{(l)}\cap {}^{sw}\mathcal{U}^{(l)}))}{\mathrm{eu}(V^{(l)}\cap {}^{xx_i}(\mathcal{U}^{(l)}\cap {}^s\mathcal{U}^{(l)}) \oplus V^{(l)}\cap {}^{xx_i}({}^s\mathcal{U}^{(l)}\cap {}^{sw}\mathcal{U}^{(l)}))}$$

$$= \frac{\mathrm{eu}({}^x[\mathfrak{g}\cap {}^{x_i}(\frac{\mathcal{U}^-}{\mathcal{U}^-\cap^{sw}\mathcal{U}^-})])}{\mathrm{eu}({}^x[\mathfrak{g}\cap {}^{x_i}(\frac{\mathcal{U}^-}{\mathcal{U}^-\cap^{s}\mathcal{U}^-}) \oplus \mathfrak{g}\cap {}^{x_i}(\frac{{}^s\mathcal{U}^-}{{}^s\mathcal{U}^-\cap^{sw}\mathcal{U}^-})])}$$
$$\cdot \prod_{l=1}^r \frac{\mathrm{eu}({}^x[V^{(l)}\cap {}^{x_i}(\frac{{}^s\mathcal{U}^{(l)}}{\mathcal{U}^{(l)}\cap^s\mathcal{U}^{(l)}}) \oplus V^{(l)}\cap {}^{x_i}(\frac{{}^{sw}\mathcal{U}^{(l)}}{{}^s\mathcal{U}^{(l)}\cap^{sw}\mathcal{U}^{(l)}})])}{\mathrm{eu}({}^x[V^{(l)}\cap {}^{x_i}(\frac{{}^{sw}\mathcal{U}^{(l)}}{\mathcal{U}^{(l)}\cap^{sw}\mathcal{U}^{(l)}})])}$$

That for each x and each $l \in \{1, \ldots, r\}$ the big two fraction in the product are equal to 1 is a consequence of the following lemma. $\square$

Lemma 30. *Let $T \subset \mathbb{B} \subset \mathbb{G}$ a maximal torus in a Borel subgroup in a reductive group (over$\mathbb{C}$), $F \subset Lie(\mathbb{G}) = \mathcal{G}$ a $\mathbb{B}$-subrepresentation. Let $(\mathbb{W}, \mathbb{S})$ be the Weyl group for $(\mathbb{G}, T)$. Let $w \in \mathbb{W}, s \in \mathbb{S}$ such that $l(sw) = l(w) + 1$, then it holds for any $x \in \mathbb{W}$*

$$^{x}\left(\frac{^{s}F}{F \cap {}^{s}F} \oplus {}^{s}\left(\frac{^{w}F}{F \cap {}^{w}F}\right)\right) \cong {}^{x}\left(\frac{^{sw}F}{F \cap {}^{sw}F}\right).$$

In particular, this holds also for $F = \mathfrak{u}^-$.

proof: Let $\Phi_F := \{\alpha \in \mathrm{Hom}(\mathfrak{t}, \mathbb{C}) \mid F_\alpha \neq 0\} \subset \underline{\Phi}$, $\Phi^+(y) := \underline{\Phi}^+ \cap y(\underline{\Phi}^-)$, $\Phi_F^+(y) := \Phi_F \cap \Phi^+(y), y \in \mathbb{W}$ where $\underline{\Phi}, \underline{\Phi}^+, \underline{\Phi}^-$ are the set of roots (of T on $\mathcal{G}$), positive roots, negative roots respectively.
The assumption $l(sw) = l(w) + 1$ implies $\Phi_F^+(sw) = s\Phi_F^+(w) \sqcup \Phi_F^+(s)$ and for $\Phi_F^-(y) := -\Phi_F^+(y)$, $\Phi_F(y) := \Phi_F^+(y) \cup \Phi_F^-(y) = \Phi_F \setminus (\Phi_F \cap y\Phi_F)$ it holds $\Phi_F(sw) = s\Phi_F(w) \sqcup \Phi_F(s)$ and for any $x \in W$ it holds $x\Phi_F(sw) = x(s\Phi_F(w) \sqcup \Phi_F(s))$. Now, the weights of ${}^{x}(\frac{^{sw}F}{F \cap {}^{sw}F})$ are $x\Phi_F(sw)$, the weights of ${}^{x}(\frac{^{s}F}{F \cap {}^{s}F} \oplus {}^{s}(\frac{^{w}F}{F \cap {}^{w}F}))$ are $x(s\Phi_F(w) \sqcup \Phi_F(s))$. $\square$

2.10 Generators for $\mathcal{Z}_G$

Let $J = \emptyset$. Recall, we denote the right $\mathbb{W}$-operation on $I = W \setminus \mathbb{W}$ by $(i, w) \mapsto iw$, $i \in I, w \in \mathbb{W}$.
For $i \in I$ we set $\mathcal{E}_i := H_G^*(E_i) = \mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_i(1), \ldots, x_i(m)]$, we write

$$w(\alpha_s) = w(\alpha_s(x_{iw^{-1}}(1), \ldots, x_{iw^{-1}}(m))) \in \mathcal{E}_{iw^{-1}}$$

for the element corresponding to the root $w(\alpha_s), s \in \mathbb{S}, w \in \mathbb{W}$ without mentioning that it depends on $i \in I$.
We define a collection of elements in $\mathcal{Z}_G$

$$\begin{aligned} 1_i &:= [Z_{i,i}^e] \\ z_i(t) &:= x_i(t) \in \mathcal{Z}_G^{\leq e} (\subset \mathcal{Z}_G) \\ \sigma_i(s) &:= [\overline{Z_{i,j}^s}] \in \mathcal{Z}_G^{\leq s}, \text{ where } is = j \end{aligned}$$

where we use that $\mathcal{E}_i \subset \mathcal{Z}_G^{\leq e} \subset \mathcal{Z}_G$ and the degree of $x_i(t)$ is 2 in $H_{[*]}^G(Z)$, see Lemm 6 and the definition of the grading (just before theorem 2.1) . It is also easy to see that $1_i \in H_{[0]}^G(Z)$ because $\deg 1_i = 2e_i - 2\dim Z_{i,i}^e = 0$. Furthermore, the degree of $\sigma_i(s)$ is

$$e_{is} + e_i - 2\dim Z_{i,is}^s = \begin{cases} 2\deg q_i(s) - 2, & \text{if } is = i \\ 2\deg q_i(s), & \text{if } is \neq i. \end{cases}.$$

Recall $\mathcal{Z}_G \hookrightarrow \operatorname{End}(\mathcal{E}_G) = \operatorname{End}(\bigoplus_{i\in I} \mathcal{E}_i)$ from [VV11], remark after Prop.3.1, p.12. Let us denote by $\widetilde{1_i}, \widetilde{z_i(t)}, \widetilde{\sigma_i(s)}$ be the images of $1_i, z_i(t), \sigma_i(s)$.

Proposition 4. *Let $k \in I$, $f \in \mathcal{E}_k$, $\alpha_s \in \underline{\Phi}^+$ be the positive root such that $s(\alpha_s) = -\alpha_s$. It holds*

$$\widetilde{1_i}(f) := 1_i * f = \begin{cases} f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$\widetilde{z_i(t)}(f) := z_i(t) * f = \begin{cases} x_i(t)f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$\widetilde{\sigma_i(s)}(f) := \begin{cases} q_i(s)\frac{s(f)-f}{\alpha_s}, & \text{if } i = is = k, \\ q_i(s)s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

for $\mathcal{U} = \operatorname{Lie}(\mathbb{U})^{\oplus r}$ this looks like

$$\widetilde{\sigma_i(s)}(f) := \begin{cases} \alpha_s^{h_i(s)}\frac{s(f)-f}{\alpha_s}, & \text{if } i = is = k, \\ \alpha_s^{h_i(s)}s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

We write $\delta_s := \frac{s-1}{\alpha_s}$, it is the BGG-operator from [Dem73], i.e. for $is = i, f \in \mathcal{E}_i$, $\sigma_i(s)(f) = q_i(s)\delta_s(f)$.

proof: Consider the following two maps

$$\Theta\colon \mathcal{E}_G \to \mathcal{E}_T \to \mathcal{E}_T \otimes K \to \bigoplus_{w\in\mathbb{W}} K\psi_w$$
$$\mathcal{E}_k \ni \quad f \mapsto \sum_{w\in Wx_k} w(f)\Lambda_w^{-1}\psi_w$$

$$C\colon \bigoplus_{w\in\mathbb{W}} K\psi_w \to \bigoplus_{w\in\mathbb{W}} K\psi_w, \quad \psi_w \mapsto [\overline{Z^s_{i,is}}] * \psi_w =$$
$$\begin{cases} (\sum_{x\in W} \Lambda^s_{xx_i,xx_i}\psi_{xx_i,xx_i} + \Lambda^s_{xx_i,x^{x_i}sx_i}\psi_{xx_i,x^{x_i}sx_i}) * \psi_w & \\ \qquad = \Lambda^s_{w,w}\Lambda_w\psi_w + \Lambda^s_{ws,w}\Lambda_w\psi_{ws}, & \text{if } w \in Wx_i,\ i = is \\ (\sum_{x\in W} \Lambda^s_{xx_i,x^{x_i}sx_i}\psi_{xx_i,x^{x_i}sx_i}) * \psi_w & \\ \qquad = \Lambda^s_{ws,w}\Lambda_w\psi_{ws}, & \text{if } w \in Wx_is,\ i \neq is \\ 0, & \text{if } w \notin Wx_is \end{cases}$$

To calculate $[\overline{Z^s_{i,is}}] * f, f \in \mathcal{E}_k$ it is enough to calculate $[\overline{Z^s_{i,is}}] * \Theta(f) = C(\Theta(f))$ because Θ is an injective algebra homomorphism.

$$C\Theta(f) = \begin{cases} \delta_{is,k}\sum_{w\in Wx_i}[w(f)\Lambda^s_{w,w} + w(sf)\Lambda^s_{w,ws}]\psi_w, & \text{if } i = is \\ \delta_{is,k}\sum_{w\in Wx_i}[w(sf)\Lambda^s_{w,ws}]\psi_w, & \text{if } i \neq is \end{cases}$$

Now, recall,

(1) If $i = is = k$

$$\begin{aligned} C\Theta(f) &= \sum_{w \in Wx_i} w[q_i(s)\frac{s(f)-f}{\alpha_s}]\Lambda_w^{-1}\psi_w \\ &= \Theta(q_i(s)\frac{s(f)-f}{\alpha_s}) \end{aligned}$$

Once we identify $\mathcal{E}_k = \mathbb{C}[\mathfrak{t}],\ k \in I$, we see that $\sigma_i(s)\colon \mathcal{E}_G \to \mathcal{E}_G$ is the zero map on the k-th summand, $k \neq i$ and on the i-th summand

$$\begin{aligned} \mathbb{C}[\mathfrak{t}] &\to \mathbb{C}[\mathfrak{t}] \\ f &\mapsto q_i(s)\frac{s(f)-f}{\alpha_s} \end{aligned}$$

(2) If $i \neq is = k$,

$$\begin{aligned} C\Theta(f) &= \sum_{w \in Wx_i} [w(sf)\Lambda^s_{w,ws}]\psi_w \\ &= \Theta(q_i(s)s(f)) \end{aligned}$$

Once we identify $\mathcal{E}_k = \mathbb{C}[\mathfrak{t}]$, we see that $\sigma_i(s)\colon \mathcal{E}_G \to \mathcal{E}_G$ is the zero map on the k-th summand, $k \neq is$ and on the is-th summand it is the map

$$\begin{aligned} \mathbb{C}[\mathfrak{t}] &\to \mathbb{C}[\mathfrak{t}] \\ f &\mapsto q_i(s)s(f) \end{aligned}$$

□

Lemma 31. *The algebra $\mathcal{Z}_G$ is generated as Λ_G-algebra by the elements*

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I.$$

proof: It follows from the cellular fibration property that $\mathcal{Z}_G$ is generated by $1_i, i \in I, \quad z_i(t), 1 \leq t \leq rk(T), i \in I, [\overline{Z^w_{i,j}}], w \in \mathbb{W}$. By corollary 2.7.0.2 it follows that one can restrict to the case $w \in \mathbb{S}$, more precisely as free $H^*_G(E)$-module it can be generated by

$$\sigma(w) := \sigma(s_1) * \cdots \sigma(s_t), w \in \mathbb{W}, w = s_1 \cdots s_t \text{ reduced expression }, \sigma(s) := \sum_{i \in I} \sigma_i(s),$$

and this basis has a unitriangular base cange to the basis given by the $[\overline{Z^w}]$.

□

2.11 Relations for $\mathcal{Z}_G$

Furthermore, we consider

$$\Phi\colon \bigoplus_{i \in I} \mathbb{C}[x_i(1), \dots x_i(n)] \cong \bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots z_i(n)],\ x_i(t) \mapsto z_i(t)$$

as the left $\mathbb{W}$-module $\mathrm{Ind}_W^{\mathbb{W}}\mathbb{C}[\mathfrak{t}]$, we fix the polynomials

$$c_i(s,t) := \Phi(\sigma_i(s)(x_i(t))) \in \bigoplus_{i\in I}\mathbb{C}[z_i(1),\dots z_i(n)], \quad i\in I,\ 1\le t\le n,\ s\in\mathbb{S}.$$

Proposition 5. *Let $\mathbb{S}\subset\mathbb{W}=Weyl(\mathbb{G},T)$ be the simple reflections. Under the following assumption for the data $(\mathbb{G},\mathbb{B},\mathcal{U}=(\mathrm{Lie}(\mathbb{U}))^{\oplus r},H,V)$, $J=\emptyset$: We assume for any $s,t\in\mathbb{S}$*

(B2) If the root system spanned by α_s,α_t is of type B_2 (i.e. $stst=tsts$ is the minimal relation), then for every $i\in I$ such that $is=i=it$ it holds $h_i(s),h_i(t)\in\{0,1,2\}$.

(G2) If the root system spanned by α_s,α_t is of type G_2 (i.e. $ststst=tststs$ is the minimal relation), then for every $i\in I$ such that $is=i=it$ it holds $h_i(s)=0=h_i(t)$.

Then the generalized quiver Hecke algebra for $(\mathbb{G},\mathbb{B},\mathcal{U}=(\mathrm{Lie}(\mathbb{U}))^{\oplus r},H,V)$ is the graded $\mathbb{C}$-algebra with generators

$$1_i, i\in I,\quad z_i(t), 1\le t\le n=rk(T), i\in I,\quad \sigma_i(s), s\in\mathbb{S}, i\in I$$

in degrees

$$\deg 1_i=0,\ \ \deg z_i(k)=2,\ \ \deg\sigma_i(s)=\begin{cases}2h_i(s)-2, & \text{if } is=i\\ 2h_i(s), & \text{if } is\neq i\end{cases}$$

and relations

*(1) (*orthogonal idempotents*)*

$$1_i1_j=\delta_{i,j}1_i,$$
$$1_iz_i(t)1_i=z_i(t),$$
$$1_i\sigma_i(s)1_{is}=\sigma_i(s)$$

*(2) (*polynomial subalgebras*)*

$$z_i(t)z_i(t')=z_i(t')z_i(t)$$

(3) (relation implied by $s^2=1$*)*

$$\sigma_i(s)\sigma_{is}(s)=\begin{cases}0 & \text{, if } is=i,\ h_i(s) \text{ is even}\\ -2\alpha_s^{h_i(s)-1}\sigma_i(s) & \text{, if } is=i,\ h_i(s) \text{ is odd}\\ (-1)^{h_{is}(s)}\alpha_s^{h_i(s)+h_{is}(s)} & \text{, if } is\neq i\end{cases}$$

*(4) (*straightening rule*)*

$$\sigma_i(s)z_i(t)-s(z_i(t))\sigma_i(s)=\begin{cases}c_i(s,t), & \text{, if } is=i\\ 0 & \text{, if } is\neq i.\end{cases}$$

*(5) (*braid relations*)*
Let $s, t \in \mathbb{S}, st = ts$, then

$$\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$$

Let $s, t \in \mathbb{S}$ not commuting such that $x := sts\cdots = tst\cdots$ minimally, $i \in I$. There exists explicit polynomials $(Q_w)_{w<x}$ in $\alpha_s, \alpha_t \in \mathbb{C}[\mathfrak{t}]$ such that

$$\sigma_i(sts\cdots) - \sigma_i(tst\cdots) = \sum_{w<x} Q_w \sigma_i(w)$$

(observe that for $w < x$ there exists just one reduced expression).

proof: For the convenience of the reader who wants to check the relations for the generators of $\mathcal{Z}_G$, we include the detailed calculations. (1), (2) are clear. Let always $f \in \mathbb{C}[\mathfrak{t}] \cong \mathcal{E}_{is}$. We will use as shortage $\delta_s(f) := \frac{s(f)-f}{\alpha_s}$ and use that these satisfy the usual relations of BGG-operators (cp. [Dem73]).

(3) If $is = i$, then

$$\begin{aligned}\sigma_i(s)\sigma_i(s)(f) &= \alpha_s^{h_i(s)}\delta_s(\alpha_s^{h_i(s)}\delta_s(f)) \\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_s^{h_i(s)})\delta_s(f) = [(-1)^{h_i(s)} - 1]\alpha_s^{h_i(s)-1}\sigma_i(s)(f).\end{aligned}$$

If $is \neq i$, then

$$\sigma_i(s)(f)\sigma_{is}(s) = \alpha_s^{h_i(s)} s(\alpha_s^{h_{is}(s)}) s(s(f)) = (-1)^{h_{is}(s)}\alpha_s^{h_i(s)+h_{is}(s)} f.$$

(4) (straightening rule)

The case $is \neq i$ is clear by definition. Let $is = i$, then the relation follows directly from the product rule for BGG-operators, which states $\delta_s(xf)) = \delta_s(x)f + s(x)\delta_s(f), \;\; x, f \in \mathbb{C}[\mathfrak{t}]$.

(5) (braid relations)
$s, t \in \mathbb{S}, st = ts, \; f \in \mathbb{C}[\mathfrak{t}]$, to prove

$$\sigma_i(s)\sigma_{is}(t)(f) = \sigma_i(t)\sigma_{it}(s)(f)$$

we have to consider the following four cases. We use the following:

$$t(\alpha_s) = \alpha_s, \; s(\alpha_t) = \alpha_t, \; h_i(s) = h_{it}(s), \; h_i(t) = h_{is}(t), \; \delta_s(\alpha_t^{h_i(t)}) = 0 = \delta_t(\alpha_s^{h_i(s)}).$$

1. $is = i, it = i$, use $\delta_s\delta_t = \delta_t\delta_s$

$$\begin{aligned}\sigma_i(t)\sigma_i(s)(f) = \alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)}\delta_s(f)) &= \alpha_s^{h_i(s)}\alpha_t^{h_i(t)}\delta_s\delta_t(f) \\ &= \alpha_s^{h_i(s)}\alpha_t^{h_i(t)}\delta_t\delta_s(f) \\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(f)) = \sigma_i(s)\sigma_i(t)(f)\end{aligned}.$$

2. $is = i, it \neq i$, use $\delta_s t = t\delta_s$

$$\begin{aligned}\sigma_i(t)\sigma_{it}(s)(f) = \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}\delta_s(f)) &= \alpha_t^{h_i(t)}\alpha_s^{h_i(s)} t\delta_s(f)\\ &= \alpha_t^{h_{is}(t)}\alpha_s^{h_i(s)}\delta_s(t(f))\\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)} t(f)) = \sigma_i(s)\sigma_i(t)(f)\end{aligned} .$$

3. $is \neq i, it = i$, follows by symmetry from the last case.
4. $is \neq i, it \neq i$.

$$\begin{aligned}\sigma_i(t)\sigma_{it}(s)(f) = \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)} s(f)) &= \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)} t(f))\\ &= \sigma_i(s)\sigma_{is}(t)(f).\end{aligned}$$

Let $st \neq ts$. There are three different possibilties, either

(A) $sts = tst$ (type A_2)

(B) $stst = tsts$ (type B_2)

(C) $ststst = tststs$ (type G_2)

We write $\mathrm{Stab}_i := \{w \in \langle s,t\rangle \mid iw = i\}$. For each case we go through the subgrouplattice to calculate explicitly the polynomials Q_w.

(A) $sts = tst$: $\langle s,t\rangle \cong S_3$, $s(\alpha_t) = t(\alpha_s) = \alpha_s + \alpha_t$. We have five (up to symmetry between s and t) subgroups to consider. Always, it holds

$$h_{is}(t) = h_{it}(s), h_{ist}(s) = h_i(t), h_{its}(t) = h_i(s)$$

which implies an equality which we use in all five cases

$$\begin{aligned}\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) &= \alpha_s^{h_i(s)}(\alpha_s + \alpha_t)^{h_{it}(s)}\alpha_t^{h_{ist}(s)}\\ &= \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)})\end{aligned}$$

A1. $\mathrm{Stab}_i = \langle s,t\rangle$, this implies $h_i(s) = h_i(t) =: h$ by definition ($x_i(\alpha_s) \in \Phi_{V(k)}$ if and only if $x_{it}(\alpha_s) = x_i(\alpha_s + \alpha_t) = x_{is}(\alpha_t) \in \Phi_{V(k)} \Leftrightarrow x_i(\alpha_t) \in \Phi_{V(k)}$) and as a consequence we get
$\alpha^h\delta_s(\alpha_t^h t(\alpha_s^h)) = 0$. This simplifies the equation to

$$\sigma_i(s)\sigma_i(t)\sigma_i(s) - \sigma_i(t)\sigma_i(s)\sigma_i(t) = \delta_s(\alpha_t^h\delta_t(\alpha_s^h))\sigma_i(s) - \delta_t(\alpha_s^h\delta_s(\alpha_t^h))\sigma_i(t)$$

note that $Q_s := \delta_s(\alpha_t^h\delta_t(\alpha_s^h)), Q_t := -\delta_t(\alpha_s^h\delta_s(\alpha_t^h))$ are polynomials in α_s, α_t.

A2. $\mathrm{Stab}_i = \langle s\rangle$ (analogue $\mathrm{Stab}_i = \langle t\rangle$). It holds $itst = its$. We use in this case

$h_{is}(t) = h_i(t), h_{ist}(s) = h_{it}(s) = h_i(t), h_{its}(t) = h_i(s).$

$$\begin{aligned}
&\sigma_i(s)\sigma_i(t)\sigma_{it}(s)(f) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)(f) \\
&= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(f)) - \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(f) \\
&= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)}))ts(f) + \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})\delta_s(ts(f)) \\
&= \quad -\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(f) = 0
\end{aligned}$$

Since $st\delta_s = \delta_t st$ and $\delta_s(\alpha_t^h t(\alpha_s)^h) = 0$.

A3. $\mathrm{Stab}_i = \langle sts \rangle$, then $ist = is, its = it$.

$$\begin{aligned}
&\sigma_i(s)\sigma_{is}(t)\sigma_{is}(s)(f) - \sigma_i(t)\sigma_{it}(s)\sigma_{it}(t)(f) \\
&= \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)}\delta_t(\alpha_s^{h_{is}(s)}s(f))) - \alpha_t^{h_{is}(s)}t(\alpha_s^{h_{is}(t)}\delta_s(\alpha_t^{h_i(s)}t(f))) \\
&= [\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s(\delta_t(\alpha_s^{h_{is}(s)})) - \alpha_t^{h_{is}(s)}t(\alpha_s^{h_{is}(t)})t(\delta_s(\alpha_t^{h_i(s)}))] \cdot f
\end{aligned}$$

using $t\delta_s t = s\delta_t s$.

A4. $\mathrm{Stab}_i = \{1\}$ (and the same for $\mathrm{Stab}_i = \langle st \rangle$)

$$\begin{aligned}
&\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t) \\
&= \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts - \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst \\
&= 0
\end{aligned}$$

(B) $stst = tsts$: $\quad < s,t > \cong D_4$(order is 8),

$$t(\alpha_s) = \alpha_s + \alpha_t, \quad st(\alpha_s) = \alpha_s + \alpha_t, \quad tst(\alpha_s) = \alpha_s$$
$$s(\alpha_t) = 2\alpha_s + \alpha_t, \quad ts(\alpha_t) = 2\alpha_s + \alpha_t, \quad sts(\alpha_t) = \alpha_t.$$

Here we have to consider ten different cases because D_4 has ten subgroups. It always holds the following

$$h_{itst}(s) = h_i(s), h_{its}(t) = h_{is}(t), h_{it}(s) = h_{ist}(s), h_{ists}(t) = h_i(t)$$

which implies

$$\alpha_s^{h_i(s)}s(\alpha_t)^{h_{is}(t)}st(\alpha_s)^{h_{ist}(s)}sts(\alpha_t)^{h_{ists}(t)} = \alpha_t^{h_i(t)}t(\alpha_s)^{h_{it}(s)}ts(\alpha_t)^{h_{its}(t)}tst(\alpha_s)^{h_{itst}(s)}$$

This will be used in all cases, it is particular easy to see that for

$$\mathrm{Stab}_i = \{1\}, \quad \mathrm{Stab}_i = \{1, ts, st, stst\}, \quad \mathrm{Stab}_i = \{1, stst\}$$

we obtain that the difference is zero from the above equality. Let us investigate the other cases. Furthermore, the following is useful to notice

$$\delta_s(t(\alpha_s)^h) = 0, \quad \delta_t(s(\alpha_t)^h) = 0$$

B1. $\mathrm{Stab}_i = \langle s, t\rangle$. We prove the following

$$\begin{aligned}\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) = Q_{st}\sigma_i(s)\sigma_i(t)\\ + \alpha_s^{h_i(s)} s(\alpha_t)^{h_i(t)} st(\alpha_s)^{h_i(s)} sts(\alpha_t)^{h_i(t)}\delta_{stst}(f)\\ + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)})\delta_{sts}(\alpha_t^{h_i(t)})\delta_t\end{aligned}$$

with $Q_{st} = \delta_s(\alpha_t^{h_i(t)})\delta_t(\alpha_s^{h_i(s)}) + s(\alpha_t^{h_i(t)})\delta_{st}(\alpha_s^{h_i(s)}) + t(\alpha_s^{h_i(s)})\delta_{ts}(\alpha_t^{h_i(t)}) = Q_{ts}$ is a polynomial in α_s, α_t. By a long direct calculation (applying the product rule for the δ_s) several times

$$\begin{aligned}&\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) = \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})))\delta_t(f)\\ &+ [\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})) + \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})))]\delta_{st}(f)\\ &+ \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)} t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_i(t)}))\delta_{tst}(f)\\ &+ \alpha_s^{h_i(s)} s(\alpha_t)^{h_i(t)} st(\alpha_s)^{h_i(s)} sts(\alpha_t)^{h_i(t)}\delta_{stst}(f)\end{aligned}$$

We have a look at the polynomials occurring in front of the δ_w:

$w = t$: by the product rule

$$\begin{aligned}&\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))) = \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})^2\delta_t(\alpha_s^{h_i(s)})\\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})\delta_s(\alpha_t^{h_i(t)})\delta_{st}(\alpha_s^{h_i(s)}) + \alpha_s^{h_i(s)} t(\alpha_s^{h_i(s)})\delta_s(\alpha_t^{h_i(t)})\delta_{ts}(\alpha_t^{h_i(t)})\\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)})\delta_{sts}(\alpha_t^{h_i(t)})\end{aligned}$$

$w = st$:

$$\begin{aligned}&\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}))\\ &\quad + \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})))\\ &= \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s\delta_s(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)})\\ &\quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)})\delta_{ts}(\alpha_t^{h_i(t)})\\ &\quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})\delta_s(\alpha_t^{h_i(t)})\delta_t(\alpha_s^{h_i(s)})\\ &\quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})\delta_s(s(\alpha_t^{h_i(t)})) s\delta_t(\alpha_s^{h_i(s)})\\ &\quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})^2\delta_{st}(\alpha_s^{h_i(s)})\\ &= \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})[\delta_s(\alpha_t^{h_i(t)})\delta_t(\alpha_s^{h_i(s)})\\ &\quad + s(\alpha_t^{h_i(t)})\delta_{st}(\alpha_s^{h_i(s)})\\ &\quad + t(\alpha_s^{h_i(s)})\delta_{ts}(\alpha_t^{h_i(t)})]\\ &= \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) Q_{st}\end{aligned}$$

using $s(\delta_s(\alpha_t^h)) = \delta_s(\alpha_t^h)$ and $\delta_s(s(\alpha_t)^h) = -\delta_s(\alpha_t^h)$.

$w = tst$:

$$\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)} t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_i(t)})) = 0$$

Now, look at $\sigma_i(s)\sigma_i(t)(f) = \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})\delta_t(f) + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})\delta_{st}(f)$, which implies

$$\alpha_s^{h_i(s)}s(\alpha_t^{h_i(t)})Q_{st}\delta_{st}(f)=Q_{st}\sigma_i(s)\sigma_i(t)(f)-\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})Q_{st}\delta_t(f)$$

replace the previous expression and compare coefficients in front of $\delta_t(f)$ again gives the polynomial

$$\begin{aligned}&\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)}\delta_t(\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})))-\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_i(t)})Q_{st}=\\&\alpha_s^{h_i(s)}s(\alpha_t^{h_i(t)})st(\alpha_s^{h_i(s)})\delta_{sts}(\alpha_t^{h_i(t)})\end{aligned}$$

We conclude

$$\begin{aligned}\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)-\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)&=Q_{st}\sigma_i(s)\sigma_i(t)-Q_{st}\sigma_i(t)\sigma_i(s)\\+\alpha_s^{h_i(s)}s(\alpha_t^{h_i(t)})st(\alpha_s^{h_i(s)})\delta_{sts}(\alpha_t^{h_i(t)})\delta_t&-\alpha_t^{h_i(t)}t(\alpha_s^{h_i(s)})ts(\alpha_t^{h_i(t)})\delta_{tst}(\alpha_s^{h_i(s)})\delta_s\end{aligned}$$

Since $\delta_{sts}(\alpha_t^h)=0=\delta_{tst}(\alpha_s^k)$ for $h,k\in\{0,1,2\}$ since the maps $\delta_{sts},\delta_{tst}$ map polynomials of degree d to polynomials of degree $d-3$ or to zero, the claim follows. In general, if we localize to $\mathbb{C}[\mathfrak{t}][\alpha_t^{-1},\alpha_s^{-1}]$ we could still have the analogue statement.

B2. $\mathrm{Stab}_i=\langle s\rangle$ (analogue $\mathrm{Stab}_i=\langle t\rangle$) and use $\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)}))=0$ to see

$$\begin{aligned}&\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)-\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\\&=\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(f))\\&\quad-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tst\delta_s(f)\\&=\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})\delta_s tst(f))\\&\quad-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tst\delta_s(f)\\&=0\end{aligned}$$

because $tst\delta_s=\delta_s tst$.

B3. $\mathrm{Stab}_i=\{1,sts\}$ (analogue $\mathrm{Stab}_i=\{1,tst\}$). It holds $its=itst, is=ist$. We have

$$\begin{aligned}&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)-\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)](f)\\=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(f))\\&\quad-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(f))\\=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})[s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)}))t(f)+st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})s\delta_t(st(f))]\\&\quad-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})[ts\delta_t(\alpha_s^{h_{itst}(s)})t(f)+tst(\alpha_s^{h_{itst}(s)})ts\delta_t(s(f))]\\=&[\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)})-t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)})]\sigma_i(t)(f)\end{aligned}$$

using $s\delta_t st=ts\delta_t s$ and $s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)}))=\alpha_t^{h_i(t)}s\delta_t(\alpha_s^{h_{ist}(s)})$.

B4. $\mathrm{Stab}_i=\{1,s,tst,stst\}$ (analogue $\mathrm{Stab}_i=\{1,t,sts,stst\}$). It holds $i=is, it=$

$its, ist = sts, itst = itsts.$

$$\begin{aligned}&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)](f)\\ =&\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})t\delta_s(\alpha_t^{h_{ists}(t)}t(f)))\\ -&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})t\delta_s(f))\\ =&[\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})t\delta_s(\alpha_t^{h_{ists}(t)}))]f\\ +&[t(\alpha_s^{h_{it}(s)})[s(\alpha_t^{h_{is}(t)})st\delta_s(\alpha_t^{h_{ists}(t)}) - \alpha_t^{h_i(t)}t\delta_s(\alpha_t^{h_{its}(t)}))]\sigma_i(s)(f)\end{aligned}$$

using $\delta_s t\delta_s t = t\delta_s t\delta_s$.

This finishes the investigation of the ten possible cases. We also like to remark that in the example in chapter 5 the case B4 only occurs for $Stab_i = \{1, t, sts, stst\}$, i.e. the other stabilizer never occurs.

(C) $ststst = tststs$: $\quad \langle s, t\rangle \cong D_6$,

$$t(\alpha_s) = \alpha_s + \alpha_t, \quad st(\alpha_s) = 2\alpha_s + \alpha_t, \quad tst(\alpha_s) = st(\alpha_s),$$
$$s(\alpha_t) = 3\alpha_s + \alpha_t, \quad ts(\alpha_t) = 3\alpha_s + 2\alpha_t, \quad sts(\alpha_t) = ts(\alpha_t).$$

It holds

$$h_{itstst}(s) = h_i(s), \quad h_{itsts}(t) = h_{is}(t), \quad h_{itst}(s) = h_{ist}(s)$$
$$h_{its}(t) = h_{ists}(t), \quad h_{it}(s) = h_{istst}(s), \quad h_i(t) = h_{istists}(t).$$

this implies

$$\begin{aligned}&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})ststs(\alpha_t^{h_{iststs}(t)}) =\\ &\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)})\end{aligned}$$

Now, D_6 has 13 subgroups. In the following cases the above equality directly implies that $\sigma_i(ststst) - \sigma_i(tststs) = 0$:

$$\mathrm{Stab}_i = \{1\}, \ \mathrm{Stab}_i = \{1, tst\}, \ \mathrm{Stab}_i = \{1, sts\},$$

$$\mathrm{Stab}_i = \{1, ststst\}, \ \mathrm{Stab}_i = \langle st\rangle = \langle ts\rangle$$

C1. $\mathrm{Stab}_i = \langle s, t\rangle$. By assumption we have $h_i(s) = 0 = h_i(t)$ in this case, therefore

$$\begin{aligned}&\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s) =\\ &\delta_s\delta_t\delta_s\delta_t\delta_s\delta_t - \delta_t\delta_s\delta_t\delta_s\delta_t\delta_s = 0\end{aligned}$$

because that is known for the divided difference operators, cp [Dem73].

C2. $\mathrm{Stab}_i = \{1, s\}$ (analogue $\mathrm{Stab}_i = \{1, t\}$). Then, $is = i, itstst = itststs$.

$$\begin{aligned}
&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{istists}(t) \\
&\quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\
=&\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})tstst(f)) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)})tstst\delta_s(f) \\
=&\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)}))tstst(f) \\
=&0
\end{aligned}$$

using $\delta_s tstst = tstst\delta_s$ and

$$\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})) = 0$$

because

$$\begin{aligned}
&s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})) \\
&\qquad =\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)}).
\end{aligned}$$

C3. $\mathrm{Stab}_i = \{1, tstst\}$ (analogue $\mathrm{Stab}_i = \{1, ststs\}$). Then $its = itst, ists = istst$.

$$\begin{aligned}
&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\
&\quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\
=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})sts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)})st(f)) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)})st(\alpha_s^{h_{itstst}(s)})sts(f)) \\
=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})sts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)}))s(f) \\
+&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})ststs(\alpha_t^{h_{iststs}(t)})sts\delta_t st(f) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)})st(\alpha_s^{h_{itstst}(s)}))s(f) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)})ts\delta_t sts(f) \\
=&[s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})sts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)})) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)}))]\sigma_i(s)(f)
\end{aligned}$$

using $ts\delta_t sts = sts\delta_t st$.

C4. $\mathrm{Stab}_i = \{1, s, tstst, ststst\}$ (analogue $\mathrm{Stab}_i = \{1, t, ststs, ststst\}$). Then $is = i, itst = its$. Observe, in this case

$$h_i(t) = h_{it}(t), \text{ and } h_{it}(s) = h_{its}(s)$$

and it holds

$$
\begin{aligned}
&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{istists}(t)\\
&\quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_i(tsts)\sigma_{itstst}(s)](f)\\
&=\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})ts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{ststs}(t)})st(f)))\\
&-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)})st(\alpha_s^{h_{itstst}(s)})st\delta_s(f))\\
&=\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})ts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)}))) \cdot f\\
&+\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})sts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)}))\delta_s(f)\\
&+\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)}))ts\delta_t st(f)))\\
&+\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})ststs(\alpha_t^{h_{iststs}(t)})\\
&\quad \cdot \delta_s ts\delta_t st(f)\\
&-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)})st(\alpha_s^{h_{itstst}(s)}))\delta_s(f))\\
&-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)})\\
&\quad \cdot ts\delta_t st\delta_s(f))\\
&=[\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})ts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)})))]f\\
&+[s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})sts\delta_t(\alpha_s^{h_{istst}(s)}s(\alpha_t^{h_{iststs}(t)}))\\
&-\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})ts\delta_t(\alpha_s^{h_{itst}(s)}s(\alpha_t^{h_{itsts}(t)}))]\sigma_i(s)(f)
\end{aligned}
$$

using $\delta_s ts\delta_t st = ts\delta_t st\delta_s$.

C5. $\mathrm{Stab}_i = \{1, sts, tst, ststst\}$. Then $is = ist, it = its$. Observe, in this case

$$h_i(s) = h_{is}(s), \text{ and } h_i(t) = h_{it}(t)$$

and it holds

$$\begin{aligned}
&[\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\
&\quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\
=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(\alpha_s^{h_{istst}(s)})st\delta_s(\alpha_t^{h_{iststs}(t)}t(f))) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})ts(\alpha_t^{h_{itsts}(t)})ts\delta_t(\alpha_s^{h_{itstst}(s)}s(f))) \\
=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(\alpha_s^{h_{istst}(s)})st\delta_s(\alpha_t^{h_{iststs}(t)})) \cdot f \\
+&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})stst\delta_s(\alpha_t^{h_{iststs}(t)})s\delta_t s(f) \\
+&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(\alpha_s^{h_{istst}(s)})sts(\alpha_t^{h_{iststs}(t)}))t\delta_s t(f) \\
+&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})ststs(\alpha_t^{h_{iststs}(t)}) \\
&\quad \cdot s\delta_t st\delta_s t(f) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})ts(\alpha_t^{h_{itsts}(t)})ts\delta_t(\alpha_s^{h_{itstst}(s)})) \cdot f \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tsts\delta_t(\alpha_s^{h_{itstst}(s)})t\delta_s t(f) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})ts(\alpha_t^{h_{itsts}(t)})tst(\alpha_s^{h_{itstst}(s)}))s\delta_t s(f) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)}tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)}) \\
&\quad \cdot t\delta_s ts\delta_t s(f) \\
=&P_e f + [\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})stst\delta_s(\alpha_t^{h_{iststs}(t)}) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})ts(\alpha_t^{h_{itsts}(t)})tst(\alpha_s^{h_{itstst}(s)}))]s\delta_t s(f) \\
+&[\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(\alpha_s^{h_{istst}(s)})sts(\alpha_t^{h_{iststs}(t)})) \\
-&\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tsts\delta_t(\alpha_s^{h_{itstst}(s)})]t\delta_s t(f)
\end{aligned}$$

using $s\delta_t st\delta_s t = t\delta_s ts\delta_t s$ where P_e is a polynomial in α_t, α_s. Then we look at

$$\begin{aligned}
\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)(f) &= \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s\delta_t(\alpha_s^{h_{ist}(s)}) \cdot f \\
&\quad + \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})s\delta_t s(f) \\
\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)(f) &= \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)}) \cdot f \\
&\quad + \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})t\delta_s t(f)
\end{aligned}$$

and we observe for the coefficient in front of $s\delta_t s(f)$ that it is divisible by $\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})$ and the one in front of $t\delta_s t$ is divisible by $\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})$. Observe $h_{ist}(s) = h_i(s), h_{its}(t) = h_i(t)$. Use the following simplifictaions

$$\begin{aligned}
&t\delta_s(\alpha_t^{h_{its}(t)}t(\alpha_s^{h_{itst}(s)})ts(\alpha_t^{h_{itsts}(t)})tst(\alpha_s^{h_{itstst}(s)}))\\
=&t\delta_s(\alpha_t^{h_i(t)}t(\alpha_s^{h_i(s)})ts(\alpha_t^{h_{is}(t)})tst(\alpha_s^{h_i(s)}))\\
=&s(\alpha_t^{h_{is}(t)})t\delta_s(\alpha_t^{h_i(t)}t(\alpha_s^{h_i(s)})st(\alpha_s^{h_i(s)}))\\
=&s(\alpha_t^{h_{is}(t)})[t\delta_s(\alpha_t^{h_i(t)}t(\alpha_s^{h_i(s)}))tst(\alpha_s^{h_i(s)})+ts(\alpha_t^{h_i(t)})tst(\alpha_s^{h_i(s)})t\delta_s st(\alpha_s^{h_i(s)})]\\
=&s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_i(s)})[\alpha_s^{h_i(s)}t\delta_s(\alpha_t^{h_i(t)})+ts(\alpha_t^{h_i(t)})t\delta_s t(\alpha_s^{h_i(s)})\\
&\quad -ts(\alpha_t^{h_i(t)})t\delta_s t(\alpha_s^{h_i(s)})]\\
=&\alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_i(s)})t\delta_s(\alpha_t^{h_i(t)})
\end{aligned}$$

and analogously

$$\begin{aligned}
&s\delta_t(\alpha_s^{h_{ist}(s)}s(\alpha_t^{h_{ists}(t)})st(\alpha_s^{h_{istst}(s)})sts(\alpha_t^{h_{istists}(t)}))\\
&=s\delta_t(\alpha_s^{h_i(s)}s(\alpha_t^{h_i(t)})st(\alpha_s^{h_{it}(s)})sts(\alpha_t^{h_i(t)}))\\
&=\alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_i(t)})s\delta_t(\alpha_s^{h_i(s)})
\end{aligned}$$

Then a simple substitution gives that the difference above is of the form

$$Q_e f+Q_{sts}\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)(f)+Q_{tst}\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)$$

for some polynomials Q_e, Q_{sts}, Q_{tst} in α_s, α_t.

Now, let A be the algebra given by generator $\widetilde{1}_i, \widetilde{z_i(t)}, \widetilde{\sigma_i(s)}$ subject to relations (1)-(5). Then, by the straightening rule and the braid relation it holds that if $w = s_1 \cdots s_k = t_1 \cdots t_k$ are two reduced expressions then

$$\sigma(\widetilde{t_1 \cdots t_k}) \in \sum_{v \leq s_1 \cdots s_k \text{ reduced subword}} \mathcal{E} * \widetilde{\sigma(v)}.$$

Therefore, once we have fixed one (any) reduced expression for each for $w \in \mathbb{W}$, it holds

$$A = \sum_{w \in \mathbb{W}} \mathcal{E} * \widetilde{\sigma(w)}.$$

Since the generators of $\mathcal{Z}_G$ fulfill the relations (1)-(5), we have a surjective algebra homomorphism

$$A \to \mathcal{Z}_G$$

mapping $\widetilde{1}_i \mapsto 1_i, \widetilde{z_i(t)} \mapsto z_i(t), \widetilde{\sigma_i(s)} \mapsto \sigma_i(s)$. Since $\mathcal{Z}_G = \bigoplus_{w \in \mathbb{W}} \mathcal{E} * \sigma(w)$ and the map is by definition $\mathcal{E}$-linear it follows that

$$A = \bigoplus_{w \in \mathbb{W}} \mathcal{E} * \widetilde{\sigma(w)}$$

and the map is an isomorphism. □

Chapter 3

Parabolic Nil Hecke algebras and parabolic Steinberg algebras

Summary. Let G be a reductive group (over $\mathbb{C}$). *For us* a Steinberg variety $Z = E \times_V E$ is a cartesian product of a collapsing of a union of homogeneous vector bundles over G-homogeneous spaces with itself. The most popular example of such a collapsing is the Springer map. The equivariant Borel-Moore homology $H_*^G(Z, \mathbb{C})$ has an associative algebra structure, we call it a **Steinberg algebra**. We call it **parabolic/of Borel type** if all the homogeneous spaces are of the form G/P for parabolic/Borel subgroups $P \subset G$. We realize parabolic Steinberg algebras as corners in Steinberg algebras of Borel type. As starting point, we study parabolic (affine) nil Hecke algebras, then we generalize this result. For finding generators and relations these observations are not helpful. We revisit an example by Markus Reineke to illustrate this. We ignore the gradings of these algebras.

3.1 The parabolic (affine) nil Hecke algebra

For any complex algebraic variety X with an action of an algebraic group G, we set (as always) $H_G^*(X) := H_G^*(X, \mathbb{C}), H_*^G(X) := H_*^G(X, \mathbb{C})$ for G-equivariant cohomology/ Borel-Moore homology with complex coefficients. We will denote by $D_G^b(X)$ the G-equivariant derived category introduced by Bernstein and Lunts [BL94].
Let G be a reductive group over $\mathbb{C}$ and $B \subset G$ be a Borel subgroup. The **(affine) nil Hecke algebra**[1] is the Steinberg algebra $\mathrm{NH} := \mathrm{H}_*^{\mathrm{G}}(\mathrm{G/B} \times \mathrm{G/B})$, .
We write

$$\mathrm{NH}^{\mathrm{P}} := \mathrm{H}_*^{\mathrm{G}}(\mathrm{G/P} \times \mathrm{G/P})$$

and call it **parabolic (affine) nil Hecke algebra**. It carries a structure as graded algebra (see [CG97], chapter 8), we ignore the grading in this article.

Lemma 32. *It holds*

$$\mathrm{NH}^{\mathrm{P}} \cong \mathrm{End}_{\mathrm{H}_{\mathrm{G}}^*(\mathrm{pt})}(\mathrm{H}_{\mathrm{G}}^*(\mathrm{G/P})).$$

[1] the adjective is been left out more recently

Recall that $H_G^*(pt) = (H_T^*(pt))^W = \mathbb{C}[\mathfrak{t}]^W$ where $T \subset P$ is a maximal torus, $\mathfrak{t}$ its Lie algebra and W the Weyl group for (G,T). Also we know that $H_G^*(G/P) \cong \mathbb{C}[\mathfrak{t}]^{W_P}$ where W_P is the Weyl group of (L,T) for the Levi subgroup $L \subset P$. We write $W^P \subset W$ for the minimal coset representatives of the cosets W/W_P.

proof: Let EG be a contractible free G-space (or an appropriate approximation of it in the sense of [BL94]). Let $X := G/P, \pi\colon X_G := X \times^G EG \to BG$ the map obtained from $X \to pt$ by applying $- \times^G EG$. By [CG97], chapter 8, we know $H_*^G(G/P \times G/P) \cong Ext^*_{D^b_G(pt)}(\pi_*\underline{\mathbb{C}}, \pi_*\underline{\mathbb{C}})$ as $H_G^*(pt)$-algebras (but not as graded ones). Since π is a proper submersion, we have (by [CM07], p.14 3rd Example)

$$\pi_*\underline{\mathbb{C}} = \bigoplus_{i\in\mathbb{Z}} R^i\pi_*\underline{\mathbb{C}}[-i]$$

in $D^b_G(pt)$. Since all fibres of π are isomorphic to X and BG is simply connected, we get that

$$\pi_*\underline{\mathbb{C}} = \bigoplus_{w\in W^P} \underline{\mathbb{C}}[-2\ell(w)],$$

where $\ell(w)$ is the length of w. Let $r = \dim_{\mathbb{C}} H^*(X) = \#W^P$. We know, that $\mathbb{C}[\mathfrak{t}]^{W_P}$ is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r. Therefore, to prove the lemma, it is enough to show

$$Ext^*_{D^b_G(pt)}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \to \mathrm{End}_{H_G^*(pt)}(H_G^*(pt)) = H_G^*(pt)$$
$$f \mapsto H(f)$$

is an isomorphism of algebras, where $H\colon D^b_G(pt) \to H_G^*(pt)-\ \mathrm{mod}$ is the functor $F \mapsto \mathbb{H}^*(F) = \mathbb{H}^*(BG, F)$ is the sheaf cohomology of the complex of constructible sheaves F (which is by definition a hypercohomology group). By [BL94], Thm 12.7.2 (i), there exists an equivalence of triangulated categories

$$D^b_{A_G} \to D^b_G(pt),$$

where $A_G = (H_G^*(pt), d = 0)$ is the (trivial) dg-algebra structure on $H_G^*(pt)$ and $D^b_{A_G}$ is the bounded derived category of dg-modules over A_G. This equivalence commutes with a functor $H\colon (-) \to H_G^*(pt)-\ \mathrm{mod}$ which is for $(-) = D^b_G(pt)$ the functor just mentioned and for $(-) = D^b_{A_G}$ it is the functor taking cohomology of the complex. We prove the claim in two steps.

(1) The triangulated equivalence induces an isomorphism of algebras

$$Ext^*_{D^b_G(pt)}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \to Ext^*_{D^b_{A_G}}(A_G, A_G).$$

First note that $\underline{\mathbb{C}}$ is quasi-isomorphic to the de Rham complex Ω^*_{BG}, we can calculate its Ext-algebra instead. Let A_{BG} be the dg-algebra $\Gamma(\Omega^*_{BG})$. The derived equivalence is by definition factoring over the global section functor $\Gamma\colon D^b_G(pt) \to D^b_{A_{BG}}$.

Notice that $H^*(A_{BG}) = A_G$. By [BL94], Prop. 12.4.4, there is a quasi-isomorphism $\phi\colon A_G \to A_{BG}$ which induces an equivalence $\phi^*\colon D^b_{A_G} \to D^b_{A_{BG}}$. The first functor gives an isomorphism of algebras $Ext^*_{D^b_G(pt)}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \to Ext^*_{D^b_{A_{BG}}}(A_{BG}, A_{BG})$. The second functor (is basically tensoring via ϕ) induces an isomorphism of algebras $Ext^*_{D^b_{A_{BG}}}(A_{BG}, A_{BG}) \to Ext^*_{D^b_{A_G}}(A_G, A_G)$.

(2) The functor H defines an isomorphism $Ext^*_{D^b_{A_G}}(A_G, A_G) \to \mathrm{End}_{H^*_G(pt)}(H^*_G(pt)) = H^*_G(pt)$.
This is a direct application of [BL94], Prop. 11.3.1 (i) since A_G is a dg-module with zero differential.

□

Observe that we have a natural $H^*_G(pt)$-module homomorphism

$$\Theta\colon \mathrm{NH^P} = \mathrm{End}_{\mathrm{H^*_G(pt)}}(\mathrm{H^*_G(G/P)}) \to \mathrm{NH} = \mathrm{End}_{\mathrm{H^*_G(pt)}}(\mathrm{H^*_G(G/B)})$$
$$f \mapsto I \circ f \circ Av$$

where $I\colon \mathbb{C}[\mathfrak{t}]^{W_P} \subset \mathbb{C}[\mathfrak{t}]$ is the natural inclusion and

$$Av\colon \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[\mathfrak{t}]^{W_P}, f \mapsto \frac{1}{\#W_P} \sum_{w \in W_P} w(f)$$

is the averaging map. It holds $Av \circ I = \mathrm{id}_{\mathbb{C}[\mathfrak{t}]^{W_P}}$, the element $e_P := I \circ Av = \Theta(1) \in \mathrm{NH}$ is an idempotent element, $\Theta(fg) = \Theta(f)\Theta(g)$ for all $f, g \in \mathrm{NH^P}$. We let $W_P \times W_P$ operate on NH via graded $H^*_G(pt)$-module homomorphisms defined by

$$(v, w) \cdot h(f) := v(h(w^{-1}(f))), \quad v, w \in W_P, h \in \mathrm{NH}, \mathrm{f} \in \mathbb{C}[\mathfrak{t}].$$

Lemma 33. *(1) The map Θ impies $\mathrm{NH^P} \cong \mathrm{e_P NH e_P}$ as $H^*_G(pt)$-algebras), we call this a* corner algebra *in* NH.

(2) $\mathrm{NH^P} = \mathrm{NH^{W_P \times W_P}}$ *as $H^*_G(pt)$-modules. Furthermore,* $\mathrm{NH^{W \times W}} = \mathbb{C}[\mathfrak{t}]^{\mathrm{W}} = \mathrm{H^*_G(pt)}$.

(3) Let $s = \#W_P$. We have an isomorphism of $\mathbb{C}[\mathfrak{t}]^W$-algebras

$$\mathrm{NH} \cong \mathrm{M_s(NH^P)},$$

in particular it is a free module over $\mathrm{NH^P}$ of rank s^2.

(4) Let $r = \#W^P$. We have an isomorphism of $\mathbb{C}[\mathfrak{t}]^W$-algebras

$$\mathrm{NH^P} \cong \mathrm{M_r}(\mathbb{C}[\mathfrak{t}]^{\mathrm{W}}),$$

in particular it is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r^2.
A basis is given by $c_{v,w}, v, w \in W^P$ with $c_{v,w}$ is a lift of $[\overline{BvP/P} \times \overline{BwP/P}] \in H_*(G/P \times G/P)$ to $H^G_*(G/P \times G/P)$, i.e. elements in the fibres of the forgetful map

(which is a surjective ring homomorphism),

$$H_*^G(G/P \times G/P) \cong M_r(\mathbb{C}[\mathfrak{t}]^W) \twoheadrightarrow M_r(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{t}]^{W_P}/I_W) \cong H_*(G/P \times G/P),$$
$$(f_{i,j})_{i,j} \mapsto (f_{i,j}(0))_{i,j}$$

where I_W is the ideal generated by the W-invariant polynomials of degree ≥ 1

proof:

(1) Θ is injective: Let $f \in \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W}(\mathbb{C}[\mathfrak{t}]^{W_P})$ with $\Theta(f) = I \circ f \circ Av = 0$ then $f \circ Av = 0$ and also $f = (f \circ Av) \circ I = 0$. Therefore we have an $H_G^*(pt)$-algebra isomorphism $\mathrm{NH^P} \cong \Theta(\mathrm{NH^P})$, where the neutral element in $\Theta(\mathrm{NH^P})$ is $\Theta(1) = e_P$. Now clearly, the following map also is an $H_G^*(pt)$-algebra isomorphism.

$$\Theta(\mathrm{NH^P}) \to e_P \mathrm{NH} \mathrm{e_P}$$
$$g = I \circ f \circ Av \mapsto Av \circ g \circ I = e_P f e_P.$$

(2) More precisely, we show $\mathrm{NH}^{\mathrm{W_P \times W_P}} = \Theta(\mathrm{NH^P})$. Each element $I \circ f \circ Av \in \mathrm{NH}, \quad \mathrm{f} \in \mathrm{NH^P}$ is $W_P \times W_P$-invariant, therefore $\mathrm{NH}^{\mathrm{W_P \times W_P}} \subset \Theta(\mathrm{NH^P})$. On the other hand, given $h \in \mathrm{NH}$ with $v(h(w^{-1}(P))) = h(P), \quad P \in \mathbb{C}[\mathfrak{t}], v, w \in W_P$, then it holds $h \circ I \circ Av = I \circ Av \circ h$ which implies $h(\mathbb{C}[\mathfrak{t}]^{W_P}) \subset \mathbb{C}[\mathfrak{t}]^{W_P}$, therefore restriction induces an element $\overline{h} \in \mathrm{NH^P}$ and by definition $h = I \circ \overline{h} \circ Av$.

(3) Let $s = \#W_P$, it is the rank of $\mathbb{C}[\mathfrak{t}]$ as module over $\mathbb{C}[\mathfrak{t}]^{W_P}$ and therefore

$$\mathrm{NH} \cong \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^{\mathrm{W}}}((\mathbb{C}[\mathfrak{t}]^{\mathrm{W_P}})^{\oplus \mathrm{s}}) \cong \mathrm{M_s}(\mathrm{NH^P}).$$

(4) Let $r := \#W^P$, it is the rank of $\mathbb{C}[\mathfrak{t}]^{W_P}$ as module over $\mathbb{C}[\mathfrak{t}]^W$ and the dimension as $\mathbb{C}$-vector space of $\mathbb{C}[\mathfrak{t}]^{W_P}/I_W$. The rest follows as in (3).

□

Also recall the following well-known result.

Proposition 6. *Let $G \supset B \supset T$ be a connected reductive group with a Borel subgroup and a maximal torus, we write (W, S) for the Weyl group of (G, T) with its simple reflections. The affine nil Hecke algebra $\mathrm{NH} = \mathrm{H_*^G(G/B \times G/B)} = \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^{\mathrm{W}}}(\mathbb{C}[\mathfrak{t}])$ is the $\mathbb{C}[\mathfrak{t}]$-algebra generated by*

$$\delta_s \colon \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[\mathfrak{t}], \quad s \in S$$
$$f \mapsto \frac{sf - f}{\alpha_s}$$

where $\alpha_s \in \underline{\Phi}^+$ (=positive roots associated to G, B, T) is the positive root with $s(\alpha_s) = -\alpha_s$. Choose generators $\mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_1, \ldots, x_n]$, we set $c_{i,s} := \delta_s(x_i) \in \mathbb{C}[\mathfrak{t}]$. Then, the algebra NH is the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n, \delta_s, s \in S$ with relations

(1) $x_i x_j = x_j x_i$,

(2) (nil Coxeter relations)
$\delta_s^2 = 0, \ \ \delta_s\delta_t\delta_s\cdots = \delta_t\delta_s\delta_t\cdots$ *whenever* $sts\cdots = tst\cdots$

(3) (straightening rule)
$\delta_s x_i - s(x_i)\delta_s = c_{i,s}$

The previous proposition also is a corollary of [Sau13], Theorem 2.1.

Remark. Our elements $\delta_s, s\in S$ correspond to the cycles $[\overline{Z^s}] \in H_*^G(G/B\times G/B)$, they are called **(BGG)-divided difference operators** or **Demazure operators** and have been introduced by Beilinson, Bernstein, Gelfand and Demazure ([BGG73a] and [Dem73]). Observe, that in the proof of [Sau13], Thm 2.1 we have seen how these operators look like after localizing to the T-fixpoints, i.e. as operators on $H_T^*(G/B)\otimes K, \ K = Quot(H_T^*(pt))$, they are given by

$$\delta_s\colon \bigoplus_{w\in W} K\psi_w \to \bigoplus_{w\in W} K\psi_w, \quad \delta_s(\lambda\psi_w) = \frac{\lambda\psi_{ws} - \lambda\psi_w}{w(\alpha_s)}, \ \ \lambda\in K.$$

This coincides with divided differences operators on $H_T^*(G/B)$ defined by Arabia in [Ara01], thm 3.3.1.
There are other very similar looking types of divided difference operators using the GKM-graph description (cp. J. Tymoczko, [Tym09])

$$H_T^*(G/P) = \{p = (p_w)_{w\in W^P} \in \mathbb{C}[\mathfrak{t}]^{\oplus\# W^P} \mid p_w - p_{sw} \in (\alpha_s), s\in S, w\in W^P\}.$$

(1) Following Kostant and Kumar in [KK86], section (4.17):

$$\delta_s^{KK}\colon H_T^*(G/B) \to H_T^*(G/B)$$
$$p \mapsto \left(\frac{p_{sw} - p_w}{w^{-1}\alpha_s}\right)_{w\in W}$$

(2) Following J. Tymoczko in [Tym09] (this is the only version defined for partial flag varieties):

$$\delta_s^T\colon H_T^*(G/P) \to H_T^*(G/P)$$
$$p \mapsto \left(\frac{p_w - s(p_w)}{\alpha_s}\right)_{w\in W^P}$$

Now, observe that following [Bri97], p.258, one has a identification as $H_G^*(pt)$-modules of $\mathrm{NH^P} = \mathrm{H_*^G(G/P\times G/P)}$ with $(H_T^*(G/P))^{W_L}$ up to a degree shift. It is an open question, if Tymoczko's divided difference operators can be helpful to find generators for the parabolic nil Hecke algebra.

Recall, Schubert polynomials are elements $c_w \in \mathbb{C}[\mathfrak{t}], w\in W$ in the fibre over $[\overline{BwB/B}], w\in W$ under the forgetful map $H_G^*(G/B) = \mathbb{C}[\mathfrak{t}] \twoheadrightarrow \mathbb{C}[\mathfrak{t}]/I_W = H^*(G/B)$. Any choice of such lifts gives a basis of $\mathbb{C}[\mathfrak{t}]$ as $\mathbb{C}[\mathfrak{t}]^W$-module, cp. [Hil82], section II.3, proof of Thm 3.1(Sheppard-Todd, Chevalley), p.77 .

Usually the interest in divided difference operators comes from that they provide a construction tool for Schubert polynomials or more generally a flow-up basis (cp. [Tym09]). A G-equivariant version of it says there exists a polynomial $p \in \mathbb{C}[\mathfrak{t}]$ such that

$$\mathbb{C}[\mathfrak{t}] = \bigoplus_{v \in W_P} \delta_v(p)\mathbb{C}[\mathfrak{t}]^{W_P}$$

as $\mathbb{C}[\mathfrak{t}]^{W_P}$-module, where for $v = s_1 \cdots s_r$ reduced expression $\delta_v := \delta_{s_1} \circ \cdots \circ \delta_{s_r}$. See again [Tym09].

Remark. We know that

$$NH = \bigoplus_{w \in W} \delta_w \mathbb{C}[\mathfrak{t}],$$

(as $\mathbb{C}[\mathfrak{t}]$-module) which implies

$$NH^P = e_P \mathrm{NHe_P} = \sum_{\mathrm{w \in W}} \mathrm{e_P} \delta_{\mathrm{w}} \mathbb{C}[\mathfrak{t}]\mathrm{e_P}$$
$$\sum_{w \in W} \sum_{v \in W_P} e_P \delta_w \circ (\delta_v(p) \cdot) e_P \mathbb{C}[\mathfrak{t}]^{W_P}$$

It is easy to see that for all $s \in W_P \cap S$, it holds $\delta_s \circ e_P = 0$, this implies that if $w = w_1 s \in W, s \in W_P, \ell(w) = \ell(w_1) + 1$, we have

$$e_P \delta_w \circ (\delta_v(p) \cdot) e_P = e_P \delta_{w_1} \circ (\delta_{sv}(p) \cdot) e_P,$$

therefore we can write

$$\mathrm{NH^P} = \sum_{\mathrm{w \in W^P}} \sum_{\mathrm{v \in W_P}} \mathrm{e_P} \delta_{\mathrm{w}} \circ (\delta_{\mathrm{v}}(\mathrm{p}) \cdot) \mathrm{e_P} \mathbb{C}[\mathfrak{t}]^{\mathrm{W_P}}.$$

By lemma 33, (4), we know that $\mathrm{NH^P}$ is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r^2, $r = \# W^P$. Since $\mathbb{C}[\mathfrak{t}]^{W_P}$ is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r, we have the following

Open question: Is $\mathrm{NH^P}$ a free module over $\mathbb{C}[\mathfrak{t}]^{W_P}$ of rank r, with basis given by $\delta_w^P := e_p \delta_w e_P, w \in W^P$? To prove this, it is enough to show that $\sum_{w \in W^P} \delta_w^P \mathbb{C}[\mathfrak{t}]^{W_P} = \bigoplus_{w \in W^P} \delta_w^P \mathbb{C}[\mathfrak{t}]^{W_P}$.

3.2 On parabolic Steinberg algebras

Definition 9. Let $(G, P_i, V, F_i)_{i \in I}$ be a tuple with G a reductive group with parabolic subgroups $P_i, i \in I$ (where I is some finite set) such that $\bigcap P_i$ contain a maximal torus T and V a G-representation with P_i-subrepresentations $F_i \subset V$. We associate to this the Steinberg variety

$$Z^P := \bigsqcup_{i,j \in I} (G \times^{P_i} F_i) \times_V (G \times^{P_j} F_j$$

and call $\mathcal{Z}_A := H_*^A(Z)$, $A \in \{pt, T, G\}$ Steinberg algebra associated to the data (where the product is given by a convolution construction defined by [CG97], section 2.7).

The (affine) nil Hecke algebra is the Steinberg algebra in the special case $\#I = 1$, $V = \{0\}$. We ask for the realtionship of parabolic Steinberg algebras to Steinberg algebras where all the parabolic groups are Borel subgroups. More precisely consider the following situation:

Let $(G, P_i, V, F_i)_{i \in I}$ be construction data as above, we denote all associated data with $()^P$, we set $E_i^P := G \times^{P_i} F_i, E^P := \bigsqcup E_i^P, \pi^P \colon E^P \to V, \overline{(g,f)} \mapsto gf$. Choose $T \subset B_i \subset P_i, i \in I$ Borel subgroups of G (where $T \subset \bigcap_{i \in I} P_i$) and consider F_i as B_i-representation, then $(G, B_i, V, F_i)_{i \in I}$ can be used to define E_i^B, E^B, π^B, Z^B analogously.
We want to compare $H_*^A(Z^P)$ with $H_*^A(Z^B)$ for $A \in \{pt, T, G\}$.

Consider the following commutative triangle

$$\begin{array}{ccc} E_i^B = G \times^{B_i} F_i & \xrightarrow{\alpha_i} & E_i^P = G \times^{P_i} F_i \\ & \searrow^{\pi^B} \quad \swarrow^{\pi^P} & \\ & V & \end{array}$$

Observe, that the fibres of $\alpha := \bigsqcup_{i \in I} \alpha_i$ over E_i^P are all isomorphic to $\alpha_i^{-1}(\overline{(e,0)}) \cong P_i/B_i$ and that the Weyl group W_i of (L_i, T) where $L_i \subset P_i$ operates on them topologically via choosing a compact form for $K_i \subset L_i$, then there exists a maximal torus $T_i \subset K_i$ such that

$$P_i/B_i = L_i/(B_i \cap L_i) \cong K_i/T_i, \quad W(K_i, T_i) = W(L_i, T) =: W_i.$$

The groups W_i operates on K_i/T_i via $nT_i \cdot kT_i = kn^{-1}T_i, n \in N_{K_i}(T_i), k \in K_i$ without fixpoints. We have the following lemma using the left W_i-operation on $\alpha_i^{-1}(\overline{(e,0)}) \cong L_i/(B_i \cap L_i)$.

Lemma 34. *Given $B \subset P$ a Borel inside a parabolic subgroup in a reductive group G, and let F be a P-representation. We write $\alpha \colon E^B := G \times^B F \to E^P := G \times^P F, \overline{(g,f)}^B \mapsto \overline{(g,f)}^P$ for the canonical map. For the constant sheaf $\underline{\mathbb{C}}$ on E^P the adjunction map $\underline{\mathbb{C}} \to R\alpha_*\alpha^*\underline{\mathbb{C}}$ is a monomorphism in $D_A^b(E^P)$, furthermore it factorizes over an isomorphism $\underline{\mathbb{C}} \to (R\alpha_*\underline{\mathbb{C}})^{W_P}$ where W_P is the Weyl group of a Levi subgroup in P.*

proof: For any variety we write $X_A := X \times^A EA$ where EA is a contractible space with a free A-operation. We denote $\alpha := \alpha_A \colon (E^B)_A \to (E^P)_A$ the associated map. It is a proper submersion with fibres all isomorphic to P/B. The decomposition theorem (in the more specific version for a proper submersion, see [CM07],p.14 3rd Example) implies

$$R\alpha_*\underline{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} R^i\alpha_*\underline{\mathbb{C}}[-i].$$

Since $R^i\alpha_*\underline{\mathbb{C}}$ is the sheaf associated to the presheaf

$$U \mapsto H^i(\alpha^{-1}(U))$$

this implies $(R^i\alpha_*\underline{\mathbb{C}})_x = H^i(\alpha^{-1}(x)) \cong H^i(P/B)$ for all $x \in (E^P)_A$. Therefore, $R^i\alpha_*\underline{\mathbb{C}}$ is a local system on $(E^P)_A$ and since $\pi_1((E^P)_A, x_0)$ (for any $x_0 \in (E^P)_A$) is trivial, it is the constant local system $\bigoplus_{w\in W_P}\underline{\mathbb{C}}[-2\ell(w)]$ because $H^*(P/B) = H^*(L/(L\cap B) = \mathbb{C}[\mathfrak{t}]/I_{W_P}$ where the last isomorphism is graded algebras and as W_P-representations by the Borel isomorphism. But since $(\mathbb{C}[\mathfrak{t}]/I_{W_P})^{W_P} = \mathbb{C}$ in degree 0, it holds $(R\alpha_*\underline{\mathbb{C}})^{W_P} \cong \underline{\mathbb{C}}$. Furthermore, it is easy to see that the unit of the adjunction is a monomorphism (since α is locally trivial). By taking the trivial W_P-operation on $\underline{\mathbb{C}}$, we can make the unit of the adjunction a W_P-linear map (because the map is locally trivial and W_P operates only on the fibre), then taking W_P-invariants proves the lemma. □

Back to the more general situation from before. Since it is shorter we write α_* instead of $R\alpha_*$ even though the second is meant. For the constant sheaf $\underline{\mathbb{C}}$ the adjunction map $\underline{\mathbb{C}} \to \alpha_*\alpha^*\underline{\mathbb{C}}$ is a monomorphism in $D_c^A(\bigsqcup_{i\in I} E_i^P)$. The previous lemma implies

$$\underline{\mathbb{C}} = \bigoplus_{i\in I}\underline{\mathbb{C}}_{E_i^P} \cong \bigoplus_{i\in I}[(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}]^{W_i} \hookrightarrow \alpha_*\alpha^*\underline{\mathbb{C}}.$$

We set $\pi_i^P := \pi^P|_{E_i^P}, \pi_i^B := \pi^B|_{E_i^B}$ and we can apply the functor π_*^P and get an *inclusion*

$$\begin{aligned} Inc\colon \pi_*^P\underline{\mathbb{C}}_{E^P} \cong \pi_*^P(\bigoplus_{i\in I}[(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}]^{W_i}) &= \bigoplus_{i\in I}[(\pi_i^P)_*(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}]^{W_i} \\ &\hookrightarrow \pi_*^P\alpha_*\alpha^*\underline{\mathbb{C}}_{E^P} = \pi_*^B\underline{\mathbb{C}}_{E^B} \end{aligned}$$

Also we can define an *averaging* map

$$Av\colon \pi_*^B\underline{\mathbb{C}}_{E^B} = \bigoplus_{i\in I}(\pi_i^B)_*\underline{\mathbb{C}}_{E_i^B} \twoheadrightarrow \bigoplus_{i\in I}[(\pi_i^P)_*(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}]^{W_i} \cong \pi_*^P\underline{\mathbb{C}}$$

which is given by the Reynolds operator for the finite group W_i (the Reynolds operator exists for arbitrary reductive groups (representations in characteristic zero), for finite groups it is equal to the averaging map.) It holds $Av \circ Inc = \mathrm{id}$ and $Inc \circ Av =: e_P$ is an idempotent endomorphism.

Proposition 7. *The map*

$$\begin{aligned} \Psi_P^B\colon H_*^A(Z^P) = Ext^*_{D_b^A(V)}(\pi_*^P\underline{\mathbb{C}}, \pi_*^P\underline{\mathbb{C}}) &\to Ext^*_{D_b^A(V)}(\pi_*^B\underline{\mathbb{C}}, \pi_*^B\underline{\mathbb{C}}) = H_*^A(Z^B) \\ f &\mapsto Inc \circ f \circ Av \end{aligned}$$

induces an isomorphism of $H_*^A(pt)$-algebras

$$H_*^A(Z^P) \cong e_P H_*^A(Z^B) e_P,$$

where $e_P = Inc \circ Av$ is an idempotent element in $H_*^A(Z^B)$.

proof: Completely analogue to the proof of lemma 2, (1). □

Recall $E^B := \bigsqcup_{i\in I} E_i^B, E^P := \bigsqcup_{i\in I} E_i^P$ and we have a commutative diagram

$$\begin{array}{ccc} H_*^A(Z^B) & \xrightarrow{\Phi^B} & \mathrm{End}_{H_*^A(pt)}(H_A^*(E^B)) \\ \Psi_P^B \uparrow & & \uparrow \Theta_P^B \\ H_*^A(Z^P) & \xrightarrow{\Phi^P} & \mathrm{End}_{H_*^A(pt)}(H_A^*(E^P)) \end{array}$$

with algebra homomorphisms Φ^B, Φ^P given by applying the global section functor and *corner* inclusions Ψ_P^B, Θ_P^B as defined before. Since the maps Φ^B, Φ^P are given by taking global sections, we see that the element $e_P \in H_*^A(Z^B)$ maps under Φ^B to the idempotent element $\varepsilon_P \in \mathrm{End}_{H_*^A(pt)}(H_A^*(E^B))$ (also defined by $\varepsilon_P = Av \circ Inc$). We know that the map Ψ_P^B can be identified with the corner inclusion $e_P H_*^A(Z^B) e_P \to H_*^A(Z^B)$, the map Θ_P^B with the corner inclusion $\varepsilon_P \mathrm{End}_{H_*^A(pt)}(H_A^*(E^B))\varepsilon_P \to \mathrm{End}_{H_*^A(pt)}(H_A^*(E^B))$. Therefore, we see that we can have an identification of algebras

$$H_*^A(Z^P) = \varepsilon_P \Phi^B(H_*^A(Z^B))\varepsilon_P.$$

Now, using theorem [Sau13], thm 2.1 combined with the previous consideration, we can calculate (in theory) parabolic Steinberg algebras. Practically, the corner description makes it even in easy cases difficult to find generators and relations, we do not know an example for it. Let us revisit an example from Markus Reineke using our description of the parabolic Steinberg algebra.

3.2.1 Reineke' s Example (cp. end of [Rei03])

Let Q be the quiver $(1 \to 2)$ and let $(d_1, d_2) \in \mathbb{N}_0^{Q_0}$. A directed partition of the Auslander-Reiten quiver of $\mathbb{C}Q$ is given by $I_1 := \{E_2 := (0 \to \mathbb{C})\}, I_2 := \{E_{1,2} := (\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C})\}, I_3 := \{E_1 := (\mathbb{C} \to 0)\}$, i.e. it is a partition of the vertices of the Auslander-Reiten quiver $\{I_t\}_t$ such that $\mathrm{Ext}^1(I_t, I_t) = 0$ and $\forall t < u$ $\mathrm{Hom}(I_u, I_t) = 0 = \mathrm{Ext}^1(I_t, I_u)$. Let $M = E_2^{d_2} \oplus E_{1,2} \oplus E_1^{d_1}$. Then, M. Reineke proved that quiver-graded Springer map corresponding to the dimension filtration $(0, (d_1 - 1, 0), (d_1 - 1, 1), (d_1, 1), (d_1, d_2))$ gives a resolution of singularities (i.e. birational projective map) for the orbit closure of M. Yet, we will consider the even easier dimension filtration $\underline{\mathbf{d}} := (0, (d_1, 1), (d_1, d_2))$. The associated Steinberg variety is

$$Z := \{(A, L_1, L_2) \in M_{d_2 \times d_1}(\mathbb{C}) \times \mathbb{P}^{d_2}(\mathbb{C}) \mid Im(A) \subset L_i, i = 1, 2\},$$

it carries an operation of $\mathbf{Gl}_{\underline{d}} := \mathbf{Gl}_{d_2} \times \mathbf{Gl}_{d_1}$ via

$$(A, L_1, L_2) \mapsto (g_2^{-1} A g_1, g_2^{-1} L_1, g_2^{-1} L_2), \quad (g_2, g_1) \in \mathbf{Gl}_{\underline{d}}.$$

We want to describe the Steinberg algebra $H_*^{\mathbf{Gl}_{\underline{d}}}(Z)$ with our method. We set

$\mathbb{G} := \mathbf{Gl}_d, d := d_1 + d_2,$

$T :=$ invertible diagonal matrices,

$\mathbb{B} :=$ invertible upper triangular matrices,

$\mathbb{P} :=$ invertible upper block matrices with diagonal block sizes $(1, d-1)$,

$\mathcal{U} = \mathrm{Lie}\,\mathbb{U}_{\mathbb{P}}$, where $\mathbb{U}_{\mathbb{P}}$ is the unipotent radical of $\mathbb{P}$,

$G := \mathbf{Gl}_{d_2} \times \mathbf{Gl}_{d_1}$ diagonally embbeded into $\mathbb{G}$,

$V = M_{d_2 \times d_1}$ embedded into the right upper corner of $\mathcal{G} = \mathfrak{gl}_d$.

as usual set $B := \mathbb{B} \cap G, P := \mathbb{P} \cap G,\ F := \mathcal{U} \cap V$.

The algebra $H_*^G(Z^B), Z^B := (G \times^B F) \times_V (G \times^B F)$ can by theorem [Sau13], thm 2.1 be described as the the algebra $1_e * \mathcal{Z} * 1_e$ for $\mathcal{Z}$ be the Steinberg algebra associated to $(\mathbb{G}, \mathbb{B}, \mathcal{U}, V)$ and $e \in W \backslash \mathbb{W}$ be the coset of the neutral element. If we set $s_i := (i, i{+}1) \in S_d$ and

$$\delta_i := \delta_{s_i} \colon \mathbb{C}[t_1, \ldots, t_d] \to \mathbb{C}[t_1, \ldots, t_d], \quad f \mapsto \frac{s_i(f) - f}{t_i - t_{i+1}}$$

Then, $H^*_G(Z^B)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}[t_1,\ldots,t_d]^{S_{d_2} \times S_{d_1}}}(\mathbb{C}[t_1, \ldots, t_d])$ generated by

$$(t_j \cdot),\ 1 \leq j \leq d, \quad \delta_i,\ i \in \{2, \ldots d_2 - 2, d_2, \ldots, d-1\},$$

$$\theta := \prod_{j=d_1+d_2+1}^{d} (t_1 - t_j)\delta_1.$$

Now, Reineke's variety equals $Z = (G \times^P F) \times_V (G \times^P F)$, by the previous section we conclude it is the corner algebra of $e_P H_*^G(Z^B) e_P$ where

$$e_P \colon \mathbb{C}[t_1, \ldots, t_d] \to \mathbb{C}[t_1, \ldots, t_d], f \mapsto \frac{1}{(d_2 - 1)! d_1!} \sum_{w \in <s_2, \ldots s_{d_2-2}, s_{d_2}, \ldots, s_{d-1}>} w(f).$$

Remark. Since

* $\delta_i e_P = 0$ for $i \neq 1$
* $\delta_i \theta = q \delta_1 \delta_i$ for $i \neq 2$ some $q \in \mathbb{C}[\mathfrak{t}]$
* $\theta \delta_2 \theta = q_1 \theta + q_2 \delta_2 \theta$ for some $q_1, q_2 \in \mathbb{C}[\mathfrak{t}]$
* there are straightening rules to bring polynomial elements in the beginning of the element,

we think that $H_*^G(Z)$ is as $\mathbb{C}[\mathfrak{t}]^{W_P}$-module generated by products of
$e_P \theta^r e_P,\ \ e_P \delta_2 \theta^t e_P,\ r, t \in \mathbb{N}_0$. I have no idea on the relations.

3.2.2 Literature

For the classical Springer map a description of the parabolic Steinberg algebra as a corner in the Steinberg algebra of Borel type has been investigated by Douglas and Röhrle in [DR09] and earlier by Borho, MacPherson [BM83].

Chapter 4

From Springer theory to monoidal categories

Summary. *Here*: Springer theory is a construction of a graded algebra (called Steinberg algebra) from a collapsing of a union of homogeneous vector bundles over homogeneous spaces. We define $\mathbb{I}$-graded Springer theory (for a monoid $\mathbb{I}$) as a collection of certain Springer Theories parametrized by the elements of $\mathbb{I}$. Associated to this we study the following monoidal categories.

(1) We sometimes have a new product on the associated Steinberg algebras, called horizontal product. The projective graded modules for such a multiplicative family of algebras are a monoidal category. In this case, there is a different description of the monoidal category in terms of *Lusztig*'s perverse sheaves[1].

(2) If there is no horizontal product, we embed the Steinberg algebra in a bigger Steinberg algebra which has a horizontal product and extend the Steinberg algebra by the images under the horizontal map of the bigger Steinberg algebra. Then projective graded modules over this new algebra have a monoidal category structure.

Of course, we would like to understand their Grothendieck rings but as the definitions of these categories are very long, we just recall the known results here. The main example due to Lusztig is quiver-graded Springer theory and the horizontal product is constructed with varieties of short exact sequences of KQ-representation in the spirit of the Hall algebra product. The Grothendieck group of the projective graded modules over the Steinberg algebra is a twisted Hopf algebra which can be identified by the negative half of the quantum group associated to the quiver.
We consider as a second example symplectic quiver-graded Springer theory in analogy to quiver-graded Springer theory. Here we describe the monoidal category of type (2).

[1] They are not perverse sheaves because we allow shifts of them

4.1 ($\mathbb{I}$-)Graded Springer theory

Definition 10. Let $(\mathbb{I}, +)$ be a semi-group, not necessarily commutative. By an **$\mathbb{I}$-graded variety** (over $\mathbb{C}$) we mean

(1) For every $i \in \mathbb{I}$ a finite-dimensional variety (X_i, p_i) over $\mathbb{C}$ together with a base point $p_i \in X_i$.

(2) For every $i, j \in \mathbb{I}$ a closed embedding $m_{i,j} \colon X_i \times X_j \to X_{i+j}$ which maps (p_i, p_j) to p_{i+j} with $m_{i,j+k} \circ (\mathrm{id}_{X_i} \times m_{j,k}) = m_{i+j,k} \circ (m_{i,j} \times \mathrm{id}_{X_k})$ for $i, j, k \in \mathbb{I}$.

We say an $\mathbb{I}$-graded variety is irreducible/connected if all $X_i, i \in \mathbb{I}$ are irreducible/connected.

Remark. Let $(X_i, p_i, m_{i,j})$ be an $\mathbb{I}$-graded variety. The set $X = \bigcup_{i \in \mathbb{I}} X_i$ where we consider $X_i \subset X_{i+j}$ via $X_i \cong X_i \times \{p_j\} \xrightarrow{m_{i,j}|_{X_i \times \{p_j\}}} X_{i+j}$ gives X the structure of an ind variety (with respect to the partial order on $\mathbb{I}$ which is induced by the addition).

A morphism of $\mathbb{I}$-graded varieties $f \colon (X_i, p_i, m_{i,j}) \to (Y_i, q_i, n_{i,j})$ consists of a collection of pointed morphisms $f_i \colon X_i \to Y_i$ such that for all $i, j \in \mathbb{I}$ the the following diagram is commutative:

$$\begin{array}{ccc} X_i \times X_j & \xrightarrow{m_{i,j}} & X_{i+j} \\ {\scriptstyle f_i \times f_j} \downarrow & & \downarrow {\scriptstyle f_{i+j}} \\ Y_i \times Y_j & \xrightarrow{n_{i,j}} & Y_{i+j} \end{array}$$

We call $X := (X_i, p_i, m_{i,j})$ an **$\mathbb{I}$-graded subvariety** of $Y := (Y_i, q_i, n_{i,j})$ if there exists a morphism $f \colon X \to Y$ with f_i is a closed embedding for all $i \in \mathbb{I}$.

Definition 11. We say an $\mathbb{I}$-graded variety G is an **$\mathbb{I}$-graded group** if all G_i are algebraic groups, $p_i = e$ are the unit elements, and all $m_{i,j}$ are morphisms of algebraic groups.
We call a morphism of $\mathbb{I}$-graded varieties $f \colon G \to H$ a morphism of $\mathbb{I}$-graded groups if all f_i are algebraic group homomorphisms. We call an $\mathbb{I}$-graded subvariety $G \subset H$ an **$\mathbb{I}$-graded subgroup** if all $G_i \subset H_i, i \in \mathbb{I}$ are algebraic subgroups. We say G operates on an $\mathbb{I}$-graded variety X if there is a morphism of $\mathbb{I}$-graded varieties $f \colon G \times X \to X$ such that f_i is an operation of the algebraic group G_i on X_i, $i \in \mathbb{I}$.
If in addition, all X_i are $\mathbb{C}$-vector spaces and f_i defines a linear operation of G_i on X_i, then we call X a linear G-representation.

For example, let $V = (V_i, 0, m_{i,j})_{i,j \in \mathbb{I}}$ be an $\mathbb{I}$-graded variety with all V_i finite dimensional $\mathbb{C}$-vector spaces and all $m_{i,j}$ linear maps which come from a restriction of an isomorphism $V_i \oplus V_j \oplus X_{i,j} \cong V_{i+j}$. In particular, the complements have to fulfill $X_{i,j} \oplus X_{i+j,k} = X_{i,j+k} \oplus X_{j,k}, i, j, k \in \mathbb{I}$. Then, $\mathbf{Gl}(V) = (\mathbf{Gl}(V_i), n_{i,j})$ is an $\mathbb{I}$-graded group (the choice of the complements $X_{i,j}$ gives homomorphisms $n_{i,j} \colon \mathbf{Gl}(V_i) \times \mathbf{Gl}(V_j) \to \mathbf{Gl}(V_{i+j})$) and V is a $\mathbf{Gl}(V)$-representation.
If $\rho \colon G \to \mathbf{Gl}(V)$ is an $\mathbb{I}$-graded group homomorphism, then V is a G-representation via ρ.

We say an I-graded group G is reductive if all G_i are reductive groups. We say an $\mathbb{I}$-graded subgroup $H \subset G$ is a parabolic/ Borel subgoup/ maximal torus if all $\iota_i \colon H_i \to G_i$ are inclusion of a parabolic/ Borel subgroup/ maximal torus. This is up to taking products the list of parabolics which we consider.

Example. (1) $\mathbb{I} = \mathbb{N}_0$, $G_n := \mathbf{Gl}_n$, the map $\mathbf{Gl}_n \times \mathbf{Gl}_m \to \mathbf{Gl}_{n+m}$ is blockwise inclusion (where we place $\mathbf{Gl}_n$ in the left upper corner, $\mathbf{Gl}_m$ in the right lower corner). Let $B_n \subset \mathbf{Gl}_n$ be the upper triangular matrices, then this defines a Borel subgroup of the $\mathbb{I}$-graded group $\mathbf{Gl} := (\mathbf{Gl}_n)_{n\in\mathbb{N}_0}$.

(2) $\mathbb{I} := \{i = (n_1, \ldots, n_r) \mid r \in \mathbb{N}_0, n_i \in \mathbb{N}_0\}$, the composition is concatenation of sequences. We write $|i| := \sum_{t=1}^r n_t$ and define $G_i := \mathbf{Gl}_{|i|}, i \in \mathbb{I}$ with the same embeddings as in (1). We define P_i to be the standard parabolic in $G_i = \mathbf{Gl}_{|i|}$ (i.e. upper triangular block matrices) with block sizes from the left upper corner to the right bottom corner given by the sequence $i = (n_1, \ldots, n_r)$. Then the block diagonal embedding $\mathbf{Gl}_{|i|} \times \mathbf{Gl}_{|j|} \to \mathbf{Gl}_{|i+j|}$ restricted to $P_i \times P_j$ gives a map $P_i \times P_j \to P_{i+j}$. This defines a parabolic subgroup of the $\mathbb{I}$-graded group $(G_i)_{i\in\mathbb{I}}$.

(3) $\mathbb{I} = \mathbb{N}_0$, $G_n := \mathbf{Sp}_{2n}$, we define $J = J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$

$$\begin{aligned}\mathbf{Sp}_{2n} &= \{g \in \mathbf{Gl}_{2n} \mid {}^t g J g = J\} \\ &= \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Gl}_{2n} \mid {}^tAC = {}^tCA, {}^tDB = {}^tBD, {}^tAD - {}^tCB = E_n\}\end{aligned}$$

The block diagonal embedding $\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \to \mathbf{Sp}_{2(n+m)}$ is given by the map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix}.$$

The standard Borel in $\mathbf{Sp}_{2n}$ is

$$B_n := \{\begin{pmatrix} A & S \\ 0 & (A^{-1})^t \end{pmatrix} \in \mathbf{Gl}_{2n} \mid A \text{ upper triangular, } A^{-1}S \text{ symmetric}\},$$

this defines a Borel subgroup of the $\mathbb{I}$-graded group $\mathbf{Sp} := (\mathbf{Sp}_{2n})_{n\in\mathbb{N}_0}$.

(4) $\mathbb{I} := \{i = (n_1, \ldots, n_r) \mid r \in \mathbb{N}_0, n_i \in \mathbb{N}_0\}$, $|i| := \sum_{t=1}^r n_t$ (as in (2)).
We define $G_i := \mathbf{Sp}_{2|i|}$ with the block diagonal embedding. We set $n := |i|$

$$P_i := \{\begin{pmatrix} A & S \\ 0 & (A^{-1})^t \end{pmatrix} \in \mathbf{Sp}_{2n} \mid A \in \mathbf{Gl}_n \text{ in a standard parabolic,}$$
$$\text{block sizes } i = (n_1, \ldots, n_r)\}$$

This defines a parabolic subgroup of the $\mathbb{I}$-graded group $(G_i)_{i\in\mathbb{I}}$.
Observe, that Levi-subgroups of $\mathbf{Sp}_{2n}$ have at most one direct factor (or summand) which is a symplectic group. The $P_i, i \in \mathbb{I}$ have a product of $\mathbf{Gl}_t$'s as Levi group and there can not exist a parabolic subgroup of $G_i, i \in \mathbb{I}$ which has symplectic groups as Levi factors (because of the map $P'_i \times P'_i \to P'_{i+i}$ we would get a parabolic with two symplectic Levi factors).

We should keep in mind that parabolic subgroups of $\mathbb{I}$-graded groups are a seldom species.

Definition 12. We call (G, P, V, F) an **$\mathbb{I}$-graded Springer theory** if

(1) G is a reductive $\mathbb{I}$-graded group, P is a parabolic subgroup of G. We always denote by T a maximal torus in P and we assume $T_i \times T_j = T_{i+j}$, $i, j \in \mathbb{I}$. We also assume, if $G_i = G_k, G_j = G_l$ then $m_{i,j} = m_{k,l}$.

(2) V is a G-representation and $F = (F_i, 0, n_{i,j}) \subset V$ is P-subrepresentation. We also fix $\mathbb{C}$-vector space complements $m_{i,j} \colon F_i \oplus F_j \oplus X_{i,j} \to F_{i+j}, i, j \in \mathbb{I}$ with $m_{i,j}|_{F_i \times F_j \times \{0\}} = n_{i,j}$ and $m_{i,j}$ is $P_i \times P_j$-linear.

Observe, that if we assume that V is a T-subrepresentation of a Lie algebra of a connected reductive group with maximal torus T, then all T_i-weight (=root) spaces are 1-dimensional, $i \in \mathbb{I}$, and the complement of $F_i \oplus F_j$ in F_{i+j} can be chosen to be the unique T_{i+j}-equivariant complement.

$\mathbb{I}$-graded Steinberg algebras. Let (G, P, V, F) be an $\mathbb{I}$-graded Springer theory. We define an equivalence relation on $\mathbb{I}$ via $i \sim j$ if and only if $G_i = G_j$ and $V_i = V_j$ as G_i-representations. We write $|i|$ for the equivalence class. By assumption we have $|i| + |j| := |i + j|$ is welldefined and gives $|\mathbb{I}| := \mathbb{I}/\sim$ the structure of a quotient semi-group of $\mathbb{I}$.
Now, for $i \sim j$ set $Z_{i,j} := E_i \times_{V_i} E_j, E_i = G_i \times^{P_i} F_i$ $\mathcal{Z}_{i,j} := H_*^{G_i}(Z_{i,j})$. We call

$$\mathcal{Z}_{|i|} := H_*^{G_i}(\bigsqcup_{i,j\in|i|} Z_{i,j}) = \bigoplus_{i,j\in|i|} \mathcal{Z}_{i,j}$$

the Steinberg algebra for $|i|$, the algebra product is given by the convolution product defined by Chriss and Ginzburg in [CG97], chapter 3. The **$\mathbb{I}$-graded Steinberg algebra** is defined as

$$\mathcal{Z}_G := \bigoplus_{|i|\in|\mathbb{I}|} H_*^{G_i}(\bigsqcup_{i,j\in|i|} Z_{i,j}).$$

We want to find two multiplications as follows.

(1) The **vertical product** is the direct sum of the algebra products on $\mathcal{Z}_{|i|}$, $i \in |\mathbb{I}|$

(2) The **horizontal product**

$$*\colon \mathcal{Z}_{|i|} \times \mathcal{Z}_{|j|} \to \mathcal{Z}_{|i|+|j|}$$

which respects the algebra multiplication but not necessarily maps the unit element to the unit element.

We tried to give a geometric construction of the horizontal product but as since in the symplectic quiver-graded Springer theory it does not exist in some cases (see later), this construction can not exist in general.
Instead, we will give the horizontal map in each case where we can find it explicitly. In the quiver-graded Springer theory it always exists and in the symplectic quiver-graded Springer theory we often have to enlarge the Steinberg algebras to define the horizontal multiplication. At least, I want to mention two examples which work for all $\mathbb{N}_0$-graded reductive groups. These groups are families of products of groups of type $A_n, B_n, C_n, D_n, n \in \mathbb{N}$.

Example. Let G be a $\mathbb{N}_0$-graded reductive group with a Borel subgroup B, a maximal torus T and Weyl group $W = (W_n)_{n\in\mathbb{N}}$.

(1) (Nil Springer theory)
We consider the $\mathbb{N}_0$-graded Springer theory $(G, B, \{0\}, \{0\})$. It holds

$$H_*^{G_n}(G_n/B_n \times G_n/B_n) = \mathrm{End}_{\mathbb{C}[\mathfrak{t}_n]^{W_n}-\mathrm{lin}}(\mathbb{C}[\mathfrak{t}_n]) =: \mathrm{NH_n},$$

with $\mathbb{C}[\mathfrak{t}_n] = \mathbb{C}[x_1, \ldots, x_n]$. The Steinberg algebra is by definition

$$\bigoplus_{n\in\mathbb{N}_0} \mathrm{NH_n}$$

Next: We explain that there is a graded algebra homomorphism

$$\mathrm{NH_n} \otimes_{\mathbb{C}} \mathrm{NH_m} \to \mathrm{NH_{n+m}},$$

which we denote by $(f, g) \mapsto f \otimes g$. It is known that $\mathrm{NH_n}$ is as a graded $\mathbb{C}$-algebra generated by

$$z_k := x_k \cdot \colon \mathbb{C}[\mathfrak{t}_n] \to \mathbb{C}[\mathfrak{t}_n], f \mapsto x_k f$$

which has degree 2 and the divided difference operators $\delta_s, s \in S^{(n)}$ of degree -2 defined by

$$f \mapsto \delta_s(f) = \frac{s(f) - f}{\alpha_s}$$

where $S^{(n)}$ is the set of simple reflections associated to G_n, B_n, T_n. Recall that we have natural maps $i \colon W_n \times W_m \to W_{n+m}$. We can extend the definition of the divided difference operator for a general reflection $t = wsw^{-1}, s \in S^{(n+m)}, w \in W^{(n+m)}$ by $\Delta_t(f) := \frac{t(f)-f}{w(\alpha_s)}, f \in \mathbb{C}[\mathfrak{t}_{n+m}]$. Then we define

$$z_k \otimes 1 \mapsto z_k, \qquad 1 \otimes z_\ell \mapsto z_{n+\ell}$$
$$\delta_s \otimes 1 \mapsto \Delta_{i(s)}, \qquad 1 \otimes \delta_s \mapsto \Delta_{i(s)}.$$

(For $G = \mathbf{Gl}$ this is quiver-graded Springer theory with quiver $Q = \bullet$.)

(2) (Classical Springer theory)
We consider the $\mathbb{N}_0$-graded Springer theory $(G, B, \operatorname{Lie} G, \operatorname{Lie} U)$ where $\operatorname{Lie} G$ is the adjoint representation of G and $U \subset B$ is the unipotent radical.
It holds $\mathcal{Z}_n = \mathbb{C}[\mathfrak{t}_\mathfrak{n}]\#W_n$ is the skew group ring defined by $(f \otimes w) \cdot (g \otimes v) := fw(g) \otimes wv,\ f, g \in \mathbb{C}[\mathfrak{t}_n], w, v \in W_n$.
The inclusion $i \colon W_n \times W_m \subset W_{n+m}$ and $\mathbb{C}[\mathfrak{t}_n] \otimes_\mathbb{C} \mathbb{C}[\mathfrak{t}_m] = \mathbb{C}[\mathfrak{t}_{n+m}]$ induce the horizontal product

$$\begin{aligned} \mathbb{C}[\mathfrak{t}_\mathfrak{n}]\#W_n \times \mathbb{C}[\mathfrak{t}_\mathfrak{m}]\#W_m &\to \mathbb{C}[\mathfrak{t}_{\mathfrak{n}+\mathfrak{m}}]\#W_{n+m} \\ (f \otimes w, g \otimes v) &\mapsto (f \otimes g) \otimes i(w, v) \end{aligned}$$

(For $G = \mathbf{Gl}$ this is quiver-graded Springer theory for the Jordan quiver (or 1-loop quiver).)

4.2 Monoidal categorification of a multiplicative sequence of algebras

Definition 13. A monoidal category $\mathcal{C} = (\mathcal{C}_0, \otimes, E, a, l, r)$ consists of a category $\mathcal{C}_0$, a functor $\otimes \colon \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$, an object E of $\mathcal{C}_0$ and natural isomorphisms $a_{XYZ} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X \colon E \otimes X \to X$, $r_X \colon X \otimes E \to X$, subject to the commutativity of the following diagrams

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle a \otimes 1} & & & & \uparrow{\scriptstyle 1 \otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\qquad\qquad a \qquad\qquad} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

$$\begin{array}{ccc} (X \otimes E) \otimes Y & \xrightarrow{a} & X \otimes (E \otimes Y) \\ & {\scriptstyle r \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes l} & \\ & X \otimes Y & \end{array}$$

We only want to study examples of additive $\mathbb{C}$-linear monoidal categories. We recall a way of constructing them which is a slight modification of the definition of a monoidal categorification associated to a multiplicative sequence of algebras from A. Davidov and A. Molev (see [DM11]). Given a sequence of associative unital $\mathbb{C}$-algebras $A_* = \{A_i \mid i \in I\}$ for a monoid $(I, +)$ with $A_0 := \mathbb{C}$ equipped with collections of multiplicative maps (i.e. they respect the algebra product but are not necessarily mapping the unit to the unit)

$$\mu_{i,j} \colon A_i \otimes_\mathbb{C} A_j \to A_{i+j}, \quad i, j \in I$$

satisfying the following associativity axiom: For all $i, j, k \in I$ the following diagram commutes

$$\begin{array}{ccc} A_i \otimes A_j \otimes A_k & \xrightarrow{\mu_{i,j}\otimes \mathrm{id}} & A_{i+j} \otimes A_k \\ {\scriptstyle \mathrm{id}\otimes\mu_{j,k}}\downarrow & & \downarrow{\scriptstyle \mu_{i+j,k}} \\ A_i \otimes A_{j+k} & \xrightarrow[\mu_{i,j+k}]{} & A_{i+j+k}. \end{array}$$

We call such a sequence *multiplicative*.

Example. (see loc. cit.) Assume all $m_{i,j}$ are algebra homomorphisms, one can find a so called *(strict) monoidal categorification* of A_* as follows: Let $\mathcal{C} = \mathcal{C}(A_*)$ be the category with objects by $[i], i \in \mathbb{I}$. And morphisms are defined as $\mathrm{End}_{\mathcal{C}}([0]) =: \mathbb{C}$, $\mathrm{Hom}_{\mathcal{C}}([i],[j]) = 0$ for $i \neq j$ in I and $\mathrm{End}_{\mathcal{C}}([i]) := A_i$. The tensor product is given by $[i]\otimes[j] := [i+j], i, j \in I$. The multiplicative structure on A_* yields the tensor product structure on morphisms.

Example. (Main example) Assume that I is monoid generated by a finite set Q_0. Let $\mathbb{I}$ be the free monoid in the set Q_0, we have a surjective map $\mathbb{I} \to I,\ i \mapsto |i|$ of monoids. We assume for every $d \in I$ the algebra A_d is a $\mathbb{Z}$-graded algebra and the maps $\mu_{d,e}$ respect the grading.
Then, for every $d \in I$ we have a set of idempotent elements $1_i \in A_d, i \in \mathbb{I}, |i| = d$ given by the iterated images of 1 of the multiplicative maps. We define the monoidal category $\mathcal{C} := \mathcal{C}(A_*)$ to have objects finite direct sums of $[i](n), i \in \mathbb{I}, n \in \mathbb{Z}$ and

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}([i](0),[j](n)) := \begin{cases} 1_i A_d 1_j, & \text{if } |i| = |j| = d \\ 0, & \text{else.} \end{cases}$$

determines all homomorphisms in the category $\mathcal{C}$. The monoidal structure is given by $[i](n) \otimes [j](m) := [i+j](n+m)$.

Notational convention. When we describe categories later, we will write for an element $x \in \mathrm{Hom}_{\mathcal{C}}([i](0),[j](n))$ a *homomorphism* $x\colon [i] \to [j]$ *of degree* n. If $y\colon [j] \to [k]$ is a homomorphism of degree m, we write $y \circ x\colon [i] \to [k]$ for the homomorphism of degree $n+m$ given by $y(n) \circ x \in \mathrm{Hom}_{\mathcal{C}}([i](0),[k](n+m))$.

Example. (Steinberg algebra) Back to our previous situation. Let (G, P, V, F) be an $\mathbb{I}$-graded Springer theory. Assume that there exist the horizontal product map, that implies that $\mathcal{Z}_{|i|}, i \in |\mathbb{I}|$ is a multiplicative sequence, $\mathcal{Z}_\emptyset := \mathbb{C}$ with the structure maps $\mu_{|i|,|j|}$ respecting the grading and mapping $1_i \otimes 1_j$ to 1_{i+j}, i.e. $\mu_{|i|,|j|}(1\otimes 1) = \sum_{i\in I_{|i|}, j \in I_{|j|}} 1_{i+j} =: 1_{|i|,|j|}$ is just an idempotent element. Then we define $\mathcal{C} := \mathcal{C}(\mathcal{Z})$ with objects finite direct sums of $[i](n), i \in \mathbb{I}, n \in \mathbb{Z}$ and

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}([i](0),[j](n)) := \begin{cases} \mathcal{Z}_{j,i}, & \text{if } |i| = |j| \\ 0, & \text{else.} \end{cases}$$

The tensor product is given by $[i](n)\otimes[j](m) := [i+j](n+m)$. Apart from the assumption on $\mathbb{I}$ being a free monoid in a set of generator of I, this is just a special case of the main example.

We are going to see.

* Quiver-graded Springer theory is a special case of the previous example and the main example.
* Symplectic quiver-graded Springer theory gives a special case of the main example.

4.2.1 Alternative description of $\mathcal{C}$ as category of projective graded modules

We study the situation of the main example (it also applies to the last example, i.e. a Steinberg algebra with a horizontal product).
Let d, e be in I, we set $1_{d,e} := \mu_{d,e}(1 \otimes 1)$. We see $A_d \otimes_{\mathbb{C}} A_e$ as a $\mathbb{Z}$-graded algebra via the degree r elements are $\bigoplus_{k+\ell=r}(A_d)_k \otimes_{\mathbb{C}} (A_e)_\ell$. For a $\mathbb{Z}$-graded ring R we write $R-\mathrm{mod}^{\mathbb{Z}}$ for the category of finitely generated $\mathbb{Z}$-graded left R-modules. There are the following induction and restriction functors:

$$\begin{aligned} \mathrm{Ind}_{d,e}^{d+e} \colon (A_d \otimes A_e)\,\mathrm{mod}^{\mathbb{Z}} &\to A_{d+e}\,\mathrm{mod}^{\mathbb{Z}} \\ X &\mapsto A_{d+e}1_{d,e} \otimes_{A_d \otimes A_e} X \end{aligned}$$

$$\begin{aligned} \mathrm{Res}_{d+e}^{d,e} \colon A_{d+e}\,\mathrm{mod}^{\mathbb{Z}} &\to (A_d \otimes A_e)\,\mathrm{mod}^{\mathbb{Z}} \\ Y &\mapsto 1_{d,e}Y \end{aligned}$$

Remark. Let $A-\mathrm{mod}^{\mathbb{Z}}$ be the category of finitely generated $\mathbb{Z}$-graded left A_d-modules , $d \in I$ with homomorphisms are maps of degree zero if both are graded A_d-modules and zero else. The induction functor defines a monoidal structure on $A-\mathrm{mod}^{\mathbb{Z}}$ as follows

$$M \circ N := \mathrm{Ind}_{d,e}^{d+e}(M \boxtimes N)$$

where $M \boxtimes N$ is the vector space $M \otimes_{\mathbb{C}} N$ with the obvious $A_d \otimes A_e$-module structure, it is in fact a $\mathbb{Z}$-graded module with degree r part given by $\bigoplus_{k+\ell=r} M_k \otimes_{\mathbb{C}} N_\ell$. The object E is given by $\mathbb{C} = A_0$ which is a $\mathbb{Z}$-graded algebra concentrated in degree zero.

Let $\mathcal{B}_d \subset A-\mathrm{mod}^{\mathbb{Z}}$ be the full subcategory of finitely generated projective graded A_d-modules.
Let $\mathcal{B} \subset A-\mathrm{mod}^{\mathbb{Z}}$ be the full subcategory with objects in $\mathcal{B}_d, d \in \mathbb{I}$.
Let $\mathcal{B}_{\mathbb{I}}$ be the full subcategory subcategory of finite direct sums of shifts of projective graded modules of the form

$$P_i := A_d 1_i,\ i \in \mathbb{I}, |i| = d.$$

Lemma 35. *The functor* Ind *respects projective graded modules. Then $\mathcal{B}$ has the structure of a monoidal category and $\mathcal{B}_{\mathbb{I}}$ is monoidal subcategory and it holds*

$$P_i(n) \circ P_j(m) = P_{i+j}(n+m).$$

proof: This follows from the obvious observation that $\mathrm{Ind}_{d,e}^{d+e}(A_d \otimes A_e) = A_{d+e}1_{d,e}$ is a projective module. The rest is by definition of the functors fulfilled. □

Remark. There is an obvious equivalence of monoidal categories

$$\begin{aligned}\mathcal{C}(A_*) &\to \mathcal{B}_{\mathbb{I}}\\ [i](n) &\mapsto P_i(n)\end{aligned}$$

Grothendieck rings

We assume we are in the situation from the previous section. Observe, that the induction functors on $\mathcal{B}$ induce a $|\mathbb{I}|$-graded multiplication on the abelian group

$$K_0(\mathcal{B}) = \bigoplus_{d\in I} K_0(\mathcal{B}_d).$$

It has the structure of a $\mathbb{Z}[q, q^{-1}]$-module where q operates as the graded shift $q{\cdot}M := M(1)$.

Remark. With the restriction functor one can sometimes define a coproduct as follows

$$K_0(\mathcal{B}) \to K_0(\mathcal{B}) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{B}), \quad \mathcal{B}_d \ni P \mapsto \sum_{e,f\colon e+f=d} \mathrm{Res}^d_{e,f} P$$

For example, the following has to be fullfilled

(1) $1_{d,e}A_{d+e}$ is a projective $A_d{\otimes}A_e$-module (because then the restriction of the restriction functor to projective modules is well-defined).

(2) A version of *Mackey's induction restriction theorem* holds (see for example [KL09] in the case of quiver-graded Springer theory). This is needed to see that the comultiplication is a twisted algebra homomorphism.

So a natural question is:
When is the algebra $K_0(\mathcal{B})$ part of a twisted $\mathbb{Z}[q, q^{-1}]$-Hopf algebra structure ?
We do not discuss this here further.
There is some more algebraic structure in the examples which we are not discussing like anti-involutions, nondegenerate bilinear forms and graded dimension vectors. For quiver-graded Springer theory you find definitions in [Lus91], for symplectic quiver-graded Springer theory look in [VV11].

4.3 Lusztig's perverse sheaves

Now, we assume $\mathcal{Z}$ to be the Steinberg algebra of an $\mathbb{I}$-graded Springer theory (G, P, V, F). We write $\pi_i\colon E_i = G_i \times^{P_i} F_i \to G_iF_i(\subset V_i), \overline{(g, f)} \mapsto gf$ for the collapsing map. By the equivariant decomposition theorem, we get that $(\pi_i)_*\underline{\mathbb{C}}[e_i]$ with $e_i = \dim_{\mathbb{C}} E_i$ is a direct sum of shifts of simple perverse sheaves on V_i.

The equivariant decomposition theorem takes place in $D^b_{G_i}(V_i)$ which is the equivariant derived category in the sense of Bernstein and Lunts, for both see [BL94][2]. Let $\mathcal{P}_{|i|} \subset D^b_{G_i}(V_i)$ be the full subcategory given by finite direct sums of shifts of direct summands of $(\pi_i)_*\underline{\mathbb{C}}[e_i]$, $i \in \mathbb{I}_{|i|}$.

We define a category $\mathcal{P}$ called *Lusztig's perverse sheaves* with objects in $\mathcal{P}_{|i|}, i \in \mathbb{I}$. Morphisms are given by morphisms in the equivariant derived category $D^b_{G_i}(V_i)$ if both objects are from this category, otherwise it is zero.

Let $\mathcal{P}_{\mathbb{I}} \subset \mathcal{P}$ be the monoidal category generated by finite direct sums of shifts of $L_i := (\pi_i)_*(\underline{\mathbb{C}}[e_i]), i \in \mathbb{I}$.

Lemma 36. *The following are equivalent.*

(1) There exists a monoidal structure on $\mathcal{P}$ mapping $\mathcal{P}_{|i|} \times \mathcal{P}_{|i|} \to \mathcal{P}_{|i+j|}$ which restricts to a monoidal structure on the category $\mathcal{P}_{\mathbb{I}}$ defined by $L_i(n) * L_j(m) := L_{i+j}(n+m)$.

(2) There exists a monoidal structure on $\mathcal{B}$ mapping $\mathcal{B}_{|i|} \times \mathcal{B}_{|i|} \to \mathcal{B}_{|i+j|}$ which restricts to a monoidal structure on the category $\mathcal{B}_{\mathbb{I}}$ defined by $P_i(n) * P_j(m) := P_{i+j}(n+m)$.

(3) There exists a horizontal product on $\mathcal{Z}$.

and in this case it holds

$$\mathcal{P}_{\mathbb{I}} \cong \mathcal{B}_{\mathbb{I}} \cong \mathcal{C}(\mathcal{Z}).$$

proof: By [Sau13], lemma 7, $\mathcal{P}_{|i|}$ is equivalent to $\mathcal{B}_{|i|}$ for every $i \in \mathbb{I}$ mapping L_i to P_i. A consequence of the equivalence is (1) $\Leftrightarrow$ (2).

Now, assume that horizontal products exist for $\mathcal{Z}$, then (2) holds true by the previous section. The existence of a monoidal structure on the category $\mathcal{B}_{\mathbb{I}}$ defined by $P_i(n) * P_j(m) := P_{i+j}(n+m)$ induces on homomorphisms the structure of a multiplicative sequence on $(\mathcal{Z}_{|i|}, i \in \mathbb{I})$, i.e. a horizontal product on $\mathcal{Z}$. ☐

Remark. In the case of a Steinberg algebra $\mathcal{Z}$ with horizontal product, we get $K_0(\mathcal{P}) = K_0(\mathcal{B})$ and $K_0(\mathcal{P}_{\mathbb{I}}) = K_0(\mathcal{B}_{\mathbb{I}})$ as $\mathbb{Z}[q, q^{-1}]$-algebras.

Let us look at one of the examples from before.

Example. (1) (nil Springer theory) The only indecomposable projective graded $\mathrm{NH_n}$-module is $P_n := \mathbb{C}[\mathsf{t}_n]$. To see this, it is given by $P_n = \mathrm{NH_n e}$ for $e\colon \mathbb{C}[\mathsf{t}_n] \xrightarrow{Av} \mathbb{C}[\mathsf{t}_n]^{W_n} \hookrightarrow \mathbb{C}[\mathsf{t}_n]$ where $Av(f) := \frac{1}{\#W_n} \sum_{w \in W_n} w(f)$
(it is easy to see that $\mathrm{NH_n e} \cong \mathrm{Hom}_{\mathbb{C}[\mathsf{t_n}]^{\mathrm{W_n}}}(\mathbb{C}[\mathsf{t_n}]^{\mathrm{W_n}}, \mathbb{C}[\mathsf{t_n}]) = \mathbb{C}[\mathsf{t_n}]$ as left $\mathrm{NH_n}$-module).
Since $\mathbb{C}[\mathsf{t}_n] = \bigoplus_{w \in W_n} \mathbb{C}[\mathsf{t}_n]^{W_n} \cdot b_w$ as graded module with deg $b_w = 2\ell_{W_n, S^{(n)}}(w)$

[2] Lusztig used the usual derived categories, but his constructions work also equivariantly and give the same Grothendieck group.

(b_w is a so called Schubert polynomial), it gives

$$\mathrm{NH_n} = \bigoplus_{\mathrm{w}\in \mathrm{W_n}} \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}_\mathrm{n}]^{\mathrm{W_n}}}(\mathbb{C}[\mathfrak{t}_\mathrm{n}]^{\mathrm{W_n}}, \mathbb{C}[\mathfrak{t}_\mathrm{n}](-2\ell(\mathrm{w}))) = \bigoplus_{\mathrm{w}\in \mathrm{W_n}} \mathrm{P_n}(-2\ell(\mathrm{w}))$$

That means that $\mathcal{B}$ is generated by finite direct sums of shifts of $P_n, n \in \mathbb{N}_0$. It holds $P_n \circ P_m = \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}_{n+m}]^{W_{n+m}}}(\mathbb{C}[\mathfrak{t}_{n+m}]^{W_n \times W_m}, \mathbb{C}[\mathfrak{t}_{n+m}])$. Since

$$\mathbb{C}[\mathfrak{t}_{n+m}]^{W_n \times W_m} = \bigoplus_{x \in (W_n \times W_m \backslash W_{n+m})} \mathbb{C}[\mathfrak{t}_{n+m}]^{W_{n+m}} b_x$$

with using the notation $\deg b_x =: 2n_x$ we conclude

$$P_n \circ P_m = \bigoplus_{x \in (W_n \times W_m \backslash W_{n+m})} P_{n+m}(-2n_x).$$

One can express n_x as the length of a minimal coset generator[3]. We conclude that $K_0(\mathcal{B})$ is as $\mathbb{Z}[q, q^{-1}]$-algebra not finitely generated (but if we tensor $- \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ we see it is finitely generated as an algebra by $[P_1]$).
On the other hand $\mathcal{B}_{\mathbb{I}}$ is the monoidal subcategory generated by free modules and it holds $\mathrm{NH_n} \circ \mathrm{NH_m} = \mathrm{NH_{n+m}}$. We conclude that $K_0(\mathcal{B}_{\mathbb{I}}) = (\mathbb{Z}[q, q^{-1}])[T]$ for $T = [\mathrm{NH_1}]$.
For $G = \mathbf{Gl}$, Khovanov describes in [Kho99] the category of finite dimensional modules over a n-NilCoxeter algebra, $n \in \mathbb{N}_0$ as a categorification of the Weyl algebra. Recall, that the n-NilCoxeter algebra is the quotient algebra obtained from $\mathrm{NH_n}$ when passing from equivariant to not equivariant Borel-Moore homology. I think it would be interesting to ask if one can extend this for other $\mathbb{N}_0$-graded groups and say something on the Nil Hecke algebras as well.

In the examples we will be interested in the following questions:

(Q1) Can we find explicit generators and relations for the monoidal categories $\mathcal{C}(\mathcal{Z})$?
If $\mathcal{Z}$ has no horizontal product, we would like to find a *minimal* multiplicative sequence A_* containing $\mathcal{Z}$ and describe $\mathcal{C}(A_*)$ instead.

(Q2) Can we describe $K_0(\mathcal{B}_{\mathbb{I}})$ in terms of generators and relations? In the examples we think of $K_0(\mathcal{B}_{\mathbb{I}})$ as an analog of the extension monoid, which is how we call the subalgebra of a Hall algebra generated by the simple modules without self-extensions, see for example [Wol09].

We did not manage to study the second question.

[3]this notion is also defined if $W_n \times W_m$ is not necessarily a parabolic subgroup of W_{n+m}, see remark on page 59

4.4 Example: Quiver-graded Springer theory

Let $Q = (Q_0, Q_1)$ be a finite quiver. We write $\mathbb{N}_0^{Q_0}$ for the dimension vectors. A word in dimension vectors is a finite sequence $(a_1, \dots, a_r)$, $a_i \in \mathbb{N}_0^{Q_0}$. We set

$$\mathbb{I} := \{(a_1, \dots, a_r) \mid a_j \in \mathbb{N}_0^{Q_0}, r \in \mathbb{N}_0\} \to |\mathbb{I}| := \mathbb{N}_0^{Q_0}$$
$$i := (a_1, \dots, a_r) \mapsto \sum_{j=1}^{r} a_j =: |i|$$

where the semi-group structure on $\mathbb{I}$ comes from word concatenation. Then, an $\mathbb{I}$-ind Springer theory (G, P, V, F) is given by the following data for $i = (a_1, \dots a_r)$, $|i| = (d_k)_{k \in Q_0}$

(1) $G_i = G_{|i|} := \prod_{k \in Q_0} \mathbf{Gl}_{d_k}$

(2) $P_i \subset G_i$ be at factor k the standard parabolic with diagonal block sizes (from left upper corner to right bottom corner) given by $((a_1)_k, \dots, (a_r)_k)$.

(3) $V_i = V_{|i|} = \prod_{(\alpha\colon k \to l) \in Q_1} \mathrm{Hom}_{\mathbb{C}-lin}(\mathbb{C}^{d_k}, \mathbb{C}^{d_l})$

(4) $F_i \subset V_i$ be stabilizer of a standard flag of type i (i.e. such that P_i is the stabilizer of that flag).

The definition of the representations (V_i, F_i) goes back to Reineke in [Rei03]. This is not a generalized Springer theory in the sense of [Sau13]. But the Borel case is.

4.4.1 Quiver-graded Springer theory - Borel case

The Borel case of quiver-graded Springer theory is defined by Lusztig, see for example [Lus91]. We set

$$\mathbb{I} := \{(i_1, \dots, i_r) \mid i_j \in Q_0, r \in \mathbb{N}_0\} \to |\mathbb{I}| := \mathbb{N}_0^{Q_0}$$
$$i := (i_1, \dots, i_r) \mapsto \sum_{j=1}^{r} i_j =: |i|$$

We can see this naturally as a sub-semigroup of the same named set from before, but for i in this smaller set we have

* We fix a numbering of the points of $Q_0 = \{k_1, \dots, k_t\}$. Then, the group $G_{|i|}$ is a Levi-group $L_{J_{|i|}}$ in $\mathbf{Gl}_n$, $n := ||i||$. We fix the following identification

  $${}^{J_{|i|}}S_n \to \mathbb{I}_{|i|} := \{j \in I \mid |j| = |i|\}$$

  which comes from the transitive right operation of S_n on $\mathbb{I}_{|i|}$ given by the following: See i as a function $i\colon \{1, \dots, n\} \to Q_0$ with $\sum_{j=1}^{n} i(j) = |i|$, $w \in S_n$, then $iw := i \circ w$. Then the stabilizer of every point is isomorphic to $\langle J_{|i|} \rangle$. Therefore, any choice of a point gives an isomorphism as above. We choose the element $i := (k_1, k_1, \dots, k_2, \dots, k_3, \dots) \in \mathbb{I}_{|i|}$.

* Let $\mathbb{B}_n \subset \mathbf{Gl}_n$, $n = ||i||$ be the invertible upper-triangular matrices and $B_{|i|} := G_{|i|} \cap B_n$. Then we can assume that $P_j = B_{|i|}$ for all $j \in I_{|i|}$ More precisely, if $w(j) \in {}^{J_{|i|}}S_n$ is the element corresponding to j, we have $P_j = G_{|i|} \cap {}^{w(j)}\mathbb{B}_n = G_{|i|} \cap \mathbb{B}_n = B_{|i|}$.

* Furthermore, in the above notation

$$F_j = V_{|j|} \cap {}^{w(j)}\mathcal{U}_n, \quad j \in \mathbb{I}$$

where $\mathcal{U}_n = \mathrm{Lie}(\mathbb{U}_n)$, $\mathbb{U}_n \subset \mathbb{B}_n$ is the unipotent radical. This is the same F_j defined in the quiver-graded Springer theory from before.

For a quiver without loops Varagnolo and Vasserot [Var09] and independently Rouquier in [Rou11] calculated the associated Steinberg algebra in terms of generators and relations and identified it with the quiver-Hecke algebra (or KLR-algebra) which was introduced in the simply laced case by Khovanov and Lauda in [KL09]. This is the version with loops (cp. main thm in [Sau13]).

Theorem 4.4.1. *Let Q be a quiver. Set $I := \mathbb{I}_{|i|} = \{j = (j_1, \dots, j_n) \mid j_k \in Q_0, \sum j_k = |i|\}$, $\mathbb{S} \subset S_n$ the set of positive roots. We define $h_i(s) := \#\{\alpha \in Q_1 \mid$ for $V := (V_{|i|})_\alpha,\ x_i(\alpha_s) \in \Phi_V\}$ with Φ_V is the set of T_n-weights of V. It holds for $i = (i_1, \dots, i_n) \in \mathbb{I}$ and $s_\ell = (\ell, \ell+1) \in S_n$*

$$h_i(s_\ell) = h_{i_{\ell+1}, i_\ell} := \#\{\alpha \in Q_1 \mid \alpha\colon i_{\ell+1} \to i_\ell\}$$

We consider $\bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots z_i(n)]$ as the left $\mathbb{W} := S_n$-module $\mathrm{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}_n]$ via $f \in \bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots z_i(n)]$, $w \in \mathbb{W}$ map to $w(f) \in \mathbb{C}[z_{iw^{-1}}(1), \dots z_{iw^{-1}}(n)]$. For every $i \in \mathbb{I}$, $s = (\ell, \ell+1) \in S$ we have $\alpha_s := z_i(\ell) - z_i(\ell+1)$ without mentioning the dependence on i if that is clear from the context. Then $\mathcal{Z}_{|i|}$ is the graded $\mathbb{C}$-algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

of degrees

$$\deg 1_i = 0,\ \deg z_i(t) = 2\ \deg \sigma_i((\ell, \ell+1)) = \begin{cases} 2h_{i_\ell, i_\ell} - 2 & , \text{ if } i_\ell = i_{\ell+1} \\ 2h_{i_{\ell+1}, i_\ell} & , \text{ if } i_\ell \neq i_{\ell+1} \end{cases}$$

subject to the relations

$$\begin{aligned} 1_i 1_j &= \delta_{i,j} 1_i, \\ 1_i z_i(t) 1_i &= z_i(t), \\ 1_i \sigma_i(s) 1_{is} &= \sigma_i(s) \\ z_i(t) z_i(t') &= z_i(t') z_i(t) \end{aligned}$$

$$\sigma_i(s)\sigma_{is}(s) = \begin{cases} 0 & , \text{ if } is = i, \ h_i(s) \text{ is even} \\ -2\alpha_s^{h_i(s)-1}\sigma_i(s) & , \text{ if } is = i, \ h_i(s) \text{ is odd} \\ (-1)^{h_{is}(s)}\alpha_s^{h_i(s)+h_{is}(s)} & , \text{ if } is \neq i \end{cases}$$

Let $s = s_\ell = (\ell, \ell+1)$

$$\sigma_i(s)z_{is}(t) - s(z_{is}(t))\sigma_i(s) = \begin{cases} \alpha_s^{h_i(s)}, & , \text{ if } is = i, t = \ell+1 \\ -\alpha_s^{h_i(s)}, & , \text{ if } is = i, t = \ell \\ 0 & , \text{ if } is \neq i \text{ or } is = i, t \notin \{\ell, \ell+1\} \end{cases}$$

Let $s = s_\ell = (\ell, \ell+1), t = s_{\ell+1}$

$$\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)$$

$$= \begin{cases} \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})s(\delta_t(\alpha_s^{h_{is}(s)})) - \alpha_t^{h_{is}(s)}t(\alpha_s^{h_{is}(t)})t(\delta_s(\alpha_t^{h_i(s)})) & , \text{ if } ists = i, \\ & \quad is \neq i, it \neq i \\ 0 & , \text{ else.} \end{cases}$$

We can now write down the horizontal product

$$\begin{aligned} \mu_{|i|,|j|} \colon \mathcal{Z}_{|i|} \otimes \mathcal{Z}_{|j|} &\to \mathcal{Z}_{|i+j|} \\ 1_i \otimes 1_j &\mapsto 1_{i+j} \\ z_i(k) \otimes 1_j &\mapsto z_{i+j}(k) \\ 1_i \otimes z_j(\ell) &\mapsto z_{i+j}(n+\ell) \\ \sigma_i((t,t+1)) \otimes 1_j &\mapsto \sigma_{i+j}((t,t+1)) \\ 1_i \otimes \sigma_j((r,r+1)) &\mapsto \sigma_{i+j}((n+r,n+r+1)) \end{aligned}$$

the condition that it respects the algebra product defines it in general. This explicitly defines a multiplicative sequence of algebras.

Furthermore, Rouquier defined in [Rou11] a monoidal category which is equivalent to the monoidal subcategory of the category $\mathcal{P}$ generated by the perverse sheaves L_k corresponding to the dimension vectors $\varepsilon_k, k \in Q_0$. This construction can easily be extended to our slightly different situation: Because we are allowing loops in the quiver, we have some more relations to consider, cp. the article Generalized quiver Hecke algebras [Sau13].

Remark. Lusztig studied the category $\mathcal{P}$ for a quiver without loops. He proved the following.

(1) ([Lus91], Prop. 7.2, p.390) There is geometric construction of the monoidal product $*$ on the category $\mathcal{P}$. Let $i = (i_1, \ldots, i_n) \in \mathbb{I}$, it holds

$$L_{i_1} * L_{i_2} * \cdots * L_{i_n} = (\pi_i)_* \underline{\mathbb{C}}[e_i] =: L_i.$$

(2) In particular, the inclusion $\mathcal{P}_{\mathbb{I}} \subset \mathcal{P}$ induces an equivalence of monoidal categories

$$(\mathcal{P}_{\mathbb{I}})^i \to \mathcal{P}$$

where $()^i$ is the idempotent completion functor, see for example [Rou11], section 3.3. For Q a Dynkin quiver it even holds $\mathcal{P}_{\mathbb{I}} = \mathcal{P}$.

(3) There is an ismorphism of $\mathbb{Z}[q, q^{-1}]$-algebras

$$K_0(\mathcal{P}_{\mathbb{I}}) = K_0(\mathcal{P}).$$

Furthermore, $K_0(\mathcal{P})$ also has the structure of a twisted $\mathbb{Z}[q, q^{-1}]$-Hopf algebra. The main result in [Lus91] is that after tensoring with $\mathbb{Q}(q)$, it can be identified with the negative half of the quantized enveloping algebra associated to the quiver.

Following Rouquier's constructions in [Rou11] we define the following category. Let $\mathcal{C}$ be the monoidal category generated by finite direct sums of shifts of objects $E_a = E_a(0), a \in Q_0$, we write $E_a(n)$ for the shift, $n \in \mathbb{Z}$, and arrows

$$\begin{aligned} z_a \colon E_a &\to E_a, && \text{of degree } 2 \\ \sigma_{a,b} \colon E_aE_b &\to E_bE_a, && \text{of degree } \begin{cases} 2h_{a,a} - 2, & \text{if } a = b \\ 2h_{b,a}, & \text{if } a \neq b \end{cases} \end{aligned}$$

where $h_{a,b} := \#\{\alpha \in Q_1 \mid \alpha \colon a \to b\}$ for $a, b \in Q_0$. We write E_a also for id_{E_a} and $E_aE_b := E_a \otimes E_b$. They are subject to relations

(1) $(s^2 = 1)$

$$\sigma_{ab} \circ \sigma_{ba} = \begin{cases} (-1)^{h_{b,a}}(E_bz_a - z_bE_a)^{h_{a,b}+h_{b,a}} & , \text{ if } a \neq b \\ -2(z_aE_a - E_az_a)^{h_{a,a}-1} \circ \sigma_{a,a} & , \text{ if } a = b, h_{a,a} \text{ odd} \\ 0 & , \text{ if } a = b, h_{a,a} \text{ even} \end{cases}$$

(2) (straightening rule)

$$\sigma_{ab} \circ z_aE_b - E_bz_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ (E_az_a - z_aE_a)^{h_{a,a}} & , \text{ if } a = b, \end{cases}$$

$$\sigma_{ab} \circ E_az_b - z_bE_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ -(E_az_a - z_aE_a)^{h_{a,a}} & , \text{ if } a = b, \end{cases}$$

(3) (braid relations) for $a, b, c \in Q_0$ we have the following inclusion of $\mathbb{C}$-algebras. Let

$\mathbb{C}[\alpha_s, \alpha_t]$ be the set of polynomials in α_s, α_t.

$$\begin{aligned} J_{a,b,c}\colon \mathbb{C}[\alpha_s,\alpha_t] &\to \operatorname{End}_{\mathcal{B}}(E_aE_bE_c)\\ \alpha_s &\mapsto z_aE_bE_c - E_az_bE_c\\ \alpha_t &\mapsto E_az_bE_c - E_aE_bz_c, \end{aligned}$$

we set $t(\alpha_s^h) := (\alpha_s+\alpha_t)^h =: s(\alpha_t^h) \in \mathbb{C}[\alpha_s,\alpha_t], h \in \mathbb{N}_0$. Then, the relation is

$$\begin{aligned} &\sigma_{ab}E_c \circ E_a\sigma_{cb} \circ \sigma_{ca}E_b - E_b\sigma_{ca} \circ \sigma_{cb}E_a \circ E_c\sigma_{ab}\\ &= \begin{cases} J_{a,a,a}(\delta_s(\alpha_t^{h_{a,a}}\delta_t(\alpha_s^{h_{a,a}}))) \circ \sigma_{aa}E_a \\ \quad -J_{a,a,a}(\delta_t(\alpha_s^{h_{a,a}}\delta_s(\alpha_t^{h_{a,a}}))) \circ E_a\sigma_{aa} & , \text{ if } a=b=c\\ J_{b,a,b}(\alpha_s^{h_{a,b}}s(\alpha_t^{h_{b,b}})s\delta_t(\alpha_s^{h_{b,a}}) - \alpha_t^{h_{b,a}}t(\alpha_s^{h_{b,b}})t\delta_s(\alpha_t^{h_{a,b}})) & , \text{ if } b=c, a\neq b,\\ 0 & , \text{ else.} \end{cases} \end{aligned}$$

For $i=(i_1,\ldots,i_n)\in\mathbb{I}$ we set $E_i := E_{i_1}E_{i_2}\cdots E_{i_n}$. Let $\mathcal{Z}_{|i|}$ be the Steinberg algebra for the quiver-graded Springer theory of dimension vector $|i|\in\Gamma$. Then, by construction there is an isomorphism of algebras

$$\begin{aligned} \mathcal{Z}_{|i|} &\to \bigoplus_{i,j\in\mathbb{I}_{|i|}} \operatorname{Hom}_{\mathcal{B}}(E_i,E_j)\\ 1_i &\mapsto \operatorname{id}_{E_i}\\ z_i(t) &\mapsto E_{i_1}E_{i_2}\cdots E_{i_{t-1}}z_{i_t}E_{i_{t-2}}\cdots E_{i_n}\\ \sigma_i(s) &\mapsto E_{i_1}\cdots E_{i_{\ell-1}}\sigma_{i_{\ell+1},i_\ell}E_{i_{\ell+2}}\cdots E_{i_n}, \quad , \text{if } s=(\ell,\ell+1)\in S_n \end{aligned}$$

Theorem 4.4.2. *([Rou11]) There is an equivalence of monoidal categories*

$$\begin{aligned} \mathcal{P}_{\mathbb{I}} &\to \mathcal{C}\\ L_i &\mapsto E_i \end{aligned}$$

which is on morphisms the isomorphism of algebras from above.

4.5 Example: Symplectic quiver-graded Springer theory

This construction works in general for (general) symplectic and (special) orthogonal groups (and products of them) rather analogously to the quiver-graded Springer theory. We study only the Borel-case and we make the choice to only treat the symplectic group case because the orthogonal group is not connected (nevertheless one can treat the Steinberg algebras in the situation with orthogonal groups with Varagnolo's and Vasserot's methods, see [VV11]).
To emphasize the analogy we use mostly the same notation as in the previous subsubsection. Before we start we recall some basics about the symplectic group.

The root system of the symplectic group The group $\mathbf{Sp}_{2n}$ (cp. example (3) from earlier) has the following maximal (split) torus

$$T_n := \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad t_i \in \mathbb{C}^* \}$$

Its Lie algebra is

$$\mathfrak{sp}_{2n} := \mathrm{Lie}(\mathbf{Sp}_{2n}) = \{A \in M_{2n\times 2n} \mid A^t J = -JA\} \to \{A \in M_{2n\times 2n} \mid A = A^t\} = S^2\mathbb{C}^{2n}$$
$$A \mapsto JA$$

which maps the adjoint representation on the left hand side to $B \cdot A := BAB^t$ on the right hand side. Let us determine the roots, i.e. T-weights of $\mathfrak{sp}_{2n}$. A general element of the Lie algebra is $\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix}$ with Y, Z symmetric.

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} (t_i x_{ij} t_j^{-1})_{ij} & (t_i y_{ij} t_j)_{ij} \\ (t_i^{-1} z_{ij} t_j^{-1})_{ij} & (-t_i^{-1} x_{ji} t_j)_{ij} \end{pmatrix}$$

If we denote by $\varepsilon_i \colon T \to \mathbb{C}^*$ the projection on the i-th diagonal entry $1 \leq i \leq n$, for two maps $\lambda, \mu \colon T \to \mathbb{C}^*$ we write $\lambda + \mu \colon T \to \mathbb{C}^*, t \mapsto \lambda(t)\mu(t), -\lambda \colon T \to \mathbb{C}^*, t \mapsto \lambda(t)^{-1}, 0 \colon T \to \mathbb{C}^*, t \mapsto 1$, we have found the roots

$$0, \varepsilon_i - \varepsilon_j, \ \ \varepsilon_i + \varepsilon_j, \ \ -\varepsilon_i - \varepsilon_j, \ \ 2\varepsilon_i, \ \ -2\varepsilon_i, \ \ 1 \leq i, j \leq n, i \neq j$$

with weight spaces (write $\mathfrak{g} := \mathfrak{sp}_{2n}$ and E_{kl} to be the basic matrix with 1 at position (k,l) and zero else)

$$\mathfrak{g}_0 = \mathrm{Lie}(T), \ \ \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}(E_{ij} - E_{j+n,i+n}), \ \ \mathfrak{g}_{\varepsilon_i + \varepsilon_j} = \mathbb{C}(E_{i,j+n} + E_{j,i+n}),$$

$$\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} = \mathbb{C}(E_{i+n,j} + E_{j+n,i}), \ \ \mathfrak{g}_{2\varepsilon_i} = \mathbb{C}E_{i,i+n}, \ \ \mathfrak{g}_{-2\varepsilon_i} = \mathbb{C}E_{i+n,i}$$

The root system is of type C_n, the Weyl group is defined as $W = N_{\mathbf{Sp}_{2n}}(T)/T \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, we fix the following set of elements in $N_{\mathbf{Sp}_{2n}}(T)$ whose left cosets generate W: For $\tau \in S_n$ we write $\tau := \begin{pmatrix} P_\tau & 0 \\ 0 & P_\tau \end{pmatrix}$ with $P_\tau = (e_{\tau(1)}, \ldots, e_{\tau(n)}) \in \mathbf{Gl}_n$,

for $\sigma_i = (0, \ldots, 0, \overline{1}, 0, \ldots, 0) \in (\mathbb{Z}/2\mathbb{Z})^n$ we write $\sigma_i = \begin{pmatrix} E_n - E_{ii} & E_{ii} \\ -E_{ii} & E_n - E_{ii} \end{pmatrix}$, $1 \leq i \leq n$.

The positive roots are $0, \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j$ with $i < j$ and $2\varepsilon_i$, the simple roots are $\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1, 2\varepsilon_n$. Let $S \subset W$ be the set of reflections defined at the simple roots, it gives a generating set of W. As usually we identify $S = \{(1,2), \ldots, (n-1,n), \sigma_n\} \subset \mathbf{Sp}_{2n}$. The Borel subgroup whose Lie algebra equals the sum of the positive weights is our standard Borel subgroup from example (3).

Definition 14. A symmetric quiver (Q, σ) consists of a finite quiver Q and two maps $\sigma\colon Q_0 \to Q_0, \sigma\colon Q_1 \to Q_1$ with $\sigma^2 = \mathrm{id}$ such that $\sigma(k \to \ell)$ is an arrow $\sigma(k) \to \sigma(l)$.
We call the vertices $Q_0^\sigma = \{k \in Q_0 \mid \sigma(k) = k\}$ black vertices and we set $Q_0 \setminus Q_0^\sigma = Q_0' \sqcup \sigma(Q_0')$ for one fixed subset $Q_0' \subset Q_0$ and call $Q_0 \setminus Q_0^\sigma$ the white vertices.

For $a, b \in Q_0$ we write

$$\begin{aligned} h_{a,b} &:= \#\{\alpha \in Q_1 \mid \alpha\colon a \to b,\ \sigma(\alpha) \neq \alpha\} \\ h_{a,\sigma} &:= \#\{\alpha \in Q_1 \mid \alpha\colon a \to \sigma(a),\ \sigma(\alpha) = \alpha\} \end{aligned}$$

and we will always assume that the symmetric quiver (Q_0, σ) fulfills

$$h_{a,\sigma} = h_{a,\sigma(a)}, \quad \forall a \in Q_0.$$

Furthermore, we define

$$\Gamma := \{a \in \mathbb{N}_0^{Q_0} \mid\ a_k = a_{\sigma(k)},\ \text{for } k \in Q_0^\sigma\ a_k \in 2\mathbb{N}_0\}.$$

Observe, that Γ is a sub-semigroup of $(\mathbb{N}_0^{Q_0})^\sigma$. For a sequence $i = (i_1, \dots, i_r),\ i_j \in Q_0$ we define $|i| := \sum_{j=1}^r (i_j + \sigma(i_j))$. We define

$$\begin{aligned} \mathbb{I} := \{(i_1, \dots, i_r) \mid i_j \in Q_0\} &\to |\mathbb{I}| := \Gamma \\ i := (i_1, \dots, i_r) &\mapsto |i| \end{aligned}$$

Then we have an $\mathbb{I}$-graded Springer theory as follows

(1) Let $|i| = \sum_{k \in Q_0} a_k \cdot k \in \Gamma$, we define

$$G_{|i|} := \prod_{k \in Q_0^\sigma} \mathbf{Sp}_{2\frac{a_k}{2}} \times \prod_{k \in Q_0'} \mathbf{Gl}_{a_k}.$$

When we have fixed a numbering of the vertices Q_0 we can define an inclusion $G_{|i|} \to \mathbb{G}_n := \mathbf{Sp}_{2n}, n = \frac{||i||}{2}$ via

$$\Phi\colon G_{|i|} \to \mathbf{Sp}_{2n}, \quad (\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix})_{1 \le j \le r}, (a_i)_{1 \le i \le l} \mapsto$$

$$\begin{pmatrix}
a_1 &&&&&&&&&&& \\
& \ddots &&&&&&&&&& \\
&& a_l &&&&&&&&& \\
&&& A_1 &&&&&& B_1 && \\
&&&& \ddots &&&&&& \ddots & \\
&&&&& A_r &&&&&& B_r \\
&&&&&& (a_1^t)^{-1} &&&&& \\
&&&&&&& \ddots &&&& \\
&&&&&&&& (a_l^t)^{-1} &&& \\
&&& C_1 &&&&&& D_1 && \\
&&&& \ddots &&&&&& \ddots & \\
&&&&& C_r &&&&&& D_r
\end{pmatrix}$$

(which is for $r > 1$ not a standard Levi subgroup.)

(2) We write $Q_1 \setminus Q_1^\sigma = Q_1' \sqcup \sigma(Q_1')$ for one (fixed) subset $Q_1' \subset Q_1$.

$$V_{|i|} := \bigoplus_{(\alpha\colon k\to\sigma(k))\in Q_1^\sigma} S^2\mathbb{C}^{a_k} \oplus \bigoplus_{(\alpha\colon k\to\ell)\in Q_1'} M_{a_\ell\times a_k}(\mathbb{C})$$

where $S^2\mathbb{C}^a := \{A \in M_{a\times a}(\mathbb{C}) \mid A = {}^tA\}$. This is (roughly) Derksen and Weyman's representation space (see [DW02]). For $k \in Q_0^\sigma \cup Q_0'$ we write G_k for the corresponding factor of $G_{|i|}$ and $G_{\sigma(k)} := G_k$. On each direct summand the operation of $G_{|i|}$ is given by

1. For $\alpha\colon k \to \sigma(k) \in Q_1^\sigma$ it is $gvg^t,\ \ v \in S^2\mathbb{C}^{a_k}, g \in G_k$.
2. For $\alpha\colon k \to \ell \in Q_1'$ it is $g_\ell^{-1}vg_k,\ \ v \in M_{a_\ell\times a_k}(\mathbb{C}), g_\ell \in G_\ell, g_k \in G_k$.

The assumption $h_{a,\sigma} = h_{a,\sigma(a)}$ ensures that we have for every type of arrow in Q_0 an associated indecomposable $G = (G_{|i|})$-direct summand of $\mathfrak{sp}$ which we used to define the representation space above. To understand this remark look at the schematic picture below.

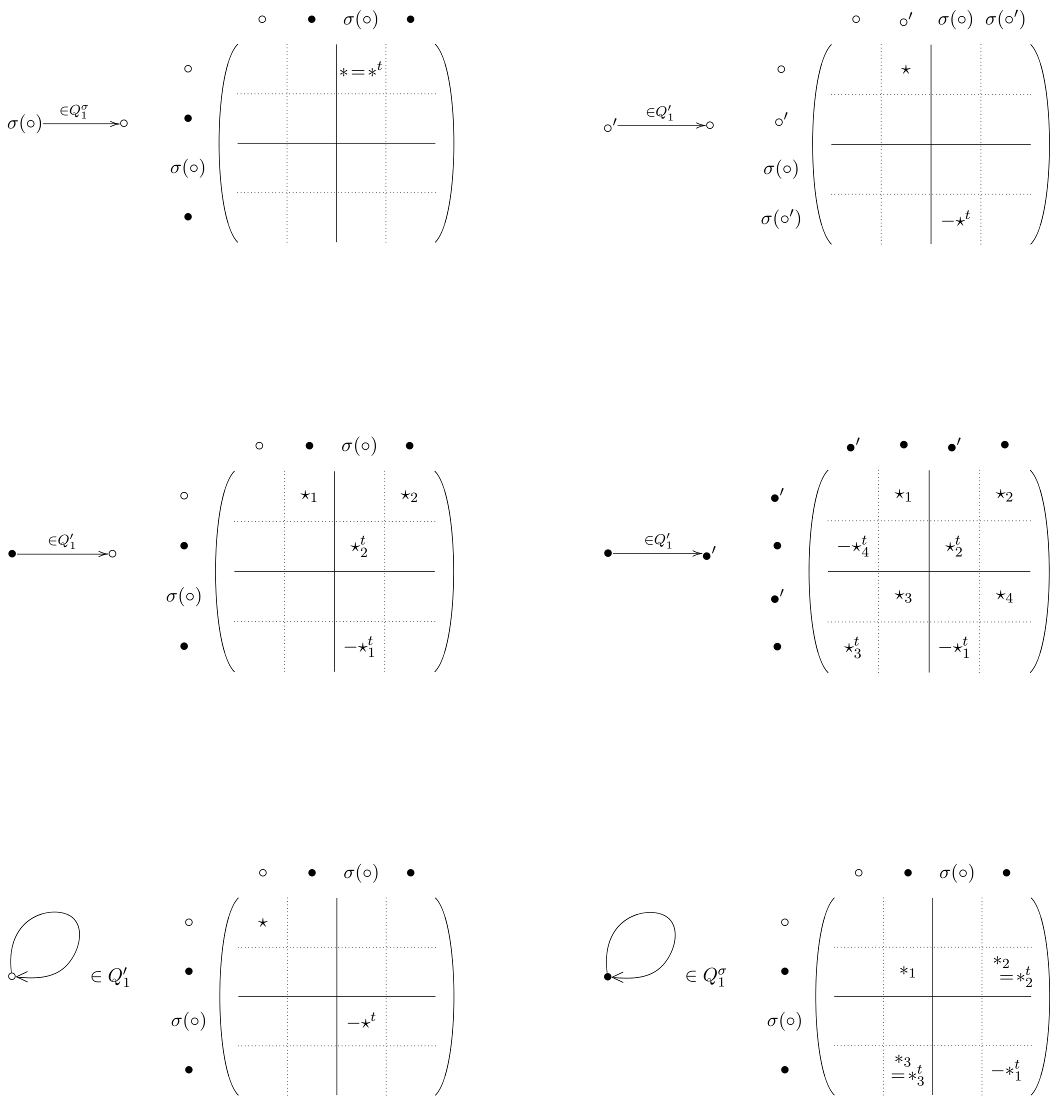
(3) Let $\mathbb{W}_n \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ be the Weyl group of $\mathbb{G}_n$ with respect to the diagonal torus. The embedding gives an inclusion of the Weyl group $W_{|i|}$ of $G_{|i|}$ into $\mathbb{W}_n$. We fix a bijection

$$W_{|i|} \setminus \mathbb{W}_n \to \mathbb{I}_{|i|} := \{j \in \mathbb{I} \mid |j| = |i|\}$$

Using the transitive right operation on $\mathbb{I}_{|i|}$ defined as follows:
See i as a function $i\colon \{1,\dots n\} \sqcup \{1^*,\dots,n^*\} \to Q_0$ with $\sum_{j=1}^n (i(j) + i(j^*)) = |i|$ with the property $i(j) = v \Leftrightarrow i(j^*) = \sigma(v)$. Then the operation of $w \in S_n$ is given by $iw := i \circ (w \sqcup w)$ and the operation of $(\mathbb{Z}/2\mathbb{Z})^n$ is given by swapping k and k^*, $1 \le k \le n$. Then the stabilizer of every point is isomorphic to $W_{|i|}$. We choose the point which is given by the numbering of $Q_0 := \{k_1, k_2, \dots\}$ which is of the form $i := (k_1, k_1, \dots, k_2, \dots) \in \mathbb{I}_{|i|}$.
Let $\mathbb{B}_n \subset \mathbb{G}_n$ the upper-triangular standard Borel in the symplectic group, $B_{|i|} := G_{|i|} \cap \mathbb{B}_n$. We will choose the unique representatives $x_i \in \mathbb{W}_n, i \in \mathbb{I}_{|i|}$ of the right cosets $W_{|i|} \setminus \mathbb{W}_n$ which satisfy $G_{|i|} \cap {}^{x_i}\mathbb{B}_n = B_{|i|}$.
We set $B_i := B_{|i|}$ as our parabolic subgroup.

(4) $F_i := V_{|i|} \cap {}^{x_i}\mathcal{U}_n$ where $\mathcal{U}_n = \mathrm{Lie}(\mathbb{U}_n), \mathbb{U}_n \subset \mathbb{B}_n$ is the unipotent radical. There is a different description of F_i in terms of elements of V_i which stabilize a complete isotropic flag (given by x_i applied to the standard flag).

Schematic pictures of the $G_{|i|}$-subrepresentations of $\mathfrak{sp}_{2n}$ associated to the arrows



Remark. Using the examples of nil Springer theory and classical Springer theory we can easily see: If (Q, σ) is such that Q has at most one loop at each vertex and no arrows between different vertices we can write down the horizontal product. Alternatively, a closer examination of the equations (1a-b), (2a-c) for the later defined elements $\tau_i(e_r)$ show you that they are in $\mathcal{Z}_{|i|}$ if and only if Q is of the described form. This is precisely the obstruction for the horizontal product being defined for $\mathcal{Z}$.

Remark. This remark is why I think there is no horizontal product. For example at the case that (Q, σ) consists of two σ-invariant loops at a black vertex i. Then, $\mathcal{Z}_{2n\cdot i}$ is generated by $z_k := x_k \cdot, 1 \leq k \leq n$, $\alpha_s^2 \delta_s, s \in S := \{(r, r+1), e_n\}$. This equals the algebra generated by $z_k, 1 \leq k \leq n, \alpha_s s, s \in S$. Now since $\mathcal{Z}_{2ni} \subset \mathrm{NH_n}$, we want a horizontal product which comes from a restriction of the horizontal product of the nil Hecke algebra. This means $(e_n, 1)$ must map to $(2x_n)e_n \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots x_{n+m}])$, but this element is not

contained in $\mathcal{Z}_{2(n+m)i}$.
I do not think that there exists a horizontal product on $\mathcal{Z}$ which is not the restriction of the one from the Nil Hecke algebra but of course a proof of this is difficult.

But as a compromise for readability I will from now on assume that the quiver Q does not contain any loops. This reduces the number of case distinctions and the length of the equations.

4.5.1 The Steinberg algebra and its horizontal product.

Theorem 4.5.1. *Let (Q,σ) be a symmetric quiver. Set $I := \mathbb{I}_{|i|} = \{j = (j_1,\dots,j_n) \mid j_k \in Q_0, \sum j_k + \sigma(j_k) = |i|\}$, $\mathbb{S} \subset S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ the set of positive roots.*

(1) (explicitly)
For $i \in I$ we set $\mathcal{E}_i := \mathbb{C}[x_i(1),\dots,x_i(n)]$. We consider $\bigoplus_{i\in I}\mathbb{C}[x_i(1),\dots x_i(n)]$ as the left $\mathbb{W} := S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-module $\operatorname{Ind}_W^{\mathbb{W}}\mathbb{C}[\mathfrak{t}_n]$ via $f \in \mathbb{C}[x_i(1),\dots x_i(n)], w \in \mathbb{W}$ maps to $w(f) \in \mathbb{C}[x_{iw^{-1}}(1),\dots x_{iw^{-1}}(n)]$. In particular, we write $\alpha_s \in \mathcal{E}_i$ for the polynomial corresponding to the simple reflection $s \in \mathbb{S}$ and $w(\alpha_s) =\in \mathcal{E}_{iw^{-1}}$, $w \in \mathbb{W}$ without mentioning that it depends on $i \in I$ when this is clear from the context.
Then $\mathcal{Z}_{|i|} \subset \operatorname{End}_{\mathbb{C}-\mathrm{lin}}(\bigoplus_{i\in I}\mathcal{E}_i)$ is the $\mathbb{C}$-subalgebra generated by

$$1_i, z_i(t), \sigma_i(s), \quad i \in I,\ 1 \le t \le n,\ s \in \mathbb{S}$$

defined by:
Let $k \in I$, $f \in \mathcal{E}_k$. It holds

$$1_i(f) := \begin{cases} f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$z_i(t)(f) := \begin{cases} x_i(t)f, & \text{if } i = k, \\ 0, & \text{else.} \end{cases}$$

$$\sigma_i(s)(f) := \begin{cases} \frac{s(f)-f}{\alpha_s}, & \text{if } i = is = k, \\ \alpha_s^{h_i(s)} s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

where $h_i(s) = \#\{\alpha \in Q_1^\sigma \sqcup Q_1' \mid \text{ for } V := (V_{|i|})_\alpha,\ \ x_i(\alpha_s) \in \Phi_V\}$ with Φ_V is the set of T-weights of V. It holds for $i = (i_1,\dots,i_n) \in \mathbb{I}$ and $s_\ell = (\ell,\ell+1) \in S_n, e_n = (0,\dots,0,1) \in (\mathbb{Z}/2\mathbb{Z})^n$

$$h_i(s_\ell) = h_{i_{\ell+1},i_\ell}, \quad h_i(e_n) = h_{\sigma(i_n),\sigma}$$

Observe, that there is a natural grading on $\mathcal{Z}_{|i|}$ by

$$\deg 1_i = 0, \deg z_i(n) = 2, \deg \sigma_i(s) = \begin{cases} -2 & , \text{ if } is = i \\ 2h_i(s) & , \text{ if } is \neq i \end{cases}$$

(2) *(in terms of generators and relations)*
Let W be the Weyl group of $(G_{|i|}, T_{|i|} = T_n)$. We consider $\bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots z_i(n)]$ as the left $\mathbb{W} := S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$-module $\operatorname{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}_n]$ defined as before. Then $\mathcal{Z}_{|i|}$ is the $\mathbb{Z}$-graded $\mathbb{C}$-algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

of the degree as in (1) and relations

$$\begin{aligned} 1_i 1_j &= \delta_{i,j} 1_i, \\ 1_i z_i(t) 1_i &= z_i(t), \\ 1_i \sigma_i(s) 1_{is} &= \sigma_i(s), \\ z_i(t) z_i(t') &= z_i(t') z_i(t), \\ \sigma_i(s)\sigma_{is}(s) &= \begin{cases} 0 & , \text{ if } is = i, \\ (-1)^{h_{is}(s)} \alpha_s^{h_i(s)+h_{is}(s)} & , \text{ if } is \neq i \end{cases} \end{aligned}$$

If $is \neq i$: $\sigma_i(s) z_{is}(t) - s(z_{is}(t))\sigma_i(s) = 0$ *for all t*

Let $s = s_\ell = (\ell, \ell+1), is = i$

$$\begin{aligned} \sigma_i(s) z_i(\ell) - z_i(\ell+1)\sigma_i(s) &= -1_i, \\ \sigma_i(s) z_i(\ell+1) - z_i(\ell)\sigma_i(s) &= 1_i, \\ \sigma_i(s) z_i(t) - z_i(t)\sigma_i(s) &= 0, \quad \text{if } t \notin \{\ell, \ell+1\} \end{aligned}$$

Let $s = e_n, is = i$

$$\begin{aligned} \sigma_i(s) z_i(n) + z_i(n)\sigma_i(s) &= -1_i, \\ \sigma_i(s) z_i(t) - z_i(t)\sigma_i(s) &= 0, \quad \text{if } t \neq n \end{aligned}$$

For $s, t \in \mathbb{S}$: $\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$ whenever $st = ts$

Let $s = s_\ell = (\ell, \ell+1), t = s_{\ell+1}$

$$\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)$$
$$= \begin{cases} \alpha_s^{h_i(s)} s(\delta_t(\alpha_s^{h_{is}(s)})) - \alpha_t^{h_{is}(s)} t(\delta_s(\alpha_t^{h_i(s)})), & \text{if } ists = i, is \neq i, it \neq i \\ 0 & , \text{ else.} \end{cases}$$

Let $s=(n-1,n), t=e_n$

$$\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)-\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)$$
$$=\begin{cases} P_t\sigma_i(t), & \textit{if } ists=i, is\neq i, it\neq i\\ P_s\sigma_i(s), & \textit{if } itst=i, it\neq i, is\neq i\\ R_t\sigma_i(t)+R_e, & \textit{if } it=i=ists, is\neq i\\ 0, & \textit{else.}\end{cases}$$

where

$$\begin{aligned} P_t &= \alpha_s^{h_i(s)}s\delta_t(\alpha_s^{h_{ist}(s)})-t(\alpha_s^{h_{it}(s)})ts\delta_t(\alpha_s^{h_{itst}(s)})\\ P_s &= \alpha_t^{h_i(t)}t\delta_s(\alpha_t^{h_{its}(t)})-s(\alpha_t^{h_{is}(t)})st\delta_s(\alpha_t^{h_{ists}(t)})\\ R_t &= t(\alpha_s^{h_{it}(s)})ts\delta_t(\alpha_s^{h_{itst}(s)})-\alpha_s^{h_i(s)}s\delta_t(\alpha_s^{h_{ist}(s)})\\ R_e &= \delta_t(\alpha_s^{h_{it}(s)})s\delta_t(\alpha_s^{h_{itst}(s)})\end{aligned}$$

Remark. Let (Q,σ) be a symmetric quiver and (Q',σ') be another symmetric quiver, such that Q' is a subquiver of Q, $Q_0=Q_0'$ and $\sigma|_{Q_1'}=\sigma'|_{Q_1'}$. Then, the explicit description of the Steinberg algebras shows that $\mathcal{Z}_{|i|}^{(Q,\sigma)}\subset\mathcal{Z}_{|i|}^{(Q',\sigma')}$ is a subalgebra. In particular, if we set $Nil(Q,\sigma):=(Q':=(Q_0,\emptyset),\sigma|_{Q_0})$, we get a symmetric subquiver and $\mathcal{Z}:=\mathcal{Z}^{(Q,\sigma)}\subset\mathcal{Z}^{Nil(Q,\sigma)}=:\mathcal{Z}^{Nil}$. Since $\mathcal{Z}^{Nil}$ has a horizontal product, it is natural to look at the restriction to $\mathcal{Z}$, see below.

The restriction to the Steinberg algebras is the map

$$\begin{aligned} \mu_{|i|,|j|}\colon \mathcal{Z}_{|i|}\times\mathcal{Z}_{|j|} &\to \mathcal{Z}_{|i+j|}^{Nil}\subset \mathrm{End}_{\mathbb{C}-\mathrm{lin}}(\bigoplus_{k\in\mathbb{I}_{|i+j|}}\mathcal{E}_k)\\ (1_i,1_j)&\mapsto 1_{i+j}\\ (z_i(k),1_j)&\mapsto z_{i+j}(k)\\ (1_i,z_j(\ell))&\mapsto z_{i+j}(n+\ell)\\ (\sigma_i((t,t+1)),1_j)&\mapsto\sigma_{i+j}((t,t+1))\\ (1_i,\sigma_j((r,r+1)))&\mapsto\sigma_{i+j}((n+r,n+r+1))\\ (1_i,\sigma_j(e_m))&\mapsto\sigma_{i+j}(e_{n+m})\\ (\sigma_i(e_n),1_j)&\mapsto\sigma_i(e_n)\otimes 1_j \quad \notin\mathcal{Z}_{|i+j|}\ (!!)\end{aligned}$$

We write $\tau_{i+j}(e_n):=\sigma_i(e_n)\otimes 1_j$, for j the empty word $j=\emptyset\in\mathbb{I}$, it holds $\tau_i(e_n)=\sigma_i(e_n)$. It is not necessarily contained in $\mathcal{Z}_{|i+j|}$. Let $f\in\mathcal{E}_k$ and recall $(i+j)e_n=(i_1,\ldots,i_{n-1},\sigma(i_n),j_1,\ldots,j_m)$

$$\tau_{i+j}(e_n)(f):=\begin{cases}\frac{e_n(f)-f}{2x_i(n)}, & \text{if } (i+j)e_n=(i+j)=k,\\ (2x_i(n))^{h_{\sigma(i_n),\sigma}}e_n(f) & \text{if } (i+j)\neq(i+j)e_n=k,\\ 0, & \text{else.}\end{cases}$$

To define a multiplicative sequence we can assume wlog. $m=1$, i.e. $j=i_{n+1}\in Q_0$, we set now $i := (i_1, \ldots, i_n, i_{n+1})$. Then we have to distinguish five cases, we will use the shortage $s = (n, n+1), t = e_{n+1}$

$$\alpha_s := (z_i(n) - z_i(n+1)), t(\alpha_s) := (z_i(n) + z_i(n+1)), \alpha_t := 2z_i(n+1), s(\alpha_t) = 2z_i(n)$$

and we leave out the dependence of i if it is clear which is meant. In each case there exists

$$\begin{aligned}
&\alpha_s^{k(i_n,i_{n+1})} t(\alpha_s)^{\ell(i_n,i_{n+1})} \alpha_t^{m(i_n,i_{n+1})} \tau_i(e_n) \\
&\quad = P(i_n, i_{n+1})\ \sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) + Q(i_n, i_{n+1})\ \sigma_i(s)\sigma_{is}(t) \\
&\quad + R(i_n, i_{n+1})\ \sigma_{is}(t)\sigma_{ist}(s) + S(i_n, i_{n+1})\sigma_i(t) + T(i_n, i_{n+1})
\end{aligned}$$

with $k(i_n, i_{n+1}), \ell(i_n, i_{n+1}) \in \mathbb{N}_0$ and
$P(i_n, i_{n+1}), Q(i_n, i_{n+1}), R(i_n, i_{n+1}),\ S(i_n, i_{n+1}),\ T(i_n, i_{n+1}) \in \mathbb{C}[z_{i+j}(n), z_{i+j}(n+1)] = \mathbb{C}[\alpha_s, \alpha_t]$ are homogeneous polynomials. We choose $k(i_n, i_{n+1}), \ell(i_n, i_{n+1}), m(i_n, i_{n+1}) \in \mathbb{N}_0$ minimal such that such an equation is fulfilled, then the polynomials $X(i_n, i_{n+1}), X \in \{P, Q, R, S, T\}$ are unique and if we see α_s, α_t as variables, then they only depend on the two vertices i_n, i_{n+1}. This can be explicitly seen as follows.

(1a) $i_n = \sigma(i_n), i_n = i_{n+1}$. It holds $f \mapsto \tau_i(e_n)(f) = \frac{e_n(f) - f}{2x_i(n)} =: \delta_{e_n}(f)$. It holds

$$\begin{aligned}
\tau_i(e_n) &= s \circ \delta_t \circ s = (\alpha_s \sigma_i(s) + 1_i)\sigma_i(t)(\alpha_s\sigma_i(s) + 1_i) \in \mathcal{Z}_{|i|} \\
&= \alpha_s t(\alpha_s)\sigma_i(s)\sigma_i(t)\sigma_i(s) + \alpha_s\sigma_i(s)\sigma_i(t) + t(\alpha_s)\sigma_i(t)\sigma_i(s) + \sigma_i(s) + 1_i
\end{aligned}$$

(This case is much easier due to our assumption that Q has no loops.)

(1b) $i_n = \sigma(i_n) \neq i_{n+1}$. It holds

$$\begin{aligned}
\tau_i(e_n) &= s\ \sigma_i(t)\ s \\
&= \alpha_s^{-h_{i_{n+1},i_n}} \sigma_i(s)\ \sigma_i(t)\ \alpha_s^{-h_{i_n,i_{n+1}}} \sigma_{is}(s) \\
&= \alpha_s^{-h_{i_{n+1},i_n}} t(\alpha_s)^{-h_{i_n,i_{n+1}}} \sigma_i(s)\sigma_i(t)\sigma_{is}(s) \\
&\quad + \alpha_s^{-1} t(\alpha_s)^{-h_{i_n,i_{n+1}}} \sum_{r+u=h_{i_n,i_{n+1}}} (-1)^r t(\alpha_s)^r \alpha_s^u
\end{aligned}$$

(2a) $i_n \neq \sigma(i_n), i_n = i_{n+1}$. Using $e_n = sts$ we have

$$\begin{aligned}
\tau_i(e_n) &= s(\alpha_t)^{h_{\sigma(i_n),\sigma}} (\alpha_s\sigma_i(s) + 1_i)\ \alpha_t^{-h_{\sigma(i_n),\sigma}} \sigma_i(t)\ \alpha_s^{-h_{i_n,\sigma}} \sigma_{it}(s) \\
&= t(\alpha_s)^{-h_{i_n,\sigma}} \alpha_s \sigma_i(s)\sigma_i(t)\sigma_{it}(s) + t(\alpha_s)^{-h_{i_n,\sigma}} \alpha_t^{-h_{\sigma(i_n),\sigma}} \\
&\quad \cdot [s(\alpha_t^{h_{\sigma(i_n),\sigma}}) - 2 \sum_{k=0}^{h_{\sigma(i_n),\sigma}-1} s(\alpha_t)^{h_{\sigma(i_n),\sigma}-1-k} \alpha_t^k] \sigma_i(t)\sigma_{it}(s)]
\end{aligned}$$

(2b) $i_n \neq \sigma(i_n), \sigma(i_n) = i_{n+1}$

$$\begin{aligned}\tau_i(e_n) &= \alpha_s^{-h_{i_{n+1},i_n}} \sigma_i(s)\sigma_{is}(t)(\alpha_s\sigma_{ist}(s) + 1_{ist}) \\ &= \alpha_s^{-h_{\sigma(i_n),\sigma}} t(\alpha_s)\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) + \alpha_s^{-h_{\sigma(i_n),\sigma}} \sigma_i(s)\sigma_{is}(t)\end{aligned}$$

(2c) $i_n \neq \sigma(i_n), i_n \neq i_{n+1}, \sigma(i_n) \neq i_{n+1}$

$$\tau_i(e_n) = \alpha_s^{-h_{i_{n+1},i_n}} t(\alpha_s)^{-h_{i_n,i_{n+1}}} \sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)$$

Let $\mathcal{Z}'_{|i|}$ be the the subalgebra of $\mathrm{End}_{\mathbb{C}-\mathrm{lin}}(\bigoplus_{i\in\mathbb{I}_{|i|}} \mathcal{E}_i)$ generated by $\mathcal{Z}_{|i|}$ and $\tau_i(e_r), i \in \mathbb{I}_{|i|}, 1 \leq r \leq n$, where n is the length of the sequence i (and $\tau_i(e_n) = \sigma_i(e_n)$). We can see this as an iterative extension by the unique solutions $\tau_i(e_r)$ of equations of the form $(*)_{i,r}$

$$\begin{aligned}&\alpha_{(r,r+1)}^{\ell(i_r,i_{r+1})} e_{r+1}(\alpha_{(r,r+1)}^{k(i_r,i_{r+1})})(2z_i(r+1))^{m(i_n,i_{n+1})}\tau_i(e_r) \\ &\quad = P(i_r, i_{r+1})\ \sigma_i(s)\tau_{is}(e_{r+1})\sigma_{ist}(s) + Q(i_r, i_{r+1})\ \sigma_i(s)\sigma_{is}(t) \\ &\quad + R(i_r, i_{r+1})\ \sigma_{is}(t)\sigma_{ist}(s) + S(i_r, i_{r+1})\sigma_i(t) + T(i_r, i_{r+1})\end{aligned}$$

where you replace α_s by $\alpha_{(r,r+1)}$, α_t by $2z_i(r+1)$ in the polynomials $X(i_r, i_{r+1})$. The right hand side is a homogeneous element in the algebra generated by $\mathcal{Z}_{|i|}$ and by the previously discussed map $(\mathcal{Z}'_{|i|})_{i\in|\mathbb{I}|}$ is a multiplicative sequence. We write $\mathcal{Z}' := \bigoplus \mathcal{Z}'_{|i|}$.

Proposition 8. *$\mathcal{Z}'_{|i|}$ is the $\mathbb{C}$-algebra generated by the symbols*

$$1_i, z_i(t), \sigma_i(s_\ell), \tau_i(e_r), \quad i \in \mathbb{I}_{|i|},\ t, r \in \{1, \dots, n\}, \ell \in \{1, \dots, n-1\}$$

subject to the relations

(1) which they fullfill in $\mathcal{Z}_{|i|}$ from theorem 4.5.1 (setting $\sigma_i(e_n) := \tau_i(e_n)$),

(2) all relations which the $\sigma_{i_1,\dots,i_r}(e_r)$ fullfill in $\mathcal{Z}_{|(i_1,\dots i_r)|}$ hold after applying $-\otimes 1_{i_{r+1},\dots,i_n}$ for the $\tau_i(e_r)$.

(3) $\tau_i(e_r)$ commutes with $1_{i_1,\dots,i_r} \otimes x$ where x is a generator of $\mathcal{Z}_{|(i_{r+1},\dots,i_n)|}$ given in theorem 4.5.1,

(4) the relations $(*)_{i,r}$ for each $i = (i_1, \dots, i_n) \in \mathbb{I}_{|i|}, r \in \{1, \dots, n-1\}$ hold.

proof: Let $A_{|i|}$ be the algebra with generators and relations as in the proposition. There is a natural surjective graded $\mathbb{C}$-algebra homomorphism $\phi_{|i|} \colon A_{|i|} \to \mathcal{Z}'_{|i|}$ mapping elements with the same names to each other, we claim that this is an isomorphism. Let $\Delta := \prod_{s\in\mathbb{S},w\in\mathbb{W}} w(\alpha_s)$ where $\mathbb{S} = \{(r, r+1), e_n, 1 \leq r \leq n-1\}$, $\alpha_{(r,r+1)} = \sum_{i\in\mathbb{I}_{|i|}} z_i(r) - z_i(r+1), \alpha_{e_n} = \sum_{i\in\mathbb{I}_{|i|}} 2z_i(n), \mathbb{W} = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. This is an element of the center of $A_{|i|}$ because it is $\mathbb{W}$-invariant.

Let $A \in \{A_{|i|}, \mathcal{Z}'_{|i|}\}$. Since in case (1a) the element $\tau_i(e_r) \in \mathcal{Z}$ we will replace these by the righthand side of the equation and exclude them if we mention elements $\tau_i(e_r)$. Let $\leq$

be the Bruhat order on $\mathbb{W}$ and $w \in \mathbb{W}$, we write $A_{\leq w}$ for the x in A such that there exists $N \in \mathbb{N}_0$ such that

$$\Delta^N x \in \mathcal{Z}_{\leq w} := \bigoplus_{v \leq w} \mathcal{Z}_v$$

where $\mathcal{Z}_w$ consists of all sums of elements of the form $1_j \ p \ \sigma(w) \ 1_i, i, j \in \mathbb{I}_{|i|}$ with $p = \sum_i p_i, \ p_i \in \mathbb{C}[z_i(1), \ldots, z_i(n)]$ and $\sigma(w) := \sum_i \sigma_i(w)$ where $\sigma_i(w)$ is a product of $\sigma_i(s_1)\sigma_{is_1}(s_2)\cdots\sigma_{is_1\cdots s_{k-1}}(s_k)$ with $s_\ell \in \{((r,r+1)) \mid 1 \leq r \leq n-1\} \cup \{e_n\}$. It holds $A_{\leq w}$ is a two-sided ideal. We define A_w to be the $\mathbb{C}$-span of $x = p \ \rho_i(t_1)\rho_{it_1}(t_2)\cdots\rho_{it_1\cdots t_{k-1}}(t_k), \rho \in \{\sigma, \tau\}$ with p homogeneous monic polynomial (monic for a fixed total monomial odering) in $\bigoplus_{i \in \mathbb{I}_{|i|}} \mathbb{C}[z_i(1), \ldots, z_i(n)]$ such that there exist $n \in \mathbb{N}_0$ with $y := \Delta^N x \in \mathcal{Z}_{\leq w}, y_w \neq 0$, we call these spanning elements w-**monomials** [4]. It follows that $A_{\leq w} = \bigoplus_{v \leq w} A_v = \bigoplus_{d \in \mathbb{Z}, v \leq w} A_v \cap A_d$ as vector spaces because w-monomials are homogeneous. We say two elements $m, m' \in A_{\leq w}$ are w-**equivalent** if there is an element in $a \in A_{<w}$ such that $m = m' + a$.
The relations $(*)_{i,r}$ are equivalent to relations $\Delta^m \tau_i(e_r) = P\sigma_i(s)\tau_{is}(e_{r+1})\sigma_{ise_{r+1}}(s)) + R$ with $s = (r, r+1)$, $P \in \bigoplus_i \mathbb{C}[z_i(1), \ldots, z_i(n)], R \in A_{<e_r}$ with Δ does not divide P, R.
We say a presentation in the generators for $x \in A$ is **Δ-reduced** (or shortly x is Δ-reduced) if x is a sum of $p\rho_i(t_1)\rho_{it_1}(t_2)\cdots\rho_{it_1\cdots t_{k-1}}(t_k)$ with $p \in \bigoplus_i \mathbb{C}[z_i(1), \ldots, z_i(n)], \rho \in \{\sigma, \tau\}$ such that the following is fullfilled: If Δ^N is a divisor of p and Δ^{N+1} is not, then for the every $\tau_i(e_r)$ in the product $\rho_i(t_1)\cdots\rho_{it_1\cdots t_{k-1}}(t_k)$ it holds $\Delta^N \tau_i(e_r) \notin \mathcal{Z}(r)$ where $\mathcal{Z}(r)$ is the subalgebra of A generated by $\mathcal{Z}_{|i|}$ and $\tau_i(e_n), \ldots, \tau_i(e_{r+1}), i \in \mathbb{I}_{|i|}$. Since every element has a Δ-reduced presentation we will from now on only consider these.
From every equivalence class of (Δ-reduced) w-monomials we fix one representative $c_{t,w} = p\rho_i(t_1)\rho_{it_1}(t_2)\cdots\rho_{it_1\cdots t_{k-1}}(t_k)$, $w \in \mathbb{W}$.
We claim: Every $x \in A$ can be written uniquely as a sum $\sum_{certain} \lambda_{t,w} c_{t,w}, \ \ \lambda_{t,w} \in \mathbb{C}$.
(In other words, the representatives $c_{t,w}$ of degree d form a $\mathbb{C}$-vector space basis of the finite dimensional vector space $A_w \cap A_d$.) The generating part is easy to see. We prove be induction wrt the length on w for every given degree d. If the length in zero, $w = e$, it holds $A_{\leq e} = \mathcal{Z}_e$ and $A_{\leq e} \cap A_d$ is spanned by monic polynomials, homogeneous of degree $\frac{d}{2}$ if d is even, zero else. They are linearly independent.
Let w be of a given length and $d \in \mathbb{Z}$, let

$$\sum_{(t,w) | c_{t,w} \in A_d} \lambda_{t,w} c_{t,w} = 0, \quad \lambda_{t,w} \in \mathbb{C}$$

We want to prove $\lambda_{t,w} = 0$ for all (t,w) such that $c_{t,w} \in A_d$. It is enough to show this for any choice of representatives $c_{t,w}$ of the equivalence classes. If there is a representative $c_{t,w}$ in an equivalence class ending on an element of the form $\rho_i(t) \in \{\sigma_i(s), is \neq i, \tau_i(e_r), ie_r \neq i\}$ we assume that we chose that one, then we compose the whole equation from the right with $\rho_{it}(t)$, the $c_{t,w}$ ending in $\rho_i(t)$ will fullfill that $c_{t,w}\rho_{it}(t)$ is w'-equivalent to a w'-monomial with $w' < w$ [5]. By induction hypothesis the coefficients of those $c_{t,w}$ must be zero.

[4]It is not necessarily $w = t_1 \cdots t_k$. For example in case (2b) it holds $\tau_i(e_r)\sigma_{ie_n}((r, r+1))$ is in A_{e_r}
[5]The case where this is not true is (2b) where $\tau_i(e_r)\sigma_{ie_r}(s)$, $s = (r, r+1)$ and $\tau_i(e_r)$ are in A_{e_r}, but

Now because of the braid relations, we are left summing over w-monomials $c_{t,w} = p\rho_i(t_1)\rho_{it_1}(t_2)\cdots\rho_{it_1\cdots t_{k-1}}(t_k)$ with $it_\ell = i$, $\forall \ell$. If $\rho_i(t_k) = \sigma_i((r, r+1))$ or $\tau_i(e_r)$ we multiply the whole equation by $z_i(r)$ (in the second case $r = n$ is allowed) from the right and use iteratively the straightening rule to write $c_{t,w}z_i(r)$ as a sum of Δ-reduced w'-monomials $w' \leq w$ and at least one $w' < w$. We proceed applying the straightening rules to all other $c_{t',w}z_i(r)$ to write them also as a sum of Δ-reduced w'-monomials $w' \leq w$. Again by induction hypothesis the coefficients in front of these have to be zero.
Now since we conclude that $\dim(\mathcal{Z}'_{|i|})_w \cap (\mathcal{Z}'_{|i|})_d = \dim(A_{|i|})_w \cap (A_{|i|})_d$, we get that the map $\phi_{|i|}$ is injective. $\square$

Remark. (from Bill Crawley-Boevey) For a $\mathbb{Z}$-graded algebra A with finite dimensional components, the category of finitely generated graded left A-modules with degree zero graded maps is abelian and has finite dimensional Hom-spaces. Therefore it is a Krull-Schmidt category.
Then, the full subcategory of finitely generated projective graded left A-modules is also Krull-Schmidt and every object is a direct summand of a finite direct sum of shifts of A, so is a direct sum of shifts of direct summands of A. Now, direct summands of A are given by idempotents in $\mathrm{Hom}(A, A) = A_0$, so are induced up from projective A_0-modules.

Remark. Let $\mathcal{B}'_{|i|}$ be the category of projective graded modules over $\mathcal{Z}'_{|i|}$. The functor $- \otimes_{\mathcal{Z}_{|i|}} \mathcal{Z}'_{|i|} \colon \mathcal{B}_{|i|} \to \mathcal{B}'_{|i|}$ is not always essentially surjective, consider for example: Let $i = (i_1, \ldots, i_n)$ with $\sigma(i_r) = i_r$, it follows that $\tau_i(e_r) = \delta_{e_r}$ and therefore $e := 1_i e_r 1_i$ is an idempotent element in $\mathcal{Z}'_{|i|}$ which, if for example $i_r \neq i_{r+1}, h_{i_r,i_{r+1}} \geq 1$, is not contained in $\mathcal{Z}_{|i|}$.
Also the induced map $K_0(\mathcal{B}_{|i|}) \to K_0(\mathcal{B}'_{|i|})$ is in general not injective. For example take $i = (i_1, \ldots, i_n)$ with $i_r \neq \sigma(i_r), i_r \neq i_{r+1}$ and $h_{i_r,i_{r+1}} \geq 1, h_{i_r,\sigma} = 0$ then $\tau_i(e_r) = 1_i e_r 1_{ie_r} \notin \mathcal{Z}_{|i|}$ and therefore we get

$$P'_i := \mathcal{Z}'_{|i|}1_i \xrightarrow{\cdot\tau_i(e_r)} \mathcal{Z}'_{|i|}1_{ie_r} = P'_{ie_r}$$

an isomorphism with inverse given by $\cdot\tau_{ie_r}(e_r)$. But P_i is not isomorphic to P_{ie_r} in $\mathcal{B}_{|i|}$.

4.5.2 Lusztig's Perverse sheaves/Projective modules corresponding to the vertices of the symmetric quiver.

Let us consider the following shifts of perverse sheaves.

$$L_k, k \in Q_0 \subset \mathbb{I}.$$

They correspond to the projective graded modules $P_k = \mathcal{Z}_{|k|}1_k, k \in Q_0$. Furthermore we set $P'_k := P_k \otimes \mathcal{Z}'_{|k|} = P_k$ because $\mathcal{Z}_{|k|} = \mathcal{Z}'_{|k|}$. Let us first describe these sheaves and their endomorphism algebras a bit more in detail.

since $ie_r s = ie_r$ it is not a counterexample

(1) $k = \sigma(k)$, $|k| = 2k$, $||k|| = 2$.

We have the Springer theory data $(G_{|k|}, B_k, V_{|k|}, F_k)$ and generalized Springer theory data $(\mathbb{G}_{||k||}, \mathbb{B}_{||k||}, , \mathcal{U}, H, V_{|k|})$ defined as follows:

$G_{|k|} = \mathbf{Sp}_2 = \mathbf{Sl}_2 = \mathbb{G}_{||k||}$ (i.e. $H = \{e\}$), the set $\mathbb{I}_{|k|} = \{1\}$.

$B_k = \mathbb{B}_{||k||} \subset \mathbf{Sl}_2$ is the upper triangular matrices, let $\mathcal{U} := \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \mid x \in \mathbb{C}\}$,

$V_{|k|} = \mathfrak{sl}_2^{\oplus h_{k,\sigma}} = \{A \in M_{2\times 2} \mid Tr(A) = 0\}^{\oplus h_{k,\sigma}}$,

where $h_{k,\sigma}$ is the number of σ-invariant loops at k. $F_k = \mathcal{U}^{\oplus h_{k,\sigma}}$ and $\pi_k \colon E_k := \mathbf{Sl}_2 \times^{B_k} \mathcal{U}^{\oplus h_{k,\sigma}} \to \mathfrak{sl}_2^{\oplus h_{k,\sigma}} = V_k$.

We then know, by the previous section that

$$\mathcal{Z}_{|k|} := H_*^{G_k}(E_k \times_{V_k} E_k) \subset \mathrm{End}_{\mathbb{C}[x]^{\mathbb{Z}/2\mathbb{Z}}}(\mathbb{C}[x]),$$

where $1 \neq t \in \mathbb{Z}/2\mathbb{Z}$ maps $x \mapsto -x$. It is the subalgebra generated by the elements $x\cdot, \sigma_k(t)$ with $\sigma_k(t)(f(x)) := (2x)^{h_{k,\sigma}} \frac{f(x)-f(-x)}{2x}$. By Chriss and Ginzburg's result (cp [CG97], chapter 8) we know

$$\mathcal{Z}_{|k|} = Ext^*_{D^b_{G_k}(V_k)}((\pi_k)_*\underline{\mathbb{C}}, (\pi_k)_*\underline{\mathbb{C}}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{End}_{\mathcal{P}}(L_k, L_k[n]).$$

But since we also know, that $\mathcal{Z}_{|k|}$ is a free (left) $\mathbb{C}[x]$-module of rank 2, it follows it is a free $\mathbb{C}[x]^{\mathbb{Z}/2\mathbb{Z}}$-module of rank 4. Since we know by the decomposition theorem that L_k is a direct sum of shifts of perverse sheaves it follows that it can contain at most two summands. For $h_{k,\sigma} \geq 1$, π_k is semi-small and it is small if and only if $h_{k,\sigma} \geq 2$: In general for $h := h_{k,\sigma} \geq 1$ let $\mathcal{N}(h) = \mathbf{Sl}_2 \cdot \mathcal{U}^{\oplus h} = \{(\lambda_1 n, \dots, \lambda_h n) \mid \lambda_i \in \mathbb{C}, n \in M_2(\mathbb{C}), n^2 = 0\}$, this is a $h+1$-dimensional variety with 0 is the only singularity. The map $\pi_k \colon \mathbf{Sl}_2 \times^{B_2} \mathcal{U}^{\oplus h} \to \mathcal{N}(h)$ is easy to be seen an isomorphism over $\mathcal{N}(h) \setminus \{0\}$ and $\pi_k^{-1}(0) \cong \mathbb{P}^1$, take as stratification $S_1 := \mathcal{N}(h) \setminus \{0\}, S_0 = \{0\}$ and let $d_1 = 0, d_0 = 1$ be the complex dimensions of the fibres over S_1, S_0 respectively. Semi-smallness is the inequality $2d_i \leq \dim \mathcal{N}(h) - \dim S_i, i = 0,1$ which is always fullfilled. Smallness is the extra condition that there exists a unique stratum where the inequality is an equality, it is fulfilled if and only if $h \geq 2$.

Semi-smallness implies that the shifts in the decomposition theorem are zero, i.e. L_k is a semi-simple perverse sheaf. Smallness (i.e. $h_{k,\sigma} \geq 2$) implies that

$$L_k = IC_{(\mathcal{N}(h), \underline{\mathbb{C}}_{\mathcal{N}(h)\setminus\{0\}})}$$

is a simple perverse sheaf (see Appendix, section **??**).

For $h_{k,\sigma} \in \{0,1\}$ we find the idempotent element $e := \frac{1}{\sqrt{2}}(2x\delta_t)$ in the endomorphism ring, which implies that L_k is a direct sum of two simple perverse sheaves (namely for $h = 1$ we write $\mathcal{N} := \mathcal{N}(1)$ it holds $L_k == IC_{(\mathcal{N}(h), \underline{\mathbb{C}}_{\mathcal{N}(h)\setminus\{0\}})} \oplus IC_{(\{0\}, \underline{\mathbb{C}})}$ where $IC_{(\{0\}, \underline{\mathbb{C}})} = i_* \underline{\mathbb{C}}_{\{0\}}$ with $i \colon \{0\} \to \mathcal{N}$ the inclusion).

For $h_{k,\sigma} = 0$
$$L_k = \underline{\mathbb{C}}_{\{0\}}[-2] \oplus \underline{\mathbb{C}}_{\{0\}}$$
and therefore
$$P'_k = P_k = \left(\mathcal{Z}_{|k|}e\right)[-2] \oplus \mathcal{Z}_{|k|}e$$
and $\mathcal{Z}_{|k|}e$ is up to shift and isomorphism the only indecomposable object in $\mathcal{B}_{|k|} = \mathcal{B}'_{|k|}$. (In the monoidal categorification we will give the generating endomorphisms different names $x\cdot \leftrightarrow z_k, \sigma_k(t) \leftrightarrow t_k$.)

(2) $k \neq \sigma(k)$, $|k| = k + \sigma(k)$, $||k|| = 2$.
$G_{|k|} = T \subset \mathbf{Sl}_2 =: G_{||k||}$ and $\mathbb{I}_{|k|} \leftrightarrow \{1, t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$.
$B_k = T = B_{\sigma(k)}$
$V_{|k|} = \mathcal{U}^{\oplus h}$ with $\mathcal{U} = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \mid x \in \mathbb{C}\}$, $h = h_{\sigma(k),\sigma}$ is the number of σ-invariant arrows $\sigma(k) \to k$ in Q_1.
$F_k = V_{|k|} = \mathcal{U}^{\oplus h}$, $F_{\sigma(k)} = F_{kt} = V_{|k|} \cap {}^t(\mathcal{U}^{\oplus h}) = \{0\}$.
It follows $\pi_k = \mathrm{id}_{\mathcal{U}^{\oplus h}}, \pi_{\sigma(k)} = i \colon \{0\} \to \mathcal{U}^{\oplus h}$ is the inclusion. Then
$$L_k = \underline{\mathbb{C}}_{\mathcal{U}^{\oplus h}}[h] = IC_{(\mathcal{U}^{\oplus h}, \underline{\mathbb{C}})}, \quad L_{\sigma(k)} = i_* \underline{\mathbb{C}}_{\{0\}} = IC_{(\{0\}, \underline{\mathbb{C}})}$$
are simple perverse sheaves. We know
$$\begin{aligned} \mathcal{Z}_{|k|} = H_*^T(F_k \sqcup \{0\} \sqcup \{0\} \sqcup \{0\}) &\hookrightarrow \mathrm{End}_{H_T^*(pt)}(H_T^*(pt) \oplus H_T^*(pt)) \\ &= \mathrm{End}_{\mathbb{C}[x]}(\mathbb{C}[x_k] \oplus \mathbb{C}[x_{\sigma(k)}]) \end{aligned}$$
is the subalgebra generated by $1_a, a \in \{k, \sigma(k)\}$ (=projections on the a-summands), $x_a \cdot, a \in \{k, \sigma(k)\}$ (multiplication with x_a on the a-summand), $\sigma_a(t), a \in \{k, \sigma(k)\}$ this is given by $\mathbb{C}[x_{\sigma(a)}] \to \mathbb{C}[x_a],\ f(x_{\sigma(a)}) \mapsto (2x_a)^h f(-x_a)$ and is zero on the other direct summand.
It holds $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{P}}(L_k \oplus L_{\sigma(k)}, L_k \oplus L_{\sigma(k)}[n]) = \mathcal{Z}_{|k|}$. (In the monoidal categorificatoin we will see $\sigma_{\sigma(a)}(t) \leftrightarrow t_a, a \in \{k, \sigma(k)\}$.)
We conclude from the decomposition theorem
$$P'_k = P_k = \mathcal{Z}_{|k|} 1_k, \quad P'_{\sigma(k)} = P_{\sigma(k)} = \mathcal{Z}_{|k|} 1_{\sigma(k)}$$
are up to isomorphism and shift the only indecomposable objects in $\mathcal{B}_{|k|} = \mathcal{B}'_{|k|}$.

We give one example of a monoidal category based on signed symmetric groups which is an important inspiration for later.

Example. Let $\mathcal{C}$ be the strict monoidal $\mathbb{C}$-linear category generated by shifts of one object $E = E(0)$, such that $\mathrm{Hom}_{\mathcal{C}}(E, E(n)) = \mathbb{C}[\tau]/(\tau^2 - 1)$ if $n = 0$ and zero else, and by an arrow $s\colon E^2 \to E^2$ of degree 0 subject to relations

$$s^2 = E^2, s \circ (\tau E) = (E\tau) \circ s, (Es) \circ (sE) \circ (Es) = (sE) \circ (Es) \circ (sE).$$

Then, the objects of this category are $E^n(r), n \in \mathbb{N}_0, r \in \mathbb{Z}$. The homomorphisms are $\mathrm{Hom}_{\mathcal{C}}(E^n, E^m(r)) = 0$ for $m \neq n$ and $\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(E^n, E^n(r)) \cong \mathbb{C}[S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n]$ where the right hand side is a graded algebra concentrated in degree 0. Recall that $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n =< s_1, \ldots, s_{n-1}, e_n >$ subject to relations

$$s_i^2 = 1,\ 0 \leq i \leq n-1 \quad e_n s_{n-1} e_n s_{n-1} = s_{n-1} e_n s_{n-1} e_n,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ 1 \leq i \leq n-2, \quad s_k s_j = s_j s_k,\ |k-j| > 1,$$

the isomorphism is given by $e_n \mapsto E^{n-1}\tau, s_i \mapsto E^{i-1} s E^{n-i-1}, 1 \leq i \leq n-1$.

4.5.3 Monoidal categorification

Following Rouquier's construction in [Rou11] we define the following. Let $\mathcal{C}$ be the monoidal category generated by direct sums of shifts of objects $E_a, a \in Q_0$, we write $E_a(n)$ for the shifted object by $n \in \mathbb{Z}$, and arrows (and their shifts)

$$z_a\colon E_a \to E_a \qquad \text{of degree } 2$$

$$t_a\colon E_a \to E_{\sigma(a)} \qquad \text{of degree } n_a := \begin{cases} -2 & , \text{ if } a = \sigma(a) \\ 2h_{\sigma(a),\sigma} & , \text{ if } a \neq \sigma(a) \end{cases}$$

$$\sigma_{a,b}\colon E_a E_b \to E_b E_a \qquad \text{of degree } m_{a,b} = \begin{cases} -2 & , \text{ if } a = b, \\ 2h_{b,a} & , \text{ if } a \neq b \end{cases}$$

for $a, b \in Q_0$, subject to the following relations (where we write E_a also for the endomorphism id_{E_a} and we always set $E_a E_b := E_a \otimes E_b$)

(1) ($s^2 =$?)

$$\sigma_{ba} \circ \sigma_{ab} = \begin{cases} (-1)^{h_{a,b}} (E_b z_a - z_b E_a)^{h_{a,b} + h_{b,a}} & , \text{ if } a \neq b \\ 0 & , \text{ if } a = b \end{cases}$$

$$t_{\sigma(a)} \circ t_a = \begin{cases} (-1)^{h_{a,\sigma}} (2z_a)^{h_{a,\sigma} + h_{\sigma(a),\sigma}} & , \text{ if } a \neq \sigma(a) \\ 0 & , \text{ if } a = \sigma(a) \end{cases}$$

(2) (straightening rule)

$$\sigma_{ab} \circ z_a E_b - E_b z_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ E_a E_a & , \text{ if } a = b, \end{cases}$$

$$\sigma_{ab} \circ E_a z_b - z_b E_a \circ \sigma_{ab} = \begin{cases} 0 & , \text{ if } a \neq b, \\ -E_a E_a & , \text{ if } a = b, \end{cases}$$

$$t_a \circ z_a - z_{\sigma(a)} \circ t_a = \begin{cases} 0 & , \text{ if } a \neq \sigma(a), \\ -E_a & , \text{ if } a = \sigma(a), \end{cases}$$

(3) (braid relations)

* Type A_2: For $a, b, c \in Q_0$ we have the following inclusion of $\mathbb{C}$-algebras. Let $\mathbb{C}[\alpha_s, \alpha_t]$ be the set of polynomials in α_s, α_t.

$$\begin{aligned} J_{a,b,c} \colon \mathbb{C}[\alpha_s, \alpha_t] &\to \operatorname{End}_{\mathcal{B}}(E_a E_b E_c) \\ \alpha_s &\mapsto z_a E_b E_c - E_a z_b E_c \\ \alpha_t &\mapsto E_a z_b E_c - E_a E_b z_c, \end{aligned}$$

we set $t(\alpha_s^h) := (\alpha_s + \alpha_t)^h =: s(\alpha_t^h) \in \mathbb{C}[\alpha_s, \alpha_t], h \in \mathbb{N}_0$. Then, the relation is

$$\begin{aligned} &\sigma_{ab} E_c \circ E_a \sigma_{cb} \circ \sigma_{ca} E_b - E_b \sigma_{ca} \circ \sigma_{cb} E_a \circ E_c \sigma_{ab} \\ &= \begin{cases} J_{bab}(\alpha_s^{h_{a,b}} s\delta_t(\alpha_s^{h_{b,a}}) - \alpha_t^{h_{b,a}} t\delta_s(\alpha_t^{h_{a,b}})) & , \text{ if } b = c, a \neq b, \\ 0 & , \text{ else.} \end{cases} \end{aligned}$$

* Type $B_2 = C_2$ for $a, b \in Q_0$ we consider the following inclusion of $\mathbb{C}$-algebras

$$\begin{aligned} J_{a,b} \colon \mathbb{C}[\alpha_s, \alpha_t] &\to \operatorname{End}_{\mathcal{B}}(E_a E_b) \\ \alpha_s &\mapsto z_a E_b - E_a z_b \\ \alpha_t &\mapsto 2 E_a z_b, \end{aligned}$$

$s(\alpha_t^h) := (2\alpha_s + \alpha_t)^h, s(\alpha_s^h) := (-1)^h \alpha_s^h, t(\alpha_s^h) := (\alpha_s + \alpha_t)^h, t(\alpha_t^h) := (-1)^h \alpha_t^h \in \mathbb{C}[\alpha_s, \alpha_t]$,
$h \in \mathbb{N}_0$. the relations are

$$\begin{aligned} &\sigma_{ba} \circ E_b t_{\sigma(a)} \circ \sigma_{\sigma(a)b} \circ E_{\sigma(a)} t_{\sigma(b)} - E_a t_{\sigma(b)} \circ \sigma_{\sigma(b)a} \circ E_{\sigma(b)} t_{\sigma(a)} \circ \sigma_{\sigma(a)\sigma(b)} \\ &= \begin{cases} J_{a,b}(P_t) \circ E_a t_{\sigma(b)}, & \text{if } a = \sigma(a), a \neq b, b \neq \sigma(b) \\ J_{a,\sigma(a)}(P_s) \circ \sigma_{\sigma(a),a}, & \text{if } a = \sigma(b), a \neq b \\ J_{a,b}(R_t) \circ E_a t_b + J_{a,b}(R_e), & \text{if } a = \sigma(a), b = \sigma(b), a \neq b \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$P_t = \alpha_s^{h_{b,a}} s\delta_t(\alpha_s^{h_{a,b}}) - t(\alpha_s^{h_{a,b}}) ts\delta_t(\alpha_s^{h_{b,a}})$$
$$P_s = \alpha_t^{h_{a,\sigma}} t\delta_s(\alpha_t^{h_{\sigma(a),\sigma}}) - s(\alpha_t^{h_{\sigma(a),\sigma}}) st\delta_s(\alpha_t^{h_{a,\sigma}})$$
$$R_t = t(\alpha_s^{h_{b,a}}) ts\delta_t(\alpha_s^{h_{a,b}}) - \alpha_s^{h_{b,a}} s\delta_t(\alpha_s^{h_{a,b}})$$
$$R_e = \delta_t(\alpha_s^{h_{b,a}}) s\delta_t(\alpha_s^{h_{a,b}})$$

* (The extra relations for $\mathcal{Z}'$) Let $a, b \in Q_0$.

(1a) $a = \sigma(a) = b$

$$t_a E_a = J_{a,a}(\alpha_s t(\alpha_s)) \circ \sigma_{aa} \circ E_a t_a \circ \sigma_{aa} + J_{a,a}(\alpha_s) \circ \sigma_{aa} \circ E_a t_a + J_{a,a}(t(\alpha_s)) \circ E_a t_a \circ \sigma_{aa} + \sigma_{aa} + E_a E_a$$

(1b) $a = \sigma(a), a \neq b$

$$J_{a,b}(\alpha_s^{h_{b,a}} t(\alpha_s)^{h_{a,b}}) \circ t_a E_b = \sigma_{ba} \circ E_b t_a \circ \sigma_{ab} + J_{a,b}(\alpha_s^{h_{b,a}-1} \sum_{r+u=h_{a,b}} (-1)^r t(\alpha_s)^r \alpha_s^u)$$

for $h_{b,a} \geq 1$, if $h_{b,a} = 0$ it is the relation

$$J_{a,b}(\alpha_s t(\alpha_s)^{h_{a,b}}) \circ t_a E_b = J_{a,b}(\alpha_s) \circ \sigma_{ba} \circ E_b t_a \circ \sigma_{ab} + J_{a,b}(\sum_{r+u=h_{a,b}} (-1)^r t(\alpha_s)^r \alpha_s^u)$$

(2a) $a \neq \sigma(a), a = b$

$$J_{a,a}(\alpha_t^{h_{\sigma(a),\sigma}} t(\alpha_s)^{h_{a,\sigma}}) \circ t_{\sigma(a)} E_a = J_{a,a}(\alpha_s) \circ \sigma_{aa} \circ E_a t_{\sigma(a)} \circ \sigma_{\sigma(a),a} + J_{a,a}(s(\alpha_t)^{h_{\sigma(a),\sigma}} - 2 \sum_{k=1}^{h_{\sigma(a),\sigma}-1} s(\alpha_t)^{h_{\sigma(a),\sigma}-1-k} \alpha_t^k) \circ E_a t_{\sigma(a)} \circ \sigma_{\sigma(a)a}$$

(2b) $a \neq \sigma(a), \sigma(a) = b$

$$J_{a,\sigma(a)}(\alpha_s^{h_{\sigma(a),\sigma}}) \circ t_{\sigma(a)} E_{\sigma(a)} = J_{a,\sigma(a)}(t(\alpha_s)) \circ \sigma_{\sigma(a),a} \circ E_{\sigma(a)} t_{\sigma(a)} \circ \sigma_{\sigma(a),\sigma(a)} + \sigma_{\sigma(a),a} \circ E_{\sigma(a)} t_{\sigma(a)}$$

(2c) $a \neq \sigma(a)$, $a \neq b$, $b \neq \sigma(a)$

$$J_{a,b}(\alpha_s^{h_{b,a}} t(\alpha_s)^{h_{a,b}}) \circ t_{\sigma(a)} E_b = \sigma_{ba} \circ E_b t_{\sigma(a)} \circ \sigma_{\sigma(a),b}$$

for $i = (i_1, \ldots, i_n) \in \mathbb{I}$ we set $E_i := E_{i_1} E_{i_2} \cdots E_{i_n}$.

Lemma 37. *The following maps define an isomorphism of multiplicative sequences of* *algebras (i.e. they are compatible with the tensor product).*

$$
\begin{aligned}
\Phi\colon \mathcal{Z}'_{|i|} &\to \bigoplus_{i,j\in\mathbb{I}_{|i|}} \operatorname{Hom}_{\mathcal{C}}(E_i, E_j)\\
1_i &\mapsto \mathrm{id}_{E_i}\\
z_i(t) &\mapsto E_{i_1}E_{i_2}\cdots E_{i_{t-1}} z_{i_t} E_{i_{t+1}}\cdots E_{i_n}\\
\sigma_i(s) &\mapsto \begin{cases} E_{i_1}\cdots E_{i_{\ell-1}}\sigma_{i_{\ell+1},i_\ell}E_{i_{\ell+2}}\cdots E_{i_n}, & \text{, if } s=(\ell,\ell+1)\in S_n\\ E_{i_1}\cdots E_{i_{n-1}}t_{\sigma(i_n)}, & \text{, if } s=e_n\end{cases}\\
\tau_i(e_r) &\mapsto E_{i_1}\cdots E_{i_{r-1}}t_{\sigma(i_r)}E_{i_{r+1}}\cdots E_{i_n}
\end{aligned}
$$

proof: We check that the map is well-defined and that the obvious inverse map is also well-defined. □

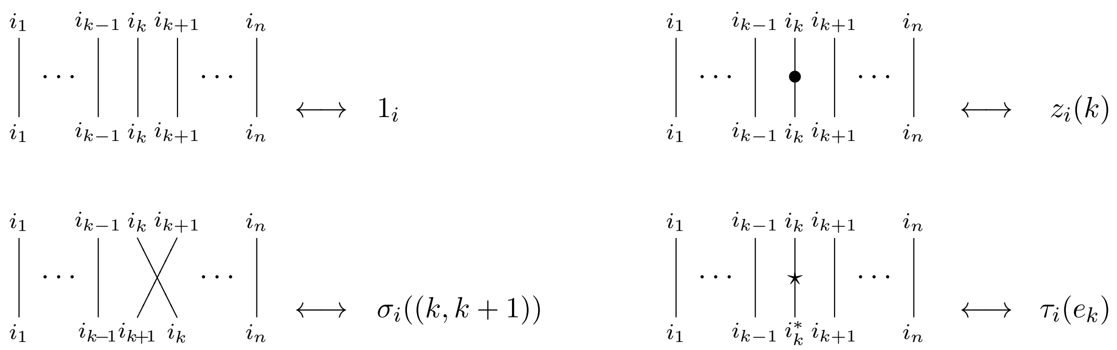
Corollary 4.5.1.1. *There is an equivalence of monoidal categories*

$$
\begin{aligned}
\mathcal{B}'_{\mathbb{I}} &\to \mathcal{C}\\
P_i &\mapsto E_i
\end{aligned}
$$

which is on homomorphisms given the map $\mathcal{Z}_{|i|}\subset\mathcal{Z}'_{|i|}\to\bigoplus_{i,j\in\mathbb{I}_{|i|}}\operatorname{Hom}_{\mathcal{B}}(E_i,E_j)$ *from the previous lemma.*

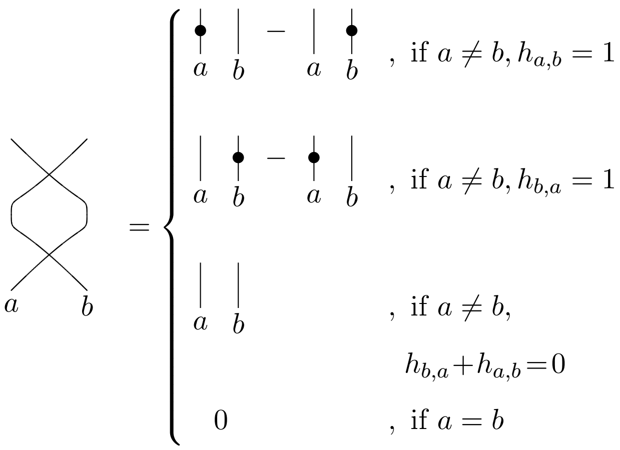
proof: see previous lemma. □

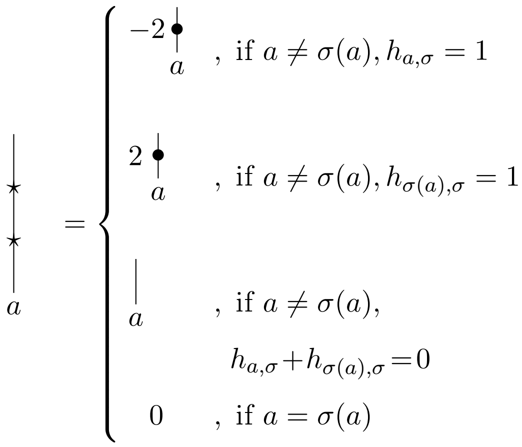
Example. Assume for simplicity $h_{a,b}+h_{b,a}\le 1, h_{a,\sigma}+h_{\sigma(a),\sigma}\le 1$ for all $a,b\in Q$. Let us describe the $\mathcal{Z}'_{|i|}$ in terms of Khovanov and Lauda's diagrams (in [KL09]). In the diagrams we write a^* instead of $\sigma(a)$, $a\in Q_0$. If the diagram has as bottom sequence i and as top sequence j, then it corresponds to an element in $\operatorname{End}(\bigoplus_i \mathcal{E}_i)$ mapping $\mathcal{E}_i$ to $\mathcal{E}_j$ and zero on the other summands.
The generators correspond to



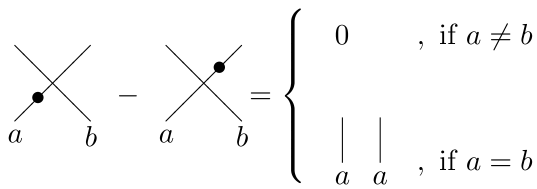
subject to the following relations.

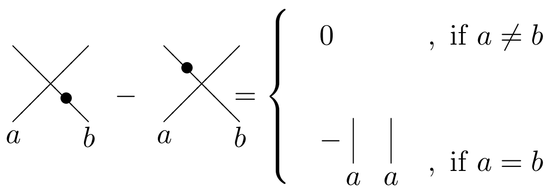
The relations implied by $s^2 = 1$.

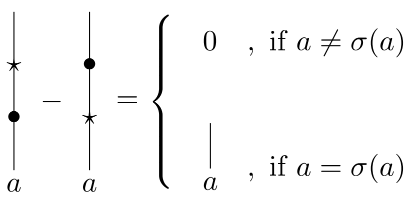




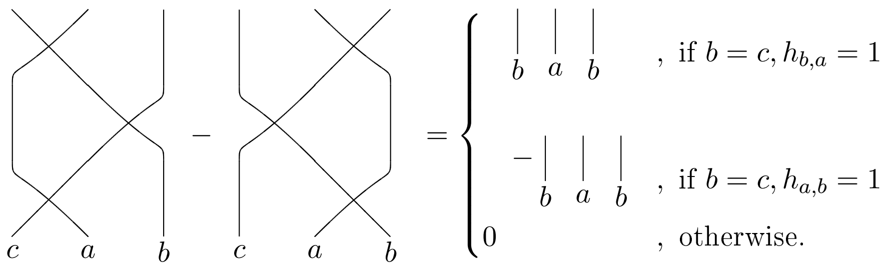
Straightening Rules.



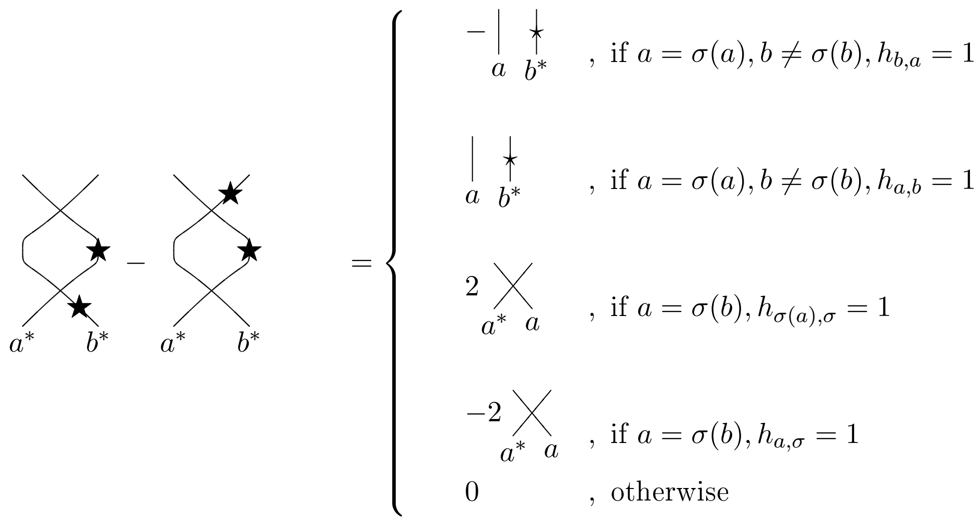




Type A_2-braid relations.

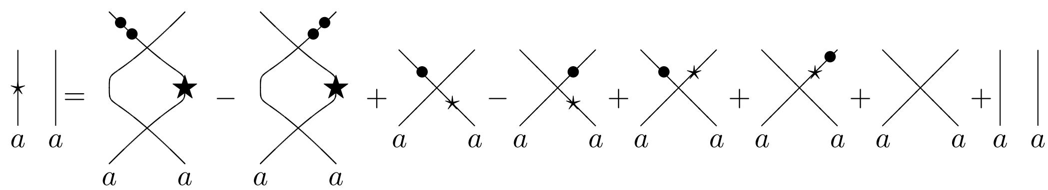


Type B_2 braid relations.

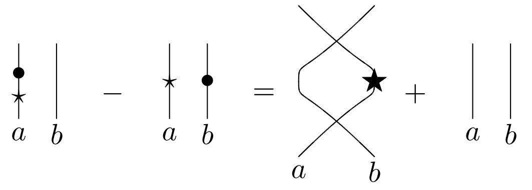


The extra relations for $\mathcal{Z}'$

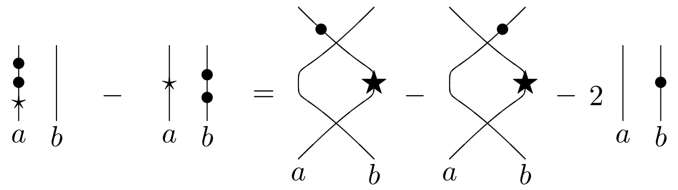
1a)



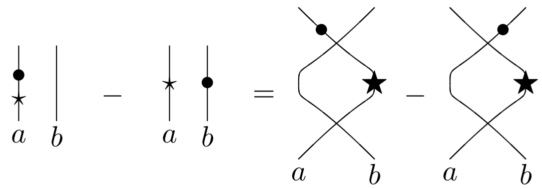
1b)



, if $a = \sigma(a), h_{b,a} = 1$

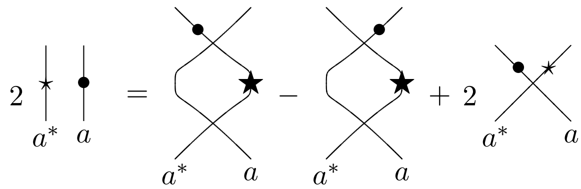


, if $a = \sigma(a), h_{a,b} = 1$

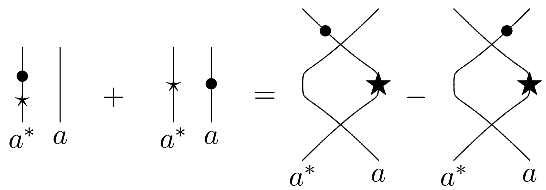


, if $a = \sigma(a), h_{a,b} = h_{b,a} = 0$

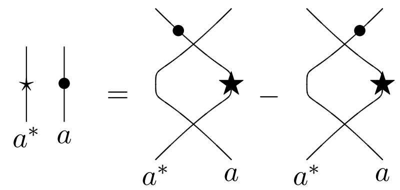
2a)



, if $h_{\sigma(a),\sigma} = 1$



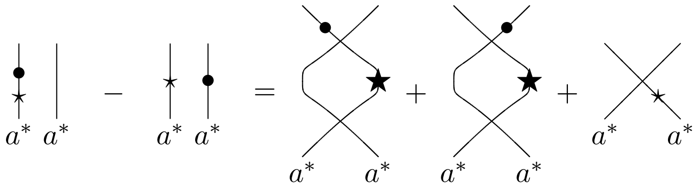
, if $h_{a,\sigma} = 1$



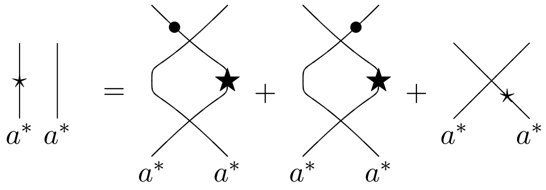
, if $a \neq \sigma(a)$,

$h_{a,\sigma} = h_{\sigma(a),\sigma} = 0$

2b)

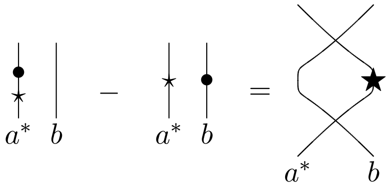


, if $h_{\sigma(a),\sigma} = 1$

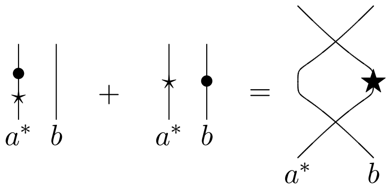


, if $a \neq \sigma(a), h_{\sigma(a),\sigma} = 0$

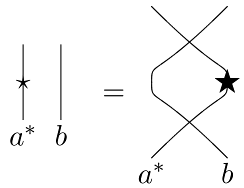
2c)



, if $a \neq \sigma(a), b \neq \sigma(a), h_{b,a} = 1$



, if $a \neq \sigma(a), b \neq \sigma(a), h_{a,b} = 1$



, if $a \neq \sigma(a), a \neq b, b \neq \sigma(a), h_{a,b} = h_{b,a} = 0$

Originally my interest was to relate the Grothendieck group of this monoidal category to Hall algebras for symmetric quivers representations.

4.6 A discussion on the search for Hall algebras for symmetric quiver representations

This has been a discussion with A. Hubery and W. Crawley-Boevey. So far, there is no definition of this Hall algebra. These are some partial answers/ algebras (and modules) which should be related to it.

(1) The Hall module. Instead of a Hall algebra we just find a module structure over the Hall algebra associated to $\widetilde{Q} := (Q_0^\sigma \sqcup Q_0', Q_1^\sigma \sqcup Q_1')$, this definition is for example in [You12].

The geometric construction by Springer theory is the same as we discussed before we extended to $\mathcal{Z}'$. The problem was to find an image for $(\sigma_i(e_n), 1_j)$, just leave out $\sigma_i(e_n)$ as generator for the first factor and you get a map $\mathcal{Z}^{\widetilde{Q}} \times \mathcal{Z}^{(Q,\sigma)} \to \mathcal{Z}^{(Q,\sigma)}$. On the Grothendieck ring of projective graded modules/ Lusztig's perverse sheaves this gives the module structure from before. The geometric construction of this operation is due to Varagnolo, Vasserot in [VV11].

(2) Our construction of the $\mathbb{Z}[q, q^{-1}]$-algebra $K_0(\mathcal{B}'_{\mathbb{I}})$. We want to see it as an analogue of the geometric construction of the composition algebra.
It is not understood yet if there is any justification for it being in this list.

(3) Restrict the choices of symmetric quivers (for example without black vertices) and find an exact category structure on symmetric quiver representations.
As a mini-example take $(Q, \sigma) = (a \to \sigma(a))$ and consider the category with objects are symmetric matrices representing linear maps $V \to V^*$ of finite dimensional vector spaces V over a fixed finite field and homomorphisms $\operatorname{Hom}(A, B)$ are commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{C} & W \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle B} \\ V^* & \xleftarrow[C^*]{} & W^* \end{array}$$

(for the representing matrices this says $A = C^t BC$). Then it is easy to write down short exact sequences (by just requiring the restriction $V \to W \to U$ is a short exact sequence of vector spaces). If this is an exact structure, then there exists an associated Hall algebra.

(4) The generic extension monoid/algebra and other geometric constructions (for example with constructible function). The generic extension monoid is defined by Reineke for quiver representations see [Rei02], its relationship to Hall algebras is investigated in [Wol09].
We explain this for $\mathbb{N}_0$-graded classical Springer theory. Let $i, j \in \mathbb{N}_0$ and $F_i := \mathcal{U}_i$ is the Lie algebra of the unipotent radical of the Borel subgroup. Recall that we had fixed T_{i+j}-equivariant complements $X_{i,j}$ of $F_i \times F_j$ in F_{i+j}. In fact, $X_{i,j}$ is even a $G_i \times G_j$-subrepresentation of $\operatorname{Lie}(G_{i+j})$. The maps

$$F_i \times F_j \longleftarrow F_i \times F_j \times X_{i,j} \longrightarrow F_{i+j}$$

induce a diagram

$$\begin{array}{ccc} & (G_i \times G_j) \cdot (F_i \times F_j \times X_{i,j}) & \\ \swarrow^{\mathcal{S} \times \mathcal{Q}} & & \searrow^{\mathcal{M}} \\ G_i F_i \times G_j F_j & & G_{i+j} F \end{array}$$

where $\mathcal{S} = pr_1, \mathcal{Q} = pr_2$ are locally free with fibre isomorphic to $X_{i,j}$ and $\mathcal{M}$ is a $G_i \times G_j$-equivariant closed immersion.

proof: Set $X := X_{i,j}$. As $m\colon F_i \times F_j \times X \to F_{i+j}$ is an isomorphism, consider

$$\begin{aligned}(G_i \times G_j)(F_i \times F_j \times X) &\to I(G_i \times G_j)F,\\ (g_i, g_j), (f_i, f_j, x) &\mapsto I(g_i, g_j)m(f_i, f_j, x)\end{aligned}$$

where $I\colon G_i \times G_j \to G_{i+j}$ is the structure map from the $\mathbb{N}_o$-graded group G. The map in the display is a closed immersion, because $F_i \times F_j \times X \to F_{i+j}$ is $I(P_i \times P_j)$-equivariant closed immersion and $I(P_i \times P_j)$ is a parabolic subgroup of $I(G_i \times G_j)$. We know that $I(G_i \times G_j)F_{i+j}$ is a closed subvariety of $G_{i+j}F_{i+j}$, because we have the commutative diagram

$$\begin{array}{ccc}(G_i \times G_j) \times^{(P_i \times P_j)} F_{i+j} & \xrightarrow{\quad J \quad} & G_{i+j} \times^{P_{i+j}} F_{i+j}\\ & \searrow \quad \swarrow & \\ & V_{i+j} = \mathrm{Lie}(G_{i+j}) & \end{array}$$

where the maps to V_{i+j} are collapsings of homogeneous bundles.
Since the images of the collapsing maps are both closed subvarieties of V_{i+j} and by the commutativity we get $I(G_i \times G_j)F_{i+j}$ is a closed subvariety of $G_{i+j}F_{i+j}$. Composing the first map with this closed immersion we get a closed immersion $(G_i \times G_j)(F_i \times F_j \times X) \to G_{i+j}F_{i+j}$. □

Then, one can define a monoid structure on the set of G_i-orbit closures on $G_iF_i, i \in \mathbb{N}_0$ as follows If $\mathcal{O} \subset G_iF_i$ is a G_i-orbit, $\mathcal{O}' \subset G_jF_j$ is a G_j-orbit then define

$$\overline{\mathcal{O}} * \overline{\mathcal{O}'} := \overline{G_{i+j}\mathcal{M}(\mathcal{S} \times \mathcal{Q})^{-1}(\overline{\mathcal{O}} \times \overline{\mathcal{O}'})}$$

To see that the right hand side is an orbit closure, notice: It is irreducible because of the properties, it is closed and G_{i+j}-equivariant by definition.
The associativity follows from the properties the complements X, we do not discuss this further.
This defines the composition monoid M_G, the composition algebra is the associated algebra KM_G with coefficients in some commutative ring K.

This is the middle of the phd

Here starts a second part of the phd. In pages it is not the middle but I spend my first 1,5 years with these topics. Essentially everything which comes from here on is concerned with quiver graded Springer maps and in particular their fibres. At that time I had not yet heard of KLR algebras.

Chapter 5

Constructing collapsings of homogeneous bundles over quiver loci

Summary. We look at orbit closures, closures of Segre classes and decomposition classes (defined by [BD01]) in various situations. We construct Springer maps (i.e. collapsings of homogeneous bundles) having these as their images and which are resolutions of singularities (or generically Galois coverings) and revisit the known results on this. First we explain a method how to get the equations for the image of a Springer map (which is in practice too unefficient). We then state the orbit lemma which deals with Springer maps which are resolutions of singularities of an orbit closure. The images of quiver-graded Springer maps for Dynkin and tame quivers are known to be the elements in the composition monoid which are described by results of Wolf and Deng, Du, here you find the quiver-graded Springer maps. (Apart from some remarks on decomposition classes, they are all known, cp. citation)

Q \ $\overline{C} = GF$	Dynkin	oriented cycle	acyclic, tame
separated orbit	5.2.3 and [Rei03]	5.2.4 and [DD05]	5.2.5
decomposition classes $\gamma = (M, ((1), \dots, (1)))$, M separated	$\emptyset$	???	5.2.5

For some type $\tilde{A}$ quiver we can use classical Springer theory for closures of arbitrary homogeneous decomposition classes.

Q	Jordan	Kronecker
regular homogeneous orbit	Lemma 49	Lemma 52
homogeneous decomposition class	Lemma 51	Lemma 53

We would like to understand the situation for arbitrary closures of decomposition classes but so far we could not find collapsing constructions.

In this section we will consider algebraic varieties over an algebraically closed field K and identify the variety as a scheme with its K-valued points.

5.0.1 Explicit equations for the image of the Springer map

Let G be a reductive group with a Borel subgroup B and F a B-subrepresentation of a G-representation V. Let $B = UT$ be the Levi decomposition with T the maximal torus and U the unipotent subgroup and let w_0 be a representative in $N_G(T) \subset G$ for the longest element in the Weyl group $N_G(T)/T$. We have $G = \overline{Uw_0B}$ because the big Bruhat cell $Bw_0B = Uw_0B$ is dense in G. Then, also $w_0Uw_0B = {}^{w_0}UB$ is dense in G, it follows

$$GF = \overline{GF} = \overline{{}^{w_0}UF}.$$

But as ${}^{w^0}U$ and F are affine space, let us say of dimension r and f respectively, one can use elimination theory to calculate the equations for GF. More precisely, consider the restriction of the multiplictaion map, let $V = K^n$ it holds

$$\begin{aligned} \rho \colon {}^{w_0}U \times F &\to V \\ x = (u,v) &\mapsto \rho((u,v)) =: (f_1(x), \dots, f_n(x)) \end{aligned}$$

Now, consider inside $K[A_1, \dots, A_r, X_1, \dots, X_f, Z_1, \dots, Z_n]$ the ideal

$$I := (f_1 - Z_1, \; \dots \; , f_n - Z_n).$$

To find generators for the ideal $J = I \cap K[Z_1, \dots, Z_n]$ one can use elimination theory: Find a Gröbner basis C for I with respect to a monomial order such that A_i, X_i are bigger than Z_j for all i, j. Let $C \cap K[Z_1, \dots, Z_n] = \{q_1, \dots, q_m\}$, then $J = (q_1, \dots, q_m)$, in other words

$$GF = \{x \in \mathbb{A}^n \mid q_1(x) = \dots = q_m(x) = 0\}$$

see [CLO92], Chap.3, 3, Thm 2. Unfortunately, this algorithm is very uneffective. Popov gives more general algorithms to calculate the equations for $\overline{GF}$ with G a linear algebraic group with a G-representation V and $F = a + L \subset V$ an affine linear subspace, i.e. a translate of a linear subvector space, cp. [Pop09].

Example. Let $1 \leftleftarrows 2$ be the Kronecker quiver, let $\underline{d} := (2,2)$, we have the $G = \mathbf{Gl}_2 \times \mathbf{Gl}_2$-representation $M_{2\times 2} \times M_{2\times 2}$. Let us take $B \subset G$ be the product of invertible lower triangular matrices, take $F := \{\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid x, y \in K\}$. We look at the following ideal I in $K[a, b, x, y, z_{11}, z_{12}, z_{21}, z_{22}, s_{11}, s_{12}, s_{21}, s_{22}]$ generated by the coefficients of the matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

Now we have to eliminate x, y, a, b, i.e. calculate $J = I \cap K[z_{11}, z_{12}, z_{21}, z_{22}, s_{11}, s_{12}, s_{21}, s_{22}]$.

First eliminate x, y by substituting $x = z_{21}, y = s_{21}$, then assume $z_{21} \neq 0, s_{21} \neq 0$ and substitute $a = \frac{z_{11}}{z_{21}} = \frac{s_{11}}{s_{21}}$ and $b = \frac{z_{22}}{z_{21}} = \frac{s_{22}}{s_{21}}$ to obtain

$$J = (z_{11}z_{22} - z_{12}z_{21}, s_{11}s_{22} - s_{12}s_{21}, z_{11}s_{21} - s_{11}z_{21}, z_{22}s_{21} - s_{22}z_{21})$$

5.1 The orbit lemma

As always we assume that all schemes are of finite type over an algebraically closed field. If we talk about orbits Gv, we always assume that the multiplication map $G \to Gv, g \mapsto gv$ is separated.

Definition 15. We call a scheme morphism $\pi\colon T \to S$ a **resolution of singularities of** S if T is smooth, π is projective, dominant and there is $U \subset S$ open and dense such that the restriction $\pi^{-1}(U) \to U$ is an isomorphism.

This lemma is a generalization of [Wol09], thm. 5.32.

Lemma 38. *Let G be a connected algebraic group, $P \subset G$ a closed subgroup, V a G-variety with a smooth P-subvariety F. Assume $GF \subset V$ has a dense G-orbit $\mathcal{O}$. Then, the fibres of $\pi\colon G \times^P F \to GF, \overline{(g,f)} \mapsto gf$ over $\mathcal{O}$ are smooth, pairwise isomorphic, irreducible of dimension* $\dim G \times^P F - \dim \mathcal{O}$.

Proof: There is an open subset $U \subset GF$ such that $U \subset \mathcal{O}$, let $U' = \bigcup_{g \in G} gU$ be an open G-invariant subset contained in $\mathcal{O}$, so $\mathcal{O} = U'$ is open in GF. As $G \times^P F$ is smooth and irreducible, it follows, $\pi^{-1}(\mathcal{O})$ is open, irreducible of dimension $\dim G \times^P F$ and smooth. Then for $v \in \mathcal{O}$ the fibre $\pi^{-1}(v)$ is smooth, irreducible of dimension $\dim G \times^P F - \dim \mathcal{O}$ because $\pi^{-1}(\mathcal{O}) \cong G \times^{\mathrm{Stab}(v)} \pi^{-1}(v)$. □

Lemma 39. *(Orbit lemma) Let G be a connected reductive group, $P \subset G$ a parabolic subgroup, V a G-variety with a closed irreducible smooth P-subvariety F. Fix $v \in V$ and denote by $\mathcal{O} \subset V$ its G-orbit. The following are equivalent*

(1) The collapsing map $\pi\colon G \times^P F \to GF, \overline{(g,f)} \mapsto gf$ is a resolution of singularities for $\overline{\mathcal{O}}$ (i.e. $G \times^P F$ irreducible, smooth, $GF = \overline{\mathcal{O}}$, π is projective and an isomorphism over $\mathcal{O}$).

(2) $\pi^{-1}(v) \neq \emptyset$ and $\dim G \times^P F = \dim \mathcal{O}$.

Proof: Clearly (1) implies (2). So, assume (2) holds. We already now that π is projective and $G \times^P F$ smooth and irreducible. By assumption it holds $\overline{\mathcal{O}} \subset GF$. This implies $\dim \overline{\mathcal{O}} \leq \dim GF \leq \dim G \times^P F$ and by assumption all are equalities. It follows $GF = \overline{\mathcal{O}}$ and $\dim GF = \dim G \times^P F$. By the previous lemma we know that the fibres over the open $\mathcal{O} \subset GF$ are smooth, irreducible and zero-dimensional again. So, the morphism is generically etale, i.e. it induces a finite separable field extension on function fields. But as the fibres are connected over an open set, the field extension has to have degree zero and π is birational. □

5.2 Quiver-graded Springer maps

The images of complete quiver-graded Springer maps carry a *monoid* structure, we call this the generic composition monoid, in the following we review the known results on it.

A short reminder on quiver-graded Springer maps. Let $Q = (Q_0, Q_1)$ be a finite quiver and $\underline{\mathbf{d}} = (0 = \underline{d}^0, \underline{d}^1, \ldots, \underline{d}^\nu =: \underline{d})$ be a sequence of $\underline{d}^k \in \mathbb{N}_0^{Q_0}$, $0 \leq k \leq \nu$. We associate a 4-tuple (G, P, V, F) as follows.

* $G = \mathbf{Gl}_{\underline{d}} := \prod_{i \in Q_0} \mathbf{Gl}_{d_i}$ and let $P = P(\underline{\mathbf{d}}) := \prod_{i \in Q_0} P(\underline{d}_i^\bullet)$ where $P(\underline{d}_i^\bullet) \subset \mathbf{Gl}_{d_i}$ is a standard parabolic stabilizing a flag $U_i^\bullet$ of dim $\underline{d}_i^\bullet = (0, \underline{d}_i^1, \ldots, \underline{d}_i^\nu = d_i)$, $i \in Q_0$.
* $V = \mathrm{R_Q(\underline{d})} := \prod_{\mathrm{i \to j \in Q_1}} \mathrm{Mat}_{\mathrm{d_j \times d_i}}$ with G operates by conjugation and $F = \{f = (f_\alpha)_{\alpha \colon i \to j} \in \mathrm{R_Q(\underline{d})} \mid \mathrm{f_\alpha(U_i^k) \subset U_j^k},\ 1 \leq \mathrm{k} \leq \nu\}$ is a P-subrepresentation.

The collapsing map of the associated fibre bundle $\pi \colon \mathrm{RF}(\underline{\mathbf{d}}) := G \times^P F \to V$, $\overline{(g, f)} \mapsto gf$ is called a *quiver-graded Springer map*.
Also, we use the following conventions: We define the *Euler form* for $\underline{d}, \underline{e} \in \mathbb{Z}^{Q_0}$ as

$$\langle \underline{d}, \underline{e} \rangle := \sum_{i \in Q_0} d_i e_i - \sum_{(i \to j) \in Q_1} d_i e_j \quad \in \mathbb{Z}.$$

For M, N two finite dimensional KQ-modules we write

$$[M, N] := \dim_K \mathrm{Hom}_{KQ}(M, N), \quad [M, N]^1 := \dim_K \mathrm{Ext}^1_{KQ}(M, N).$$

5.2.1 The generic composition monoid

We recall Reineke's definition of the composition monoid.

Definition 16. Let Q be a finite quiver, let $\underline{d}, \underline{e} \in \mathbb{N}_0^{Q_0}$ and $X \subset \mathrm{R_Q(\underline{d})}$ a $\mathbf{Gl}_{\underline{d}}$-equivariant subset and $Y \subset \mathrm{R_Q(\underline{e})}$ a $\mathbf{Gl}_{\underline{e}}$-invariant subset, we define

$$X * Y := \{M \in \mathrm{R_Q(\underline{d} + \underline{e})} \mid \exists \text{ ses } 0 \to \mathrm{y} \to \mathrm{M} \to \mathrm{x} \to 0, \text{ with } \mathrm{y \in Y, x \in X}\}$$

If X, Y are additionally closed in $\mathrm{R_Q(\underline{d})}$, $\mathrm{R_Q(\underline{e})}$ respectively and irreducible, then $X * Y$ is closed in $\mathrm{R_Q(\underline{d} + \underline{e})}$, irreducible and $\mathbf{Gl}_{\underline{d}+\underline{e}}$-equivariant. This defines an associative product on

$$\mathcal{M} = \mathcal{M}(Q) := \bigcup_{\underline{d} \in \mathbb{N}_0^{Q_0}} \{X \subset \mathrm{R_Q(\underline{d})} \mid \mathrm{X}\ \ \mathbf{Gl}_{\underline{\mathrm{d}}} - \text{equivariant, closed, irreducible }\}.$$

We write $\underline{d}$ for the element $\mathrm{R_Q(\underline{d})} \in \mathcal{M}(\mathrm{Q})$, the unit is given by $1 := \underline{0} \in \mathbb{N}_0^{Q_0}$. For a KQ-module M we write $[M] := \overline{\mathcal{O}_M} \in \mathcal{M}(Q)$. For $i \in Q_0$ let E_i be the simple module supported at i and all maps are zero, for these modules we leave out the brackets, we write $E_i := \mathcal{O}_{E_i} \in \mathcal{M}(Q)$, sometimes we write $e_i := \underline{\dim}\, E_i$ for the same element.

One can now look at the submonoid $\mathcal{CM}(Q)$ generated by the simple modules without self-extensions.

Theorem 5.2.1. *([Rei02]) Let* $\operatorname{char} K = 0$*. Let* $X \subset \mathrm{R_Q}(\underline{\mathrm{d}}), \mathrm{Y} \subset \mathrm{R_Q}(\underline{\mathrm{e}})$ *be closed irreducible subvarieties,* X $\mathbf{Gl}_{\underline{d}}$*-equivariant,* Y $\mathbf{Gl}_{\underline{e}}$*-equivariant. Then*

$$\operatorname{codim} X + \operatorname{codim} Y - \langle \underline{e}, \underline{d} \rangle_{KQ} \;\leq\; \operatorname{codim} X * Y \;\leq\; \operatorname{codim} X + \operatorname{codim} Y + [Y, X]^1$$

where $[Y, X]^1 := \min\{[y, x]^1 \mid x \in X, y \in Y\}$*. If* $[X, Y]^1 = 0$*, or* $X = \mathrm{R_Q}(\underline{\mathrm{d}})$ *and* $Y = \mathrm{R_Q}(\underline{\mathrm{e}})$*, the second inequality is equality.*
If $[Y, X] := \min\{[y, x] \mid y \in Y, x \in X\} = 0$*, then the first inequality is equality.*

By definition quiver loci are precisely the products of elements of the form $\underline{d} \in \mathcal{M}(Q)$ with $\underline{d} \in \mathbb{N}_0^{Q_0}$, more precisely for a dimension filtration $\underline{\mathbf{d}} = (\underline{d}^0, \ldots, \underline{d}^\nu)$ it holds

$$\mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} = (\underline{d}^\nu - \underline{d}^{\nu-1}) * \cdots * (\underline{d}^2 - \underline{d}^1) * \underline{d}^1.$$

For acyclic quivers it holds $\underline{d} \in \mathcal{CM}(Q)$ for all $\underline{d} \in \mathbb{N}_0^{Q_0}$. More precisely, let us call a representation M **nilpotent**, if there is an $N \geq 2$ such that for all sequences $(\alpha_1, \cdots, \alpha_N) \in Q_1^N$ with start point of α_i is the end point of α_{i-1}, $2 \leq i \leq N$, it holds $M_{\alpha_N} M_{\alpha_{N-1}} \cdots M_{\alpha_1} = 0$.

Now, the elements of the composition monoid for Q an oriented cycle, of Dynkin or extended Dynkin type are described by the next result (of Stefan Wolf). We need some knowledge on the AR-quiver of an extended Dynkin quiver Q.
For an extended Dynkin quiver Q, there exist a unique $\delta \in \mathbb{N}_0^{Q_0}$ which is minimal such that $\langle \delta, \delta \rangle = 0$. For a KQ-module M we call the value $\delta(M) := \langle \delta, \underline{\dim} M \rangle$ the **defect of M**.
A (finite-dimensional) indecomposable KQ-module M is either **preprojective** if $\delta(M) < 0$, **preinjective** if $\delta(M) > 0$ or **regular** if $\delta(M) = 0$. So each module M can be written as $M = M_P \oplus M_R \oplus M_I$. The full subcategory of regular modules (i.e. objects are modules M with $M_P = 0 = M_I$) is an abelian category that breaks into a direct sum $R(\alpha)$ with $\alpha \in \mathbb{P}^1(K)$. We say that $R(\alpha)$ is **homogeneous** if the category has only one simple, else we say it is **inhomogeneous**. There are at most three points $\alpha \in \mathbb{P}^1(K)$ such that $R(\alpha)$ is inhomogeneous, let $\mathbb{H} \subset \mathbb{P}^1(K)$ be their complement.
For $\alpha \in \mathbb{H}$ and $t \in \mathbb{N}$ there is a unique indecomposable module $U(\alpha, t)$ in $R(\alpha)$ such that its length is t (here length is the length of a filtration in $R(\alpha)$ with simple subquotients). A **partition** $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ of n is a sequence of r decreasing positive integers λ_i such that $|\lambda| = \sum_{i=1}^r \lambda_i = n$. A **Segre symbol** $\sigma = (\lambda^{(1)}, \ldots, \lambda^{(s)})$ of n is sequence of partitions $\lambda^{(i)} = (\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots \geq \lambda_{r_i}^{(i)} > 0)$ such that $(|\lambda^{(1)}| \geq |\lambda^{(2)}| \geq \cdots \geq |\lambda^{(s)}|)$ is a partition of n. A **decomposition symbol** $\gamma = (M, \sigma)$ is a pair with M is (the isomorphism class of) a module without homogeneous direct summands and σ is a Segre symbol. The **decomposition class** $D(\gamma)$ to a decomposition symbol γ consists of all modules N such that there exists $\alpha_1, \ldots, \alpha_s \in \mathbb{H}$ pairwise different elements such that N

it is isomorphic to

$$M \oplus \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{r_i} U(\alpha_i, \lambda_j^{(i)}).$$

We will consider $D(\gamma) \subset \mathrm{R_Q}(\underline{\mathrm{d}})$ for the appropriate choice of a dimension vector. Bongartz and Dudek prove in [BD01], thm 1, that the various decomposition classes in $\mathrm{R_Q}(\underline{\mathrm{d}})$ form a stratification into irreducible smooth $\mathbf{Gl}_{\underline{d}}$-invariant that have smooth rational geometric quotients.

Remark. Let $\gamma = (M = P \oplus L \oplus I, \sigma)$ be a decomposition symbol with P preprojective, I preinjective and L inhomogeneous regular. Using that certain extension groups vanish it holds $D(\gamma) = \mathcal{O}_{P \oplus L} * D((0,\sigma)) * \mathcal{O}_I$. We use Reineke's theorem above (the first inequality is equality). Let $\underline{d} = \underline{\dim}(P \oplus L), D((0,\sigma)) \subset \mathrm{R_Q}(\underline{\mathrm{e}}), \underline{\dim}\,\mathrm{I} = \underline{\mathrm{f}}$ we get

$$\operatorname{codim} D(\gamma) = [P \oplus L, P \oplus L]^1 + \operatorname{codim} D((0,\sigma)) + [I,I]^1 - \langle \underline{d}, \underline{e} \rangle - \langle \underline{d}, \underline{f} \rangle - \langle \underline{f}, \underline{e} \rangle.$$

If $\sigma = (\underbrace{(1),(1),\ldots,(1)}_{\ell-\text{times}})$, then $\operatorname{codim} D((0,\sigma)) = 0$.

Now, in an inhomogeneous $\mathcal{R} = \mathcal{R}(\alpha)$ we have have $E_0, \ldots, E_{n-1}$, $n \geq 1$ pairwise orthogonal simple modules, we calculate in the indices modulo n, i.e. for $m = qn + r, r \in \{0 \ldots, n-1\}$ we set $E_m := E_r$. Each indecomposable module in $\mathcal{R}$ has the form $E_i[\ell]$ for one $i \in \{1, \ldots, n\}, \ell \in \mathbb{N}_0^{Q_0}$ it is unquely characterized by a filtration

$$0 = E_i[0] \subset E_i = E_i[1] \subset E_i[2] \subset \cdots \subset E_i[\ell]$$

with $E_i[s+1]/E_i[s] \cong E_{i+s}$ for $0 \leq s \leq \ell - 1$. We have a bijection between the isomorphism classes of objects in $\mathcal{R}$ and the set

$$\Pi := \{\pi := (\pi^{(0)}, \ldots, \pi^{(n-1)}) \mid \pi^{(j)} = (\pi_1^{(j)} \geq \cdots \pi_{t_j}^{(j)}) \text{ partition}, 0 \leq j \leq n-1\},$$

given by $\pi \mapsto E[\pi] := \bigoplus_{j=0}^{n-1} \bigoplus_{i=1}^{t_j} E_j[\pi_i^{(j)}]$. We call an element $M \cong E[\pi], \pi \in \Pi$ **separated** if for each $k \geq 0$ there is an $j_k \in \{0, \ldots, n-1\}$ such that $\pi_i^{(j_k)} \neq k$ for all $i \in \{1, \ldots, t_{j_k}\}$.

We call a regular module separated if it is a direct sum of separated modules (for possibly different inhomogeneous tubes).

Theorem 5.2.2. *(1) Let Q be an oriented cycle of type $\widetilde{A_n}$ (with $n+1$ vertices). Then*

$$\mathcal{CM}(Q) = \{[M] \mid \; M \in \mathrm{R_Q}(\underline{\mathrm{d}}), \underline{\mathrm{d}} \in \mathbb{N}_0^{\mathrm{Q}_0}, \mathrm{M} \textit{ nilpotent and separated } \}.$$

(2) Let Q be of Dynkin type. Then

$$\mathcal{CM}(Q) = \{[M] \mid \; M \in \mathrm{R_Q}(\underline{\mathrm{d}}), \underline{\mathrm{d}} \in \mathbb{N}_0^{\mathrm{Q}_0}\}.$$

(3) Let Q be of extended Dynkin type and acyclic and $\underline{\delta}$ is the dimension vector of one (and all) simple regular representation. Then

$$\mathcal{CM}(Q) = \{\overline{D(\gamma)} \mid \gamma = (M, \sigma = (\underbrace{(1), (1), \ldots, (1)}_{\ell-times}), M_R \textit{ separated}, \ell \in \mathbb{N}_0^{Q_0})\}.$$

Proof: (1) and (2): It is wellknown that for each dimension vector $\underline{d} \in \mathbb{N}_0^{Q_0}$ there are only finitely $\mathbf{Gl}_{\underline{d}}$-orbits of nilpotent elements in $\mathrm{R_Q}(\underline{\mathrm{d}})$ (in (2) all representations are nilpotent). Therefore each closed irreducible $\mathbf{Gl}_{\underline{d}}$-equivariant subset is the closure of an orbit. Now, it is a result of Deng and Du, cp. [DD05], that the orbit closures of separated elements are precisely the orbit closures. (3) In [Wol09], corollary 4.29. Stefan Wolf proved that each element of the composition monoid can be written as $\overline{\mathcal{O}_{P\oplus L}} * \underline{\delta}^{*\ell} * \overline{\mathcal{O}_I}$ with P preprojective, L inhomogeneous regular, I preinjective. The description as the closure of the decomposition class follows easy from remark 5.2.1. The claim that M_R has to be separated follows from the normal form from Stefan Wolf, see [Wol09]. $\square$

Remark. Let Q be an extended Dynkin quiver.

a) Closures of decomposition classes $\overline{D(\gamma)}$ with $\gamma = (M, \sigma = (\underbrace{(1), (1), \ldots, (1)}_{\ell-\text{times}}), M_R$ separated are unions of decomposition classes.
To see this: Observe that if $D(\gamma) \cap \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} \neq \emptyset$ then $D(\gamma) \subset \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$. Set $\gamma' \leq \gamma$ if $D(\gamma') \cap \overline{D(\gamma)} \neq \emptyset$. If $\overline{D(\gamma)} = \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$, then $\bigcup_{\gamma' \leq \gamma} D(\gamma') \subset \overline{D(\gamma)}$ and clearly the other inclusion holds.

b) Let $\underline{\mathbf{d}} = (\underline{d}^0, \ldots, \underline{d}^\nu = \underline{d})$, $\underline{d}^k \in \mathbb{N}_0^{Q_0}, \underline{d}_i^k \leq \underline{d}_i^{k+1}$, then $\mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$ is a union of decomposition classes and a closure of one.
To see this: Let $\underline{\mathbf{d}}'$ run through all complete dimension filtrations such that $\underline{\mathbf{d}}$ can be obtained by $\underline{\mathbf{d}}'$ by omitting some elements in the sequence, then

$$GF_{\underline{\mathbf{d}}} = \bigcup_{\underline{\mathbf{d}}'} GF_{\underline{\mathbf{d}}'}$$

because every decreasing filtration of modules $0 \subset M_1 \subset \cdots \subset M_\nu = M$ can be refined to a composition series. Then use remark a).

Questions: Let Q be a Dynkin or extended Dynkin quiver and given an element in $\mathcal{CM}(Q)$, how do we find a dimension filtration $\underline{\mathbf{d}}$ such that the orbit closure / closure of decomposition class is dense? Can we find one such that the Springer map $\mathbf{Gl}_{\underline{d}} \times^{P_{\underline{\mathbf{d}}}} F_{\underline{\mathbf{d}}} \to GF_{\underline{\mathbf{d}}}$ has nice properties, i.e. generically finite or even a resolution of singularities?

5.2.2 When is the quiver-graded Springer map a resolution of singularities of an orbit closure?

We use the orbit lemma to find answers for Dynkin and extended Dynkin quivers. In the end of this subsection, we also look at the quiver-graded Springer maps with closures of decomposition classes as the image.

Remark. Let $M \in \mathrm{R_Q}(\underline{d})(\mathrm{K})$ and $\underline{\mathbf{d}} = (0, \underline{d}^1, \dots, \underline{d}^\nu =: \underline{d})$ with $\underline{d}^k \in \mathbb{N}_0^{Q_0}, \underline{d}_i^k \leq \underline{d}_i^{k+1}$. Then, obviously $\mathrm{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} = \overline{\mathcal{O}_M}$ if and only if the following two conditions hold

(i) $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \neq \emptyset$.

(ii) $[M, M]^1 = \operatorname{codim}(\underline{d}^\nu - \underline{d}^{\nu-1}) * \cdots * (\underline{d}^2 - \underline{d}^1) * \underline{d}^1$.

For both statements there are (partial) earlier results which one can apply for Dynkin quivers, more generally for preprojective or preinjective modules.

ad (i) In [Wol09] Stefan Wolf introduced reflection functors in the setting of quiver flags and used them to give for a preprojective representation M an equivalent purely combinatorial condition for $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \neq \emptyset$ (see Cor. 6.22 in [Wol09]), we recall his result. After the choice of an admissible ordering $(a_1, \dots, a_n)$ of Q_0 one can define the Coxeter-transform C^+. Take $r \in \mathbb{N}_0$ such that $(C^+)^r M = 0$. Let $\underline{\mathbf{d}}$ be a filtration of $\underline{\dim}\, M$. Then, $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \neq \emptyset$ if and only if the following two condition are fulfilled.

1) $(C^+)^r \underline{\mathbf{d}} = 0$.

2) For every intermediate sequence w of admissible sinks, $S_w^+ \underline{\mathbf{d}}$ is a filtration of $\underline{\dim}\, S_w^+ M$ where S_w^+ is a composition of reflection functors (for details see loc. cit).

ad (ii) In [Rei02] Markus Reineke gives a formula to calculate codimensions of products in the composition monoid, in general that is difficult. In the special case of two factors ([Rei02] Thm 2.7), $\underline{d}, \underline{e} \in \mathbb{N}_0^{Q_0}$

$$\operatorname{codim} \underline{d} * \underline{e} = [\underline{e}, \underline{d}]^1,$$

there is an algorithm to calculate $[\underline{e}, \underline{d}]^1 := [\mathrm{R_Q}(\underline{e}), \mathrm{R_Q}(\underline{d})]^1$ (I do not know where to find this in the literature).

Let us recall what the orbit lemma for the quiver-graded Springer maps says.

Lemma 40. *(a) (cp. [Wol09], thm. 5.32) Assume that* $\mathrm{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} = \overline{\mathcal{O}_M}$*. Then, the quiver flag varieties* $\mathrm{Fl_Q}\binom{N}{\underline{\mathbf{d}}}$ *with* $N \in \mathcal{O}_M(K)$ *are pairwise isomorphic smooth and irreducible of dimension*

$$\dim \mathrm{RF}(\underline{\mathbf{d}}) - \dim \mathrm{R_Q}(\underline{d}) + [\mathrm{M}, \mathrm{M}]^1.$$

(b) Let $M \in \mathrm{R_Q}(\underline{d})(\mathrm{K})$ *and* $\underline{\mathbf{d}}$ *a filtration of* $\underline{d}$*. Then, it holds* $\pi\colon \mathrm{RF}(\underline{\mathbf{d}}) \to \mathrm{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$ *is a resolution of singularities of* $\overline{\mathcal{O}_M}$ *if and only if the following two conditions are fulfilled:*

(D1) $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \neq \emptyset$*;*

(D2) $[M, M]^1 = \dim \mathrm{R_Q}(\underline{d}) - \dim \mathrm{RF}(\underline{\mathbf{d}})$

It follows $[M,M]^1 = \operatorname{codim}(\underline{d}^\nu - \underline{d}^{\nu-1}) * \cdots * (\underline{d}^2 - \underline{d}^1) * \underline{d}^1$ *and the restriction* $\pi^{-1}(\mathcal{O}_M) \to \mathcal{O}_M$ *is an isomorphism.*
The condition (D2) is equivalent to

$$[M,M] = \langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle_{KQ\otimes K\mathbb{A}_{\nu+1}} := \sum_{k=0}^{\nu} \langle \underline{d}^k, \underline{d}^k \rangle_{KQ} - \sum_{k=0}^{\nu-1} \langle \underline{d}^k, \underline{d}^{k+1} \rangle_{KQ}$$

which is often easier to check. When Q is without oriented cycles the right hand side defines the Tits form for the algebra $KQ \otimes \mathbb{C}\mathbb{A}_{\nu+1}$ (cp. [Wol09], Appendix).

Remark. Take a sequence of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_\nu = M$ of dimension vector $\underline{\mathbf{d}}$ and define $M_{\nu-k} := N_{k+1}/N_k$, $M' := M_1 \oplus \cdots \oplus M_\nu$. We have

$$\begin{aligned}
[M',M']^1 &= \sum_{k\le l}[M_k,M_l]^1 + \sum_{k>l}[M_k,M_l] \\
&\quad - \sum_{l=1}^{\nu-1}\Big[\sum_{i\in Q_0}\Big(\sum_{k=1}^{\nu-l}\dim_i M_{k+l}\Big)\dim_i M_l - \sum_{(i\to j)\in Q_1}\Big(\sum_{k=1}^{\nu-l}\dim_i M_{k+l}\Big)\dim_j M_l\Big] \\
&= \sum_{k\le l}[M_k,M_l]^1 + \sum_{k>l}[M_k,M_l] - \sum_{l=1}^{\nu-1}\langle \underline{d}^l, \underline{d}^{l+1} - \underline{d}^l\rangle \\
&= \sum_{k\le l}[M_k,M_l]^1 + \sum_{k>l}[M_k,M_l] + \langle \underline{d}, \underline{d}\rangle - \langle \underline{\mathbf{d}}, \underline{\mathbf{d}}\rangle \\
&= \sum_{k\le l}[M_k,M_l]^1 + \sum_{k>l}[M_k,M_l] - (\dim \mathrm{R_Q}(\underline{\mathrm{d}}) - \dim \mathrm{RF}(\underline{\mathbf{d}}))
\end{aligned}$$

So we can replace (D2) by:

(D2)′ There is a filtration of M of dimension $\underline{\mathbf{d}}$ such that for the associated direct sum of subquotients M' :

$$[M',M']^1 - [M,M]^1 = \sum_{k\le l}[M_k,M_l]^1 + \sum_{k>l}[M_k,M_l].$$

Definition 17. Let $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ and $\underline{\mathbf{d}}$ a filtration of $\underline{d} = \underline{\dim}\, M$. Then we call $(M, \underline{\mathbf{d}})$ a **resolution pair** if (D1) and (D2) are fulfilled. We call a resolution pair $(M, \underline{\mathbf{d}})$ **split** if there is a filtration of M of dimension $\underline{\mathbf{d}}$ such that $[M,M]^1 = [M',M']^1$, where M' is the associated direct sum of subquotients.

From the previous remark, we get the following which is inspired by Reineke's observation in [Rei03], Lemma 2.3.

Lemma 41. *Let $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$, $\underline{\mathbf{d}}$ a filtration of $\underline{\dim}\, M$. The following are equivalent:*

(i) $(M, \underline{\mathbf{d}})$ *is a split resolution pair.*

(ii) There is a direct sum decomposition $M = M_1 \oplus M_2 \oplus \cdots \oplus M_\nu$ with

- $\underline{\dim}\, M_{\nu-k} = \underline{d}^{k+1} - \underline{d}^k, 0 \le k \le \nu - 1,$

- $\forall k \leq l$: $[M_k, M_l]^1 = 0$, $\forall k > l$: $[M_k, M_l] = 0$.

*In this case, we say that M has a **directed decomposition**.*

5.2.3 Resolution pairs for Dynkin quivers

Reineke gave a method in [Rei03], how to find such decompositions. Assume that Q is a quiver, M a KQ-module. Then the indecomposable nonzero direct summands (without multiplicities) are contained in a finite subset $\mathcal{R}_M$ of the Auslander Reiten quiver of KQ. We assume, one can find a **directed partition** $I_* = (I_1, \dots, I_s)$ of $\mathcal{R}_M = \bigcup_{t=1}^{s} I_t$, i.e. it holds

(dP1) $\forall t, X, Y \in I_t$: $[X, Y]^1 = 0$.

(dP2) $\forall t < u, X \in I_t, Y \in I_u$: $[Y, X] = 0 = [X, Y]^1$.

For $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ we have a decomposition $M = \bigoplus_{1 \leq t \leq s} \bigoplus_{X \in I_t} X^{a_X}$ with $a_X \in \mathbb{N}_0$. Define $M_t := \bigoplus_{X \in I_t} X^{a_X}$, then $M = M_1 \oplus M_2 \oplus \dots \oplus M_\nu$ gives an example of a directed decomposition.
If M is a directing module, it is always possible to find such an $\mathcal{R}_M$.
In particular for any preprojective or preinjective module M there is a filtration $\underline{\mathbf{d}}$ (depending on M), such that $(M, \underline{\mathbf{d}})$ is a split resolution pair, where we take rigid modules as their own directed decomposition.

Also one can use Reineke's result 5.2.1 to find resolution pairs, as follows.

Lemma 42. *Let Q be a finite quiver, K an algebraically closed field of characteristic zero. Assume it holds $[M] = E_{i_m}^{r_m} * \dots * E_{i_1}^{r_1}$ with pairwise different $i_1, \dots, i_m \in Q_0$, then*

$$(M, \underline{\mathbf{d}} := (0, r_1 e_{i_1}, r_1 e_{i_1} + r_2 e_{i_2}, \dots, \sum_{j=1}^{m} r_j e_{i_j}))$$

is a resolution pair.

Proof: It holds

$$\begin{aligned} \langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle &= \sum_{k=1}^{m} \langle \sum_{j=1}^{k} r_j e_{i_j}, \sum_{j=1}^{k} r_j e_{i_j} \rangle - \sum_{k=1}^{m-1} \langle \sum_{j=1}^{k} r_j e_{i_j}, \sum_{j=1}^{k+1} r_j e_{i_j} \rangle \\ &= [M, M] - [M, M]^1 - \sum_{k=1}^{m-1} \langle \sum_{j=1}^{k} r_j e_{i_j}, r_{k+1} e_{i_{k+1}} \rangle \end{aligned}$$

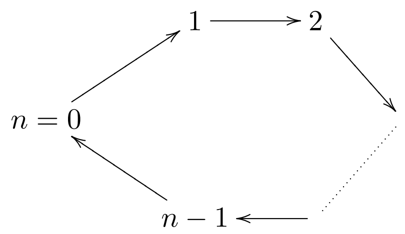
therefore $\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle = [M, M]$ if and only if $\operatorname{codim} \overline{\mathcal{O}_M} = [M, M]^1 = -\sum_{1 \leq j < k \leq m} \langle r_j e_{i_j}, r_k e_{i_k} \rangle$. Now, it holds $[E_{i_k}^{r_k}, E_{i_m}^{r_m} * \dots * E_{i_{k+1}}^{r_{k+1}}] = 0$ because the vertices $i_1, \dots, i_m$ are pairwise different. We can apply the first equality in 5.2.1 successively

$$\begin{aligned}\operatorname{codim}\overline{\mathcal{O}_M} &= \operatorname{codim} E_{i_m}^{r_m} * \cdots * E_{i_2}^{r_2} - \langle r_1 e_{i_1}, \sum_{k=2}^{m} r_k e_{i_k}\rangle \\ &= \operatorname{codim} E_{i_m}^{r_m} * \cdots * E_{i_3}^{r_3} - \langle r_2 e_{i_2}, \sum_{k=3}^{m} r_k e_{i_k}\rangle - \langle r_1 e_{i_1}, \sum_{k=2}^{m} r_k e_{i_k}\rangle \\ &= \cdots = -\sum_{j=1}^{m-1}\langle r_j e_{i_j}, \sum_{k=2}^{m} r_k e_{i_k}\rangle = -\sum_{1\le j<k\le m}\langle r_j e_{i_j}, r_k e_{i_k}\rangle\end{aligned}$$

□

5.2.4 Resolution pairs for the oriented cycle

For this subsubsection we assume Q is the following quiver



The Auslander-Reiten quiver of the nilpotent representations of Q is a tube of rank n, the simples are $E_0, \dots, E_{n-1}$, we will calculate modulo n in the vertices, i.e. $Q_0 = \mathbb{Z}/n$. For $i \in \mathbb{Z}/n$ and $\ell \in \mathbb{N}$ we denote by $E_i[\ell]$ the unique indecomposable with length l and socle E_i. It holds $[E_i, E_{i+1}]^1 = 1$ and $\mathrm{top} E_i[\ell] = E_{i-\ell+1}$. Then we can easily find resolution pairs for $E_i[\ell]$.

Example. Let $E_1, \dots, E_n$ be the simple modules in the mouth of a tube of rank $n \geq 2$ (for KQ with Q extended Dynkin quiver and K algebraically closed). Let $E_i[\ell]$ be the unique indecomposable regular module with a filtration

$$0 = E_i[0] \subset E_i = E_i[1] \subset E_i[2] \subset \cdots \subset E_i[\ell]$$

with $E_i[s+1]/E_i[s] \cong E_{i+s}$ for $0 \leq s \leq \ell - 1$. Then,

$$(E_i[\ell], \underline{\mathbf{d}} := (\underline{\dim}\, E_i[0], \underline{\dim}\, E_i[1], \dots, \underline{\dim}\, E_i[\ell]))$$

is a resolution pair that is not split. We check condition (D2). It holds

$$[E_i[\ell], E_i[\ell]] = 1 + \lfloor \frac{\ell - 1}{n} \rfloor$$

which can be seen by looking at nonzero path from $E_i[l]$ to itself in the Auslander-Reiten quiver of the tube. It also holds

$$\begin{aligned}\langle \underline{\mathbf{d}}, \underline{\mathbf{d}}\rangle &= \sum_{s=0}^{\ell}\langle \underline{\dim}\, E_i[s], \underline{\dim}\, E_i[s]\rangle - \sum_{s=0}^{\ell-1}\langle \underline{\dim}\, E_i[s], \underline{\dim}\, E_i[s+1]\rangle \\ &= \langle \underline{\dim}\, E_i[\ell], \underline{\dim}\, E_i[\ell]\rangle - \sum_{s=0}^{\ell-1}\langle \underline{\dim}\, E_i[s], \underline{\dim}\, E_{i+s}\rangle \\ &= 1 + \lfloor \frac{l-1}{n} \rfloor\end{aligned}$$

to see the last equality use the following

$$\langle \underline{\dim}\, E_i[\ell], \underline{\dim}\, E_i[\ell]\rangle = \begin{cases} 1 & , \ell \not\equiv 0 \bmod n \\ 0 & , \ell \equiv 0 \bmod n \end{cases},$$

$$\langle \underline{\dim}\, E_i[s], \underline{\dim}\, E_{i+s}\rangle = \begin{cases} -1 & , s \equiv n-1 \bmod n \\ 0 & , s \not\equiv n-1 \bmod n \end{cases}$$

see [Rin84], 3.1, p.119. Therefore (D2) holds.

Remember, that we associated to a sequence $\pi = (\pi^{(0)}, \cdots, \pi^{(n-1)})$ with $\pi^{(i)} = (\pi_1^{(i)} \geq \pi_2^{(i)} \geq \cdots \geq \pi_{t_i}^{(i)})$ partitions a module $M(\pi) := \bigoplus_{i=0}^{n-1} \bigoplus_{k=1}^{t_i} E_i[\pi_k^{(i)}]$. Now, we define the submodule of top $= E_i$ summands as

$$M_i(\pi) := \bigoplus_{(j,k) \mid j = i + \pi_k^{(j)} - 1} E_j[\pi_k^{(j)}].$$

Furthermore let $\lambda^{(i)} := (\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots \lambda_{s_i}^{(i)})$ be the partition obtained from a reordering the sequence $(\pi_k^{(j)} \colon j = i + \pi_k^{(j)} - 1)$.
If there exists $r_i := \max\{k \mid \lambda_k^{(i)} > \lambda_1^{(i+1)}\}$, write $M(\pi) = L \oplus \bigoplus_{k \leq r_i} E_{i-\lambda_k^{(i)}-1}[\lambda_k^{(i)}]$, then

$$[M(\pi)] = E_i^{r_i} * [L \oplus \bigoplus_{k \leq r_i} E_{i-\lambda_k^{(i)}-1}[\lambda_k^{(i)} - 1]] \quad \in \mathcal{M}(Q)$$

this follows from [DD05], Prop. 3.7 directly. According to loc. cit., thm 4.1, for every separated $M(\pi)$ there exist $i_1, \ldots, i_m \in \mathbb{Z}/n$ and $r_i \in \mathbb{N}$ such that $[M(\pi)] = E_{i_m}^{r_m} * \cdots * E_{i_1}^{r_1}$ obtained by the method from before. But then by loc. cit. , thm 5.5, there is a unique flag

$$0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M(\pi)$$

such that $M_j/M_{j-1} \cong E_{i_j}^{r_j}, 1 \leq j \leq m$. This together with the factorization of $[M(\pi)]$ imply that

$$(M(\pi), \underline{\mathbf{d}} = (0, r_1 e_{i_1}, r_1 e_{i_1} + r_2 e_{i_2}, \ldots, \sum_{j=1}^{m} r_j e_{i_j}))$$

is a resolution pair. A priori this produces a lot of resolution pairs for $M(\pi)$, if we take in every step of the factorization the minimal $i \in \{0, \ldots, n\}$ such that r_i exists, then we can produce an algorithm to get a uniquely determined dimension filtration for $M(\pi)$. Of course this works as well for any inhomogeneous tube of rank n, but then the E_i are simples in the mouth and not equal to the simples indexed by the vertices of the quiver.

Example. $n = 3$ and $\pi = (\pi^{(0)} = (1,1), \pi^{(1)} = (3), \pi^{(2)} = (2,1))$, this means $M(\pi) = E_0^2 \oplus E_1[3] \oplus E_2[2] \oplus E_2$. The partitions at the tops are $\lambda^{(0)} = (1,1), \lambda^{(1)} = (2), \lambda^{(2)} = (3,1)$. We find $r_2 = 1$, i.e. we replace $E_1[3]$ by $E_1[2]$,

$$M(\pi) = E_2 * M((1,1),(2),(2,1)).$$

Then, the sequence of tops for $M((1,1),(2),(2,1))$ is $\lambda^{(0)} = (2,1,1), \lambda^{(1)} = (2), \lambda^{(2)} = (1)$, we find $r_1 = 1$, i.e. we replace $E_2[2]$ by E_2,

$$M(\pi) = E_2 * E_1 * M((1,1),(1),(2,1)).$$

The sequence of tops for $M((1,1),(1),(2,1))$ is $\lambda^{(0)} = (2,1,1), \lambda^{(1)} = \emptyset, \lambda^{(2)} = (1,1)$, we find $r_0 = 3$ and replace $E_1[2] \oplus E_0^2$ by E_1 and

$$M(\pi) = E_2 * E_1 * E_0^3 * M(\emptyset,(1),(1,1)).$$

The sequence of tops of $M(\emptyset,(1),(1,1))$ is $\lambda^{(0)} = \emptyset, \lambda^{(1)} = (1), \lambda^{(2)} = (1,1)$, we find $r_2 = 2$ and

$$M(\pi) = E_2 * E_1 * E_0^3 * E_2^2 * E_1.$$

and $(M(\pi), \underline{\mathbf{d}} = (0, e_1, e_1+2e_2, e_1+2e_2+3e_0, 2e_1+2e_2+3e_0, 2e_1+3e_2+3e_0))$ is a resolution pair.

5.2.5 Resolution pairs for extended Dynkin quivers

We define two additions for dimension filtrations $\underline{\mathbf{d}} = (0 = \underline{d}^0, \underline{d}^1, \ldots, \underline{d}^\nu =: \underline{d})$, $\underline{\mathbf{e}} = (0 = \underline{e}^0, \underline{e}^1, \ldots, \underline{e}^\mu =: \underline{e})$

$$\underline{\mathbf{d}} + \underline{\mathbf{e}} := \begin{cases} (\underline{d}^0, \underline{d}^1 \ldots, \underline{d}^{\nu-\mu}, \underline{d}^{\nu-\mu+1} + \underline{e}^1, \ldots, \underline{d}^\nu + \underline{e}^\mu) & \nu \geq \mu \\ (\underline{e}^0, \underline{e}^1 \ldots, \underline{e}^{\mu-\nu}, \underline{e}^{\mu-\nu+1} + \underline{d}^1, \ldots, \underline{e}^\mu + \underline{d}^\nu) & \nu < \mu \end{cases}$$

$$\underline{\mathbf{d}} \oplus \underline{\mathbf{e}} := (\underline{d}^0, \underline{d}^1, \ldots, \underline{d}^\nu, \underline{d}^\nu + \underline{e}^1, \ldots, \underline{d}^\nu + \underline{e}^\mu)$$

Observe $\mathbf{Gl}_{\underline{d}+\underline{e}} F_{\underline{\mathbf{d}} \oplus \underline{\mathbf{e}}} = \mathbf{Gl}_{\underline{e}} F_{\underline{\mathbf{e}}} * \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$.

Lemma 43. *Assume $(M, \underline{\mathbf{d}}), (N, \underline{\mathbf{e}})$ are resolution pairs.*

(1) Then $(M \oplus N, \underline{\mathbf{d}} + \underline{\mathbf{e}})$ is a resolution pair if and only if $\langle \underline{\mathbf{d}}, \underline{\mathbf{e}} \rangle + \langle \underline{\mathbf{e}}, \underline{\mathbf{d}} \rangle = [M,N] + [N,M]$ (see corollary 40). In particular, $(M^{\oplus n}, n\underline{\mathbf{d}} = (n\underline{d}^0, n\underline{d}^1, \ldots, n\underline{d}^\nu))$, $n \in \mathbb{N}$ are also resolution pairs.

(2) Then $(M \oplus N, \underline{\mathbf{d}} \oplus \underline{\mathbf{e}})$ is a resolution pair if and only if $[M,N] = 0 = [N,M]^1$.

Proof: It holds $[M\oplus N, M\oplus N] = \langle \underline{\mathbf{d}}, \underline{\mathbf{d}}\rangle + \langle \underline{\mathbf{e}}, \underline{\mathbf{e}}\rangle + [M,N] + [N,M]$.
For (1) just use $\langle \underline{\mathbf{d}}+\underline{\mathbf{e}}, \underline{\mathbf{d}}+\underline{\mathbf{e}}\rangle = \langle \underline{\mathbf{d}}, \underline{\mathbf{d}}\rangle + \langle \underline{\mathbf{e}}, \underline{\mathbf{e}}\rangle + \langle \underline{\mathbf{d}}, \underline{\mathbf{e}}\rangle + \langle \underline{\mathbf{e}}, \underline{\mathbf{d}}\rangle$ and the claim follows from lemma 40.
For (2) use $\langle \underline{\mathbf{d}}\oplus\underline{\mathbf{e}}, \underline{\mathbf{d}}\oplus\underline{\mathbf{e}}\rangle = \langle \underline{\mathbf{d}}, \underline{\mathbf{d}}\rangle + \langle \underline{\mathbf{e}}, \underline{\mathbf{e}}\rangle + \langle \underline{e}^{\mu}, \underline{d}^{\nu}\rangle_{KQ}$ and the claim follows from lemma 40. □

As a special case of the previous lemma one obtains: Let M, N KQ-modules and assume $[M,N] = 0 = [N,M]^1$ and let $M = M_1 \oplus \cdots \oplus M_\nu, N = N_1 \oplus \cdots \oplus N_\mu$ directed decompositions. Then, the decomposition $N \oplus M = N_1 \oplus \cdots \oplus N_\mu \oplus M_1 \oplus \cdots \oplus M_\nu$ is directed again.

Corollary 5.2.2.1. *Let Q be an acyclic extended Dynkin quiver and let $M = P \oplus L \oplus I$ with P preprojective, I preinjective and L regular inhomogeneous separated. Then, by subsection 5.2.3 we find dimension filtrations $\underline{\mathbf{d}}_P$ and $\underline{\mathbf{d}}_I$ such that $(P, \underline{\mathbf{d}}_P)$ and $(I, \underline{\mathbf{d}}_I)$ are resolution pairs. By subsection 5.2.4 we find a dimension filtration $\underline{\mathbf{d}}_L$ such that $(L, \underline{\mathbf{d}}_L)$ is a resolution pair.*
It holds $[I,L] = 0 = [L,I]^1, [L\oplus I, P] = 0 = [P, L\oplus I]^1$ and then by lemma 43 we get $(M, (\underline{\mathbf{d}}_I \oplus \underline{\mathbf{d}}_L) \oplus \underline{\mathbf{d}}_P)$ is a resolution pair.

What about decomposition classes?

Example. Let Q be an extended Dynkin quiver and $\delta \in \mathbb{N}_0^{Q_0}$ the dimension vector of one (and all) regular simple modules. It is easy to find a nice Springer map for the closure of the decomposition class $D(\gamma)$ with $\gamma = (0, \sigma = ((1), \dots, (1)))$. Set $\underline{\mathbf{d}} = (0, \underline{\delta}, \dots, \ell\underline{\delta} =: \underline{d})$, then $\pi_{\underline{\mathbf{d}}} \colon \mathbf{Gl}_{\underline{d}} \times^{P_{\underline{\mathbf{d}}}} F_{\underline{\mathbf{d}}} \to \overline{D(\gamma)} = \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$ restricts over $D(\gamma)$ to a S_ℓ-Galois covering.
(proof: It holds $D(\gamma)$ is dense in $\mathrm{R_Q}(\underline{\mathrm{d}})$ and obviously every element in $D(\gamma)$ has a filtration of submodules of dimension $\underline{\mathbf{d}}$, therefore $\overline{D(\gamma)} = \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}$. Now, every element in $D(\gamma)$ is a direct sum of ℓ regular simple modules of dimension vector δ, it has the obvious ℓ-points in the fibre which are given by leaving out one of the direct summands in each step. We need to see that there is no preprojective submodule of dimension $r\delta$, $1 \leq r \leq \ell - 1$, but such a submodule would have defect zero, therefore all its direct summands have defect zero and have to be regular.)

Lemma 44. *Let $\underline{\mathbf{d}} = (0 = \underline{d}^0, \dots, \underline{d}^\nu =: \underline{d}), \underline{\mathbf{e}} := (0 = \underline{e}^0, \dots, \underline{e}^\mu =: \underline{e})$ be two dimension filtrations. If $\pi_{\underline{\mathbf{d}}} \colon \mathbf{Gl}_{\underline{d}} \times^{P_{\underline{\mathbf{d}}}} F_{\underline{\mathbf{d}}} \to \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}, \pi_{\underline{\mathbf{e}}} \colon \mathbf{Gl}_{\underline{e}} \times^{P_{\underline{\mathbf{e}}}} F_{\underline{\mathbf{e}}} \to \mathbf{Gl}_{\underline{e}} F_{\underline{\mathbf{e}}}$ are generically quasi-finite (i.e. over an open subset the fibres are finite sets) and $[\mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}, \mathbf{Gl}_{\underline{e}} F_{\underline{\mathbf{e}}}] = 0$, then $\pi_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}} \colon \mathbf{Gl}_{\underline{d}+\underline{e}} \times^{P_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}}} F_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}} \to \mathbf{Gl}_{\underline{d}+\underline{e}} F_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}}$ is also generically quasi-finite.*

Proof: The assumption of generically quasi-finiteness gives

$$\dim \mathrm{R_Q}(\underline{\mathrm{d}}) - \dim \mathrm{RF} = \operatorname{codim} \mathbf{Gl}_{\underline{\mathrm{d}}} \mathrm{F}_{\underline{\mathbf{d}}}, \dim \mathrm{R_Q}(\underline{\mathrm{e}}) - \dim \mathrm{RF}(\underline{\mathbf{e}}) = \operatorname{codim} \mathbf{Gl}_{\underline{\mathbf{e}}} \mathrm{F}_{\underline{\mathbf{e}}}$$

and the assumption $[\mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}}, \mathbf{Gl}_{\underline{e}} F_{\underline{\mathbf{e}}}] = 0$ implies by Reineke's result in 5.2.1 that

$$\operatorname{codim} \mathbf{Gl}_{\underline{d}+\underline{e}} F_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}} = \operatorname{codim} \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} + \operatorname{codim} \mathbf{Gl}_{\underline{e}} F_{\underline{\mathbf{e}}} - \langle \underline{d}, \underline{e}\rangle.$$

Then we have

$$\begin{aligned}\dim \mathrm{R_Q}(\underline{\mathrm{d}}+\underline{\mathrm{e}}) - \dim \mathrm{RF}(\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}) &= -\langle \underline{d}+\underline{e}, \underline{d}+\underline{e}\rangle_{KQ} + \langle \underline{\mathbf{d}}\oplus\underline{\mathbf{e}}, \underline{\mathbf{d}}\oplus\underline{\mathbf{e}}\rangle \\ &= -\langle \underline{d},\underline{d}\rangle_{KQ} + \langle \underline{\mathbf{d}},\underline{\mathbf{d}}\rangle - \langle \underline{e},\underline{e}\rangle_{KQ} + \langle \underline{\mathbf{e}},\underline{\mathbf{e}}\rangle - \langle \underline{d},\underline{e}\rangle_{KQ} \\ &= (\dim \mathrm{R_Q}(\underline{\mathrm{d}}) - \dim \mathrm{RF}(\underline{\mathbf{d}})) + (\dim \mathrm{R_Q}(\underline{\mathrm{e}}) - \dim \mathrm{RF}(\underline{\mathbf{e}})) - \langle \underline{\mathrm{d}},\underline{\mathrm{e}}\rangle_{\mathrm{KQ}} \\ &= \operatorname{codim} \mathbf{Gl}_{\underline{d}+\underline{e}} F_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}}\end{aligned}$$

implying $\dim \mathrm{RF}(\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}) = \dim \mathbf{Gl}_{\underline{d}+\underline{e}} F_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}}$ which gives $\pi_{\underline{\mathbf{d}}\oplus\underline{\mathbf{e}}}$ is generically finite. □

Corollary 5.2.2.2. *Let Q be an extended Dynkin quiver. Let $\gamma = (M, \sigma = ((1),(1),\ldots,(1))$ with M_R separated be a decomposition symbol. Then, there exists a dimension filtration $\underline{\mathbf{d}}$ such that $\pi_{\underline{\mathbf{d}}}\colon \mathbf{Gl}_{\underline{d}} \times^{P_{\underline{\mathbf{d}}}} F_{\underline{\mathbf{d}}} \to \mathbf{Gl}_{\underline{d}} F_{\underline{\mathbf{d}}} = \overline{D(\gamma)}$ is finite over $D(\gamma)$.*

Proof: The existence of a dimension filtration $\underline{\mathbf{d}}$ such that $\pi_{\underline{\mathbf{d}}}$ is quasi-finite over $D(\gamma)$ is an immediate consequence, therefore it is enough to prove that the morphism is projective over $D(\gamma)$. $F := F_{\underline{\mathbf{d}}} \cap D(\gamma)$ is a closed $P_{\underline{\mathbf{d}}}$-equivariant subset of $D(\gamma)$, this implies that the collapsing map $\mathbf{Gl}_{\underline{d}} \times^{P_{\underline{d}}} F \to \mathbf{Gl}_{\underline{d}} F = D(\gamma)$ is projective but it is also clear that it is just the restriction of $\pi_{\underline{\mathbf{d}}}$ over $D(\gamma)$. □

This finishes our investigation of quiver-graded Springer theory in this chapter. If we want to find resolutions of singularities for closures of decomposition classes, we need different set-ups, let us start with the easiest ones which are the well-studied Segre classes. As in most of this subsection, we restrict to the case that the reductive group is a (Levi subgroup in a) general linear group. Before we start we need a technical tool, the tube polynomial.

5.3 Springer maps for homogeneous decomposition classes

5.3.1 Tube polynomials

Let Q be an affine quiver (i.e. of type $\tilde{A}, \tilde{D}$ or $\tilde{E}$) and K an algebraically closed field. We write $K[\mathrm{R_Q}(\underline{\mathrm{d}})]$ for the ring of regular functions on the affine space $\mathrm{R_Q}(\underline{\mathrm{d}})$, recall that this is a polynomial ring (in $\sum_{i\to j\in Q_1} d_i d_j$ variables). We call a polynomial $t \in (K[\mathrm{R_Q}(\underline{\mathrm{d}})])[\mathrm{S},\mathrm{T}]$ **tube polynomial** if

(T1) for any not regular homogeneous module $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ it holds $t_M = 0 \in K[S,T]$,

(T2) for a regular homogeneous module $M \cong \bigoplus_{i=1}^r (\bigoplus_{j=1}^{r_i} U_{\alpha_i}[\lambda_j^{(i)}])$, where $\alpha_i = [s_i : t_i] \in \mathbb{P}^1(K), \lambda_j^{(i)} \in \mathbb{N}$ and $U_{\alpha_i}[\lambda_j^{(i)}]$ is the indecomposable module in the tube $\mathcal{T}_{\alpha_i}$ of length $\lambda_j^{(i)}$, it holds

$$t_M = c_M \prod_{i=1}^r (s_i T - t_i S)^{\sum_{j=1}^{r_i} \lambda_j^{(i)}} \in K[S,T]$$

for some $c_M \in K \setminus \{0\}$.

If a tube polynomial t exists, then t_M (i.e. the evaluation as M) is unique up to multiplication by a constant $c_M \in K \setminus \{0\}$ for all $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$, we usually speak of the tube polynomial for M. This is inspired by the following example.

Example. Let Q be the 1-loop (or Jordan) quiver. Every module is regular homogeneous and $K = \{[x : 1] \in \mathbb{P}^1(K) \mid x \in K\}$ as the parametrizing set for the tubes, i.e. the isomorphism class is given by $U_{[x:1]}[n] \cong xE_n + J_n \in M_n(K)$ where $J_n = (0, e_1, e_2, \ldots, e_{n-1})$ with $e_1, \ldots, e_n$ the standard (or any) basis in K^n. Then, for $M \in \mathrm{R_Q(\underline{d})(K)} = \mathrm{M_d(K)}$ we set $t_M := \det(MT - E_nS) \in K[S, T]$, using that $t_M = t_{SMS^{-1}}$ for all $S \in \mathbf{Gl}_d$ and then looking at the Jordan normal form of M gives that this function $t \in K[\mathrm{R_Q(\underline{d})}][\mathrm{S,T}]$ defines a tube polynomial. (In the affine chart $\{T = 1\}$ the characteristic polynomial is a tube polynomial.)

Recall the following

Remark. Let Q be a quiver of type $\tilde{A}_m$, i.e. a cyclic quiver with $m+1$ vertices and $p \geq 0$ arrows in one direction and $q \geq 1$ in the other direction, $p \leq q$. The homogeneous tubes are parametrized by the following sets

(0) If $m = 0$ (Jordan quiver) by $\mathbb{H}_Q := \mathbb{P}^1(K) \setminus \{[0\colon 1]\}$.

(1a) If $m = 1$ and Q the Kronecker quiver by $\mathbb{H}_Q := \mathbb{P}^1(K)$,

(1b) If $m = 1$ and Q the oriented cycle by $\mathbb{H}_Q := \mathbb{P}^1(K) \setminus \{[0 : 1], [1 : 0]\}$

(2a) If $m \geq 2$ and $p = 1$ by $\mathbb{H}_Q := \mathbb{P}^1(K) \setminus \{[0 : 1]\}$,

(2b) If $m \geq 2$ and $p \geq 2$ or $p = 0$ by $\mathbb{H}_Q := \mathbb{P}^1(K) \setminus \{[0 : 1], [1 : 0]\}$.

Recall that the adjoint matrix for $A \in M_n(K)$ is a matrix $A^a \in M_n(K)$ with $AA^a = A^aA = \det(A)E_n$.

Lemma 45. *Let Q be a quiver of type $\tilde{A}_m$, $m \geq 1$, and $p \geq 0$ arrows in one direction and $q \geq 1$ in the other direction, $p \leq q$. We number the vertices clockwise (by $\mathbb{Z}/(m+1)$). If $m \geq 2$ and $M \in \mathrm{R_Q(\underline{d})(K)}$, we write $M_{i,j} = M_{j,i}$ for the linear map associated to the arrow between two neighboring points i and j. We set $\varepsilon\colon \mathbb{Z}/(m+1) \to \{1, a\}$*

$$\varepsilon(i) := \begin{cases} 1 & , \text{ if } i \to i+1 \text{ is clockwise,} \\ a & , \text{ if } i+1 \to i \text{ is counterclockwise.} \end{cases}$$

The following define tube polynomials for the given Q and $\underline{d}$.

(1) For $m = 1$, $\underline{d} = (n, n)$, for $M = (L, R) \in \mathrm{R_Q(\underline{d})(K)} = \mathrm{M_n(K)} \times \mathrm{M_n(K)}$ we set

$$t_M = \begin{cases} \det(LT - RS), & \text{if } Q \text{ is Kronecker,} \\ \det(LR)\det(LRT - S), & \text{if } Q \text{ is an oriented cycle,} \end{cases}$$

(2a) If $m \geq 2, p = 1$ wlog let $1 \to 2$ be the arrow with opposite orientation to the other(s) and $\underline{d} = (n + c, n + c, n, \ldots, n)$ for some $n \in \mathbb{N}, c \in \mathbb{N}_0$, where

$$c_M := \det(M_{1,2}) \det(M_{1,m+1} M_{1,2}^a M_{3,2} M_{4,3} \cdots M_{m+1,m})$$
$$t_M := c_M \det(M_{3,2} M_{4,3} \cdots M_{m+1,m} M_{1,m+1} T - M_{1,2} S)$$

(2b) For $m \geq 2, p \geq 2$, $\underline{d} = (n, n \ldots, n)$

$$c_M := \prod_{i:\, \varepsilon(i)=1} \det M_{i,i+1}$$
$$t_M := c_M \det(M_{m+1,1}^{\varepsilon(m+1)} M_{m,m+1}^{\varepsilon(m)} \cdots M_{1,2}^{\varepsilon(1)} T - (\prod_{i:\, \varepsilon(i)=a} \det M_{i,i+1}) E_n S)$$

These are the all dimension vectors of regular homogeneous modules for quivers of type $\tilde{A}_m$.

Proof:

(1a) Let Q be the Kronecker quiver $1 \rightrightarrows 2$. First, recall that for every $r \in \mathbb{N}_0$ there exists up to isomorphism one preprojective indecomposable module P_r of dimension $(r+1, r)$ and one preinjective indecomposable module I_r of dimension $(r, r+1)$ given by

$$P_r := (L_r := \begin{pmatrix} E_r & 0 \end{pmatrix}, R_r := \begin{pmatrix} 0 & E_r \end{pmatrix}), \quad I_r := (\begin{pmatrix} E_r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ E_r \end{pmatrix}).$$

For any $r \in \mathbb{N}$ there are isomorphism classes of regular indecomposables of dimension (r, r) given by

$$U_{[x:y]}[r] \cong \begin{cases} (xE_r + J_r, yE_r), & y \neq 0 \\ (xE_r, J_r), & y = 0 \end{cases}$$

where $J_r = (0, e_1, e_2, \ldots, e_{r-1}) \in M_r(K)$ and $[x : y] \in \mathbb{P}^1(K)$.

(T2) If (L, R) is regular, then it is isomorphic to a module of the form $(L', R') \in B_{\underline{d}} = B_n(K) \times B_n(K)$ and clearly $t_{L,R} = t_{L',R'}$ is a homogeneous polynomial of degree n.

(T1) Assume (L, R) is not regular, i.e. it has at least one indecomposable preprojective direct summand (and also a preinjective one for dimension reason). Wlog we assume $L = \begin{pmatrix} L_r & 0 \\ 0 & X \end{pmatrix}$, $\quad R = \begin{pmatrix} R_r & 0 \\ 0 & Y \end{pmatrix}$ we get $LT - RS = \begin{pmatrix} TE_r - SJ_r & -Se_r & 0 \\ 0 & 0 & TX - SY \end{pmatrix}$. Then, in the ring $M_n(K(S,T))$ where $K(S,T)$ is the quotient field of $K[S,T]$ multiply the whole matrix by $\frac{1}{T}$, then apply the following column operations: $\frac{S}{T}$-times the r-th to the $r+1$-th, $\frac{S^2}{T^2}$-times the $(r-1)$-th to the $(r+1)$-th,..., $\frac{S^n}{T^n}$-times the first to the $(r+1)$-th. The result is a matrix with the $r+1$-th column is zero, therefore $\det(LT - RS) = 0$ in $M_n(K(S,T))$, but then it is also zero in the subring $M_n(K[S,T])$.

(1b) Now, let Q be the oriented cycle with two vertices.

(T1) A module (L, R) is regular regular homogeneous if and only if $(L, R) \in \mathbf{Gl}_n(K) \times \mathbf{Gl}_n(K)$. Therefore (T1) holds.

(T2) If (L, R) is homogeneous regular, then it is isomorphic to a module (E, A) with $A \in B_n$, it follows that t_M is homogeneous of degree n. The zeroes are the supporting tubes.

(2a) (T2) If M is regular homogeneous it is isomorphic to one of the form with $M_{1,2} = E_{n+c}, M_{3,2} = \begin{pmatrix} A \\ 0 \end{pmatrix} \in M_{(n+c)\times n}(K), M_{4,3} = E_n, \ldots, M_{m,m-1} = E_n, M_{m+1,m} = E_n, M_{1,(m+1)} = \begin{pmatrix} E_n & 0 \end{pmatrix} \in M_{n\times(n+c)}(K)$, for some $A \in B_n$ with diagonal entries $(\underbrace{s_1, \ldots, s_1}_{n_1}, \underbrace{s_2, \ldots, s_2}_{n_2}, \ldots, \underbrace{s_r, \ldots, s_r}_{n_r})$. Then

$$\begin{aligned} t_M &= \det(A)\det\left(\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) T - E_{n+c}S\right) = (-1)^c S^s \det(AT - E_nS) \\ &= [(-1)^c \det(A)]S^c \prod_{i=1}^{r}(s_iT - S)^{n_i}, \end{aligned}$$

this shows (T2).

(T1) Let M be not regular homogeneous and assume $t_M \neq 0$, then $M_{1,2},\ M_{3,2}, \ldots, M_{m+1,m}, M_{1,m+1}$ and $M_{1,m+1}M_{1,2}^a M_{3,2}$ are invertible matrices. In particular, it implies that we get a submodule N of dimension vector $(n, \ldots, n)$ which is homogeneous regular supported on tubes $\{[1 : s_i] \mid 1 \leq i \leq r, s_i \in K \setminus \{0\}\}$. The quotient M/N is regular homogeneous supported in $\{[0 : 1]\}$, as there is no extension between the different tubes, we get $M = N \oplus M/N$ is regular homogeneous. This shows (T1).

(2b) (T1) A module is regular homogeneous if and only if all matrices $M_{i,i+1}$ are invertible. This implies (T1). If $M_{i+1,i}$ is not invertible, then $M^a_{i,i+1}$ is not invertible and it follows by definition that $t_M = 0$ for all modules which are not regular homogeneous.

(T2) Now, assume M is regular homogeneous, as all matrices $M_{i,i+1}$ are invertible. We see that for all $M' \cong M$ there is a $c \in K \setminus \{0\}$ such that $t_M = ct_{M'}$ for an $c \in K \setminus \{0\}$. We can assume wlog $M_{1,2} = A \in B_n, M_\alpha = E_n$, for all $\alpha \neq 2 \to 1, \alpha \neq 1 \to 2$ and we have $t_M = \det(A)\det(AT - E_nS)$ and we see that (T2) is fulfilled.

□

Tube polynomials for arbitrary tame quivers.

This paragraph is roughly due to oral communication by M. Reineke (on 10th of December 2012).

Let Q be a quiver, $\underline{d} \in \mathbb{N}_0^{Q_0}$, $A := KQ$. We recall from [CB92], for every $N \in \mathrm{R_Q}(\underline{\mathrm{d}})$ we have a *standard resolution* by projective modules given by

$$0 \to \bigoplus_{(i\to j)\in Q_1} Ae_j \otimes_K e_iN \to \bigoplus_{i\in Q_0} Ae_i \otimes_K e_iN \to N \to 0$$

We can apply the functor $(-, M) = \operatorname{Hom}_{KQ}(-, M)$ and obtain a four term exact sequence of finite dimensional K-vector spaces

$$0 \to (N,M) \to \left(\bigoplus_{i \in Q_0} Ae_i \otimes_K e_i N, M \right) \xrightarrow{\phi_{N,M}}$$
$$\left(\bigoplus_{(i \to j) \in Q_1} Ae_j \otimes_K e_i N, M \right) \to (N,M)^1 \to 0$$

If $\langle \underline{\dim} N, \underline{\dim} M \rangle := [N,M] - [N,M]^1 = 0$, we consider $\phi_{N,M}$ as a vector space endomorphism of a, let us say, r-dimensional vector space.

Now, assume Q is a tame quiver and $N = U_{[s:\,t]}[1]$ is a simple regular homogeneous module in a tube parametrized by $[s:\,t] \in \mathbb{P}^1(K)$, up to isomorphism N is the only module with this property. For M regular homogeneous module supported on a single tube parametrized by $[s'\colon t'] \neq [s\colon t]$, then it holds $[N,M] = [N,M]^1 = 0$, which means $\phi_{N,M}$ is an isomorphism.

Now we replace $[s\colon t]$ in N by the indeterminants S and T, then

$$\Phi(S,T) := \phi_{N,-}\colon \mathrm{R_Q(\underline{d})} \to \mathrm{M_r(K[S,T])},\ \mathrm{M} \mapsto \phi_{\mathrm{N,M}}.$$

Since $\det \Phi(S,T)$ is a polynomial map $\mathrm{R_Q(\underline{d})} \to \mathrm{K[S,T]}$, we can see the coefficients of the polynomial as regular functions, so $\det\Phi(S,T) \in (K[\mathrm{R_Q(\underline{d})}])[\mathrm{S,T}]$, we write $\det\Phi_M(S,T) := \det \phi_{N,M}$. It is clear that $\det\Phi_M(S,T)$ is a homogeneous polynomial in S, T.

Lemma 46. *Let Q be a tame quiver and let $\underline{n} \in \mathbb{N}_0^{Q_0}$ be with $\langle \underline{n}, \underline{n} \rangle = 0$, let*

$$t := c \cdot \det(\Phi(S,T)) \in (K[\mathrm{R_Q(\underline{d})}])[\mathrm{S,T}]$$

where $c \in K[\mathrm{R_Q(\underline{d})}]$ given by $c_M := \prod_{\alpha \in Q_1} \prod_{I,J} \det((N_\alpha)_{I,J})$ where for $\alpha\colon i \to j$ the sets $I \subset \{1, \dots n_i\}, J \subset \{1, \dots n_j\}$ with $\#I = \max(0, n_i - n_j), \#J = \max(0, n_j - n_i)$ and $(M_\alpha)_{I,J}$ is the minor matrix of M_α given by deleting the rows in I and the columns in J. Then, t is a tube polynomial for Q and $\underline{n}$.

Proof:

(T1) If M has a regular inhomogeneous summand, it holds $c_M = 0$ (this can be seen case by case). We claim: If M is a direct sum of preprojectives and or preinjectives, then $\det\Phi_M(S,T) = 0$. Equivalently, there exist an infinite set of $[s\colon t]$ such that $[N,M] + [N,M]^1 > 0$ where N is the simple regular homogeneous module N in the tube parametrized by $[s\colon t]$.

It is enough to see that indecomposable preinjective or indecomposable preprojective M fullfill it.

Assume $M = Ae_i$, we know $\dim e_i M = n_i \neq 0$, then since the standard resolution is a minimal we conclude $[M, N] > 0$ for every N as before. If M is preprojective, not projective, we have $M = (\tau^{-1})^k Ae_i$ for some $k \in \mathbb{N}$ and τ^{-1} is the inverse of the Auslander-Reiten transpose and $[M, N] = \left[\tau^k M, N\right] = [Ae_i, N] > 0$, the first equality is because the maps are not factorizing through a projective.
We use the Auslander-Reiten formula (see [ASS06b], Thm 2.13, p.117) to see

$$[N, M]^1 = [M, \tau N] = [M, N] > 0$$

If M is injective, the same argument with the standard resolution by injectives (obtained from the standard resolution by projectives) gives $[N, M] > 0$. If M is indecomposable preinjective, we have $M = \tau^k I$ for an indecomposable injective I and $[N, M] = \left[N, \tau^k M\right] = [N, I] > 0$, where the first equality is because the map does not factor over a projective.

(T2) Let M be a regular homogeneous module in $\mathrm{R_Q}(\underline{\mathrm{d}})$, $M \cong \bigoplus_{i=1}^{r}(\bigoplus_{j=1}^{r_i} U_{\alpha_i}[\lambda_j^{(i)}])$, where $\alpha_i = [s_i : t_i] \in \mathbb{P}^1(K), \lambda_j^{(i)} \in \mathbb{N}$ and $U_{\alpha_i}[\lambda_j^{(i)}]$ is the indecomposable module in the tube $\mathcal{T}_{\alpha_i}$ of length $\lambda_j^{(i)}$. We claim

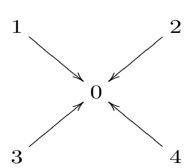
$$t_M = c_M \prod_{i=1}^{r} (s_i T - t_i S)^{\sum_{j=1}^{r_i} \lambda_j^{(i)}} \in K[S, T]$$

It holds $c_M \neq 0$ (by a case distinction) and $\det \Phi_M(S, T) \neq 0$ because it is not zero when we evaluate S, T at a point $[s \colon t] \notin \{\alpha_i \mid 1 \leq i \leq r\}$. In fact, it is easy to see that $\{\alpha_i \mid 1 \leq i \leq r\}$ are the only zeroes of this function (because the dimension of the kernel of the four term short exact sequence is $[U_{\alpha_i}[1], M] > 0$). The rest follows from the observation that for a regular homogeneous module $M_1 \oplus M_2 = M$

$$\det\Phi_M(S, T) = (\det\Phi_{M_1}(S, T)) \cdot (\det\Phi_{M_2}(S, T)),$$

this can be seen by using $\phi_{N, M_1 \oplus M_2} = \phi_{N, M_1} \oplus \phi_{N, M_2}$ by definition of the maps. □

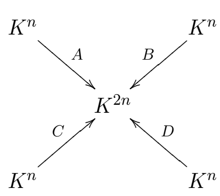
Example. (from Markus Reineke). Let Q be



, and $\underline{n} := \begin{pmatrix} n & & n \\ & 2n & \\ n & & n \end{pmatrix}$.

We name the coordinates of a representation as follows

$M =$



We have $P_0 = S_0$, $\underline{\dim} P_1 = \left(\begin{smallmatrix}1 & & 0\\ & 1 & \\ 0 & & 0\end{smallmatrix}\right)$, $\underline{\dim} P_2 = \left(\begin{smallmatrix}0 & & 1\\ & 1 & \\ 0 & & 0\end{smallmatrix}\right)$, $\underline{\dim} P_3 = \left(\begin{smallmatrix}0 & & 0\\ & 1 & \\ 1 & & 0\end{smallmatrix}\right)$, $\underline{\dim} P_4 = \left(\begin{smallmatrix}0 & & 0\\ & 1 & \\ 0 & & 1\end{smallmatrix}\right)$ determine the indecomposable projective modules. Then the tube polynomial is given by $c \cdot \det\Phi(S,T)$ with $\det\Phi(S,T) =$

$$\det((P_1 \oplus P_2 \oplus P_3 \oplus P_4, M) \xrightarrow{\phi_{[S:\,T]}} (P_0^2, M)) = \det \begin{pmatrix} A & 0 & C & TD \\ 0 & B & C & SD \end{pmatrix}$$
$$\in (K[(A_{i,j}), (B_{i,j}), (C_{i,j}), (D_{i,j})])[S,T]$$

5.3.2 Springer maps for the Jordan quiver

Let us shortly recall Grothendieck's simultaneous resolution of singularities.

Let G be a reductive group, T be a maximal torus contained in a Borel subgroup $B \subset G$ and $\mathrm{Lie}(B) = \mathfrak{b} \subset \mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebras considered as G-representation with a B-subrepresenation via the adjoint operation. Let $\mathfrak{g}^{sr}, \mathfrak{b}^{sr}$ be the sets of regular semi-simple elements and W the Weyl group associated to (G,T). It holds $\mathfrak{g}^{sr}$ is G-equivariant and dense open in $\mathfrak{g}$ and it carries a natural operation of W. Let $B = TU$ with U the unipotent radical and $\mathfrak{n} = \mathrm{Lie}(U)$ its Lie algebra. We call $\mathcal{N} = G\mathfrak{n} \subset \mathfrak{g}$ the nilpotent cone, $\mathcal{N}$ contains a unique dense G-orbit $\mathcal{O}$ consisting of smooth points. Recall, that a resolution of singularities is a projective, dominant, birational map from a smooth variety to a possibly not smooth variety.

Theorem 5.3.1. *([CG97], sections 3.1, 3.2) The Springer map $\pi\colon \widetilde{\mathfrak{g}} := G \times^B \mathfrak{b} \to \mathfrak{g}$ is a G-equivariant, projective map which restricts to*

(1) a W-Galois covering $\pi^{rs}\colon \pi^{-1}(\mathfrak{g}^{sr}) = G \times^B \mathfrak{b}^{sr} \to \mathfrak{g}^{sr}$, i.e. the morphism is etale and a principle W-bundle, and

(2) a resolution of singularities $\pi_{nil}\colon \pi^{-1}(\mathcal{N}) = G \times^B \mathfrak{n} \to \mathcal{N}$ for $\mathcal{N}$, i.e. $\mu^{-1}(\mathcal{O}) \to \mathcal{O}$ is an isomorphism. Also $\pi^{-1}(\mathcal{N}) \cong T^*(G/B)$ and μ can be identified with the moment map.

We can then look at the following more general situation. Let $S \subset W$ be the simple reflections determined by G, B, T. For any $J \subset S$ we set $P_J := B\langle J\rangle B$. Let $P_J = L_J U_J$ be its Levi-decomposition and $\mathfrak{u}_J := \mathrm{Lie}(U_J)$ be the Lie algebra of the unipotent part. There exists G-orbit $\mathcal{O}_J \subset \mathfrak{g}$ such that $\overline{\mathcal{O}_J} = G\mathfrak{u}_J$ and every point in $\mathcal{O}_J$ is smooth in $G\mathfrak{u}_J$.

(1) Are all nilpotent orbits of the form $\mathcal{O}_J$ for some $J \subset S$?

(2) Is the map $\pi_J\colon G \times^{P_J} \mathfrak{u}_J \to \overline{\mathcal{O}_J}$ a resolution of singularities?

I do not know the general answer, a partial answers for (1) is given by Carter in his book [Car85a], section 5.7-5.9, by showing all distinguished nilpotent orbits are of the form $\mathcal{O}_J$. He also gives a classification of nilpotent orbits in terms of weighted Dynkin Diagrams, cp. section 5.6. A different way of putting (1) is asking for Richardson orbits, because a generator for the dense P_J-orbit in $\mathfrak{u}_J$ will also be a generator for the dense G-orbit in $G\mathfrak{u}_J$.

We will at this point restrict to $G = \mathbf{Gl}_n$, $B = B_n$ where we will see that the π_J give a resolutions of singularities for all nilpotent orbit closures (see lemma 47 and 48).

Let $S := \{(1,2), \dots, (n-1,n)\}$ for any subset $J \subset S$ we have

$$P_J = \{A \in \mathbf{Gl}_n \mid \forall i > j : \\ a_{ij} = 0 \text{ whenever there is } k \in \{j, j+1, \dots, i\} \text{ s.t. } (k, k+1) \notin J\},$$

obviously, we can instead determine J by the sizes of the diagonal blocks in P_J.

Recall, that the nilpotent orbits in $M_n(K)$ are in bijection with the partitions of n, via $\lambda = (\lambda_1 \geq \cdots \lambda_s > 0)$ maps to $\mathbf{Gl}_n N_\lambda =: \mathcal{O}_\lambda$ where $N_\lambda \colon K^n \to K^n$, $e_{\lambda_1} \mapsto e_{\lambda_1 - 1} \mapsto \cdots \mapsto e_1 \mapsto 0, e_{\lambda_1+\lambda_2} \mapsto e_{\lambda_1+\lambda_2-1} \mapsto \cdots \mapsto e_{\lambda_1+1} \mapsto 0, \dots, e_n \mapsto e_{n-1} \mapsto \cdots \mapsto e_{1+\sum_{i=1}^{s-1}\lambda_i} \mapsto 0$. For any nilpotent endomorphism N we write $JNF(N) = \lambda$ for the partition λ such that $\mathbf{Gl}_n N = \mathcal{O}_\lambda$. Recall that for any partition λ there is a dual partition λ^t given by the partition associated to the transposition of the Young diagram of λ (i.e. the i-th row is the i-th column in the transposed, $i \geq 1$). For a sequence $J = (n_1, \dots, n_r)$ with $\sum_i n_i = n$ we write $q(J)$ for the partition obtained from reordering J. We have maps

$$\Pi \colon \{J \subset S\} \rightleftarrows \{\lambda = (\lambda_1 \geq \cdots \lambda_s > 0) \mid \sum \lambda_i = n\} \colon J$$

with $\Pi(J) := (q(J))^t$ and J maps a partition λ to the set $J(\lambda)$ such that $P_{J(\lambda)}$ has block sizes given by $(\lambda_1^t, \dots, \lambda_r^t)$ where λ^t is the dual partition of λ. Obviously it holds $\Pi \circ J = \mathrm{id}$, in particular Π is surjective and J is injective.

Lemma 47. *It holds* $\mathbf{Gl}_n \mathfrak{u}_J = \overline{\mathcal{O}_{\Pi(J)}}$ *and therefore also* $\overline{\mathcal{O}_\lambda} = \mathbf{Gl}_n \mathfrak{u}_{J(\lambda)}$.

Proof: We look at $J = (n_1, \dots, n_r)$, wlog. $n_1 \geq n_2 \geq \cdots \geq n_r$ because a permutation of the blocks does not change $\mathbf{Gl}_n \mathfrak{u}_J$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be the dual partition for $(n_1, \dots, n_r)$, we also set $m_i :=$ maximal rank of an element in $\mathfrak{u}_J^i$, it is easy to see $m_i = \sum_{k=i+1}^r n_k$. The dense nilpotent orbit in $\mathbf{Gl}_n \mathfrak{u}_J$ is $\mathcal{O}_N$ with $N \in M_n(K), \mathrm{rk}(N^i) = m_i$. We need to see that N_λ fulfills that, i.e. to see $\mathrm{rk}\, N_\lambda^i = m_i$ for $1 \leq i \leq n$. We know $n_i = \dim\ker N_\lambda^i - \dim\ker N_\lambda^{i-1}, 1 \leq i \leq r$ that implies $\dim\ker N_\lambda^i = \sum_{k=1}^i n_k$. It follows

$$m_i = \sum_{k=i+1}^{r} n_k = n - \sum_{k=1}^{i} n_k = \mathrm{rk}\, N_\lambda^i$$

□

Lemma 48. *The collapsing map* $\pi_J \colon \mathbf{Gl}_n \times^{P_J} \mathfrak{u}_J \to \mathbf{Gl}_n \mathfrak{u}_J$ *is a resolution of singularities.*

Proof: Let N_λ be the dense orbit in $Gl_n \mathfrak{u}_J$ where the block sizes of $\mathfrak{u}_J$ are given by $\lambda^t = (n_1 \geq n_2 \geq \cdots \geq n_r)$. We will see $\mathbf{Gl}_n / P_J$ as the set of partial flags $U^\bullet = (0 = U^0 \subset U^1 \subset \cdots U^r = K^n)$ with $\dim U^i = \sum_{k=1}^i n_i$, and $\mathbf{Gl}_n \times^{P_J} \mathfrak{u}_J = \{(x, U^\bullet) \in M_n(K) \times \mathbf{Gl}_n/P_J \mid x(U^i) \subset U^{i-1}, 1 \leq i \leq r\}$. By lemma 6.3.0.2 it is enough to see that the fibre over N_λ is precisely one point, more precisely we will prove

$$\pi_J^{-1}(N_\lambda) = \{(0 \subset \ker N_\lambda \subset \ker N_\lambda^2 \subset \cdots \subset \ker N_\lambda^r = K^n)\}.$$

So, take any flag $U^\bullet \in \pi_J^{-1}(N_\lambda)$, obviously $N_\lambda(U^1) = \{0\}, \dim U^1 = \dim \ker N_\lambda$ implies $U^1 = \ker N_\lambda$, $N_\lambda(U^2) \subset U^1 = \ker N_\lambda, \dim U^2 = \dim \ker N_\lambda^2$ implies $U^2 = \ker N_\lambda^2$, etc. $\square$

Now, we want a similar result for arbitrary orbit closures. Let $\sigma = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\lambda^{(i)} = (\lambda_1^{(i)} \geq \cdots \geq \lambda_{t_i}^{(i)})$ partitions and $\left|\lambda^{(1)}\right| \geq \cdots \geq \left|\lambda^{(r)}\right|$ such that $\sum_{i=1}^r |\lambda(i)| = n$, we call σ a **Segre symbol**. We write $m_i := \left|\lambda^{(i)}\right|, 1 \leq i \leq r$ and set

$$J(\sigma) = ((\lambda^{(1)})_1^t, (\lambda^{(1)})_2^t, \ldots, (\lambda^{(1)})_{s_1}^t, (\lambda^{(2)})_1^t, \ldots, (\lambda^{(r)})_{s_r}^t),$$

as usual $P_{J(\sigma)}$ for the associated standard parabolic. Let $a_1, \ldots, a_r \in K$ be pairwise different.

$$F_{a_1,\ldots,a_r;\sigma} := \{ \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_r \end{pmatrix} \in M_n(K) \mid A_i - a_i E_{m_i} \in \mathfrak{u}_{J(\lambda^{(i)})}, 1 \leq i \leq r\}$$

$F_{a_1,\ldots,a_r;\sigma}$ is an an affine space and a $P_{J(\sigma)}$-subvariety of $M_n(K)$.

Lemma 49. *Let $A \in M_n(K)$ with characteristic polynomial $\chi_A = \prod_{i=1}^r (T - a_i)^{m_i}$ where $a_i \neq a_j$ for $i \neq j$, $m_1 \geq m_2 \geq \cdots \geq m_r$ and $JNF(A|_{\ker(A-a_iE_n)^{m_i}} - a_i \operatorname{id}_{\ker(A-a_iE_n)^{m_i}}) = \lambda^{(i)}, 1 \leq i \leq r$. Then, it holds $\mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma} = \overline{\mathcal{O}_A}$ and $\pi_{a_1,\ldots,a_r;\sigma} \colon \mathbf{Gl}_n \times^{P_{J(\sigma)}} F_{a_1,\ldots,a_r;\sigma} \to \mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma}$ is a resolution of singularities.*

Proof: Obviously $\mathcal{O}_A \subset \mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma}$ and $\mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma}$ is closed it holds $\overline{\mathcal{O}_A} \subset \mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma}$. If $B \in \mathbf{Gl}_n F_{a_1,\ldots,a_r;\sigma}$, then obviously $\chi_B = \chi_A$ and also from the definition it follows

$$\dim \ker(B - a_i E_n)^j \geq \dim \ker(A - a_i E_n)^j \text{ for all } 1 \leq i \leq r, j \in \mathbb{N},$$

that implies $B \in \overline{\mathcal{O}_A}$. We can identify

$$\mathbf{Gl}_n \times^{P_{J(\sigma)}} F_{a_1,\ldots,a_r;\sigma} = \{(B, U^\bullet) \in M_n(K) \times \mathbf{Gl}_n/P_{J(\sigma)} \mid (B - a_i E_n)(U^j) \subset U^{j-1} \; \forall \sum_{k=1}^{i} s_k < j \leq \sum_{k=1}^{i-1} s_k,\ 1 \leq i \leq r-1\}$$

Then one can see that $\pi^{-1}_{a_1,\ldots,a_r;\sigma}(A)$ is just the single flag

$$\begin{aligned} &0 \subset \ker(A - a_1E) \subset \ker(A - a_1E)^2 \subset \cdots \subset \ker(A - a_1E)^{m_1} \\ &\quad \subset \ker(A - a_1E)^{m_1} + \ker(A - a_2E) \subset \ker(A - a_1E)^{m_1} + \ker(A - a_2E)^2 \subset \cdots \subset K^n. \end{aligned}$$

Then apply lemma 6.3.0.2 $\square$

Now we want to forget about the particular eigenvalues but keep the sizes of the partitions, for a Segre Symbol $\sigma = (\lambda^{(1)}, \dots, \lambda^{(r)})$ we define the **Segre class** to be

$$S(\sigma) := \{A \in M_n(K) \,|\chi_A = \prod_{i=1}^{r} (T-a_i)^{n_i}, a_i \neq a_j, \; \forall i \neq j, n_i = \left|\lambda^{(i)}\right|,$$
$$JNF(A|_{\ker(A-a_iE_n)^{n_i}}) = \lambda^{(i)}\}.$$

It is known that $S(\sigma)$ is a locally closed, irreducible smooth subvariety of $M_n(K)$, admitting rational quotients and that the set of all Segre classes in $M_n(K)$ gives a Whitney stratification (see [HTT08] for the definition and [BD01] for Segre classes being one) which implies in particular that the boundary property holds, which says that the closure of any stratum is a union of strata. We write $\sigma \leq \sigma'$ if $S(\sigma) \subset \overline{S(\sigma')}$. What was the nilpotent cone $\mathcal{N}_n$ for orbits will be replaced by the so called **equipotent cone** $\mathcal{E}_n$ for Segre classes, we define it via

$$F_n := \{A = (a_{ij})_{i,j} \in M_n(K) \mid a_{ij} = 0, a_{ii} = a_{jj} \; \forall i > j\}$$
$$\mathcal{E}_n := \mathbf{Gl}_n F_n \subset M_n(K),$$

F_n is a B_n-subrepresentation of $M_n(K)$, $\mathcal{E}_n$ is a closed and irreducible subset of $M_n(K)$. The map *projection on its eigenvalues up to reorder* is an algebraic morphism $\chi\colon M_n(K) \to \mathbb{C}^n/S_n$, see for example [CG97], 3.1.14, p.132-135 for the Sl_n-case. Therefore by restriction we have an algebraic map $\chi\colon \mathcal{E}_n \to \mathbb{C}$ mapping an equipotent matrix on its only eigenvalue. Thus, we find a morphism $\phi\colon \mathcal{E}_n \to \mathcal{N}_n \times \mathbb{A}^1, \; A \mapsto (A - \chi(A)E_n, \chi(A))$ which is $\mathbf{Gl}_n$-equivariant when you empose the trivial operation on $\mathbb{A}^1$. As the morphism $\mathcal{N}_n \times \mathbb{A}^1 \to \mathcal{E}_n, \; (N, a) \mapsto N + aE_n$ is an inverse, we see that ϕ is an isomorphism of $\mathbf{Gl}_n$-varieties.

Let us first look at equipotent Segre classes, i.e. Segre classes $S(\sigma) \subset \mathcal{E}_n$ equivalently $\sigma = (\lambda)$ for a single partition λ. Let $\mathcal{O}_\lambda \subset \mathcal{N}_n$ be the $\mathbf{Gl}_n$-orbit consisting of nilpotent matrices with JNF $= \lambda$ (for $\lambda = (n)$ we have $\overline{\mathcal{O}_\lambda} = \mathcal{N}_n, \overline{S((\lambda))} = \mathcal{E}_n$). Under the isomorphism ϕ we obviously have $S((\lambda)) = \mathbb{A}^1 \times \mathcal{O}_\lambda$. Now, for $\sigma = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\underline{a} := (a_1, \dots, a_r) \in K^r$ we define

$$q_\sigma(\underline{a}) := \begin{pmatrix} a_1E_{m_1} + N_{\lambda^{(1)}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_rE_{m_r} + N_{\lambda^{(r)}} \end{pmatrix}.$$

Let $U = \{\underline{a} \in K^r \mid a_i \neq a_j \; \forall i \neq j\}$, we have a dominant morphism $\mathbf{Gl}_n \times U \to S(\sigma), \; (g, \underline{a}) \mapsto {}^g q_\sigma(\underline{a})$. As $\mathrm{Stab}_{\mathbf{Gl}_n}(q_\sigma(\underline{a})) = \mathrm{Stab}_{\mathbf{Gl}_n}(q_\sigma(\underline{b})) =: H$ for all $\underline{a}, \underline{b} \in U$ we get an induced morphism

$$\Phi\colon \mathbf{Gl}_n/H \times U \to S(\sigma), \quad (gH, \underline{a}) \mapsto {}^g q_\sigma(\underline{a}),$$

it is $\mathbf{Gl}_n$-equivariant where the $\mathbf{Gl}_n$-operation on $\mathbf{Gl}_n \times U$ is $g \cdot (h, \underline{a}) := (gh, \underline{a})$.

Lemma 50. *Φ is a I-Galois covering, where $I_\sigma := \langle (i, i+1) \in S_r \mid \lambda^{(i)} = \lambda^{(i+1)} \rangle$. Therefore $S(\sigma)$ is smooth of dimension* $\dim \mathbf{Gl}_n - \dim H + r$ *and for $I_\sigma = \{e\}$ it holds Φ*

is an isomorphism.

Proof: It is easy to see that $\Phi^{-1}(q_\sigma(\underline{a})) = I_\sigma \cdot \underline{a} \subset K^r$ where S_r operates on K^r via permuting the coordinates. That implies that Φ has constant fibres with a simply transitive I_σ-operation. $\square$

This leads to the definition

$$F_{(\lambda)} := \{A \in M_n(K) \mid \exists\, a \in K\colon A - aE_n \in \mathfrak{u}_{J(\lambda)}\}$$

which is a $P_{J(\lambda)}$-subrepresentation of $M_n(K)$ and more generally for a Segre symbol $\sigma = (\lambda^{(1)}, \dots, \lambda^{(r)})$,

$$F_\sigma := \{\begin{pmatrix} A_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_r \end{pmatrix} \in M_n(K) \mid \exists a_1, \dots, a_r \in K\colon A_i - a_i E_{m_i} \in \mathfrak{u}_{J(\lambda^{(i)})},\ 1 \le i \le r\},$$

which is a $P_{J(\sigma)}$-subrepresentation of $M_n(K)$.

Lemma 51. *Let $\sigma = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a Segre symbol, $I_\sigma := \langle (i, i{+}1) \in S_r \mid \lambda^{(i)} = \lambda^{(i+1)} \rangle$, it holds $\mathbf{Gl}_n F_\sigma = \overline{S(\sigma)}$ and $\pi_\sigma\colon \mathbf{Gl}_n \times^{P_{J(\sigma)}} F_\sigma \to \mathbf{Gl}_n F_\sigma$ restricts over $S(\sigma)$ to a I_σ-Galois covering.*

Proof: Obviously, $S(\sigma) \subset \mathbf{Gl}_n F_\sigma$ and therefore $\overline{S(\sigma)} \subset \mathbf{Gl}_n F_\sigma$.
Let $B \in \mathbf{Gl}_n F_\sigma$ with r different eigenvalues, then there exists $A \in S(\sigma)$ such that $B \in \overline{\mathcal{O}_A} \subset \overline{S(\sigma)}$ (see proof of lemma 49). As $\{B \in \mathbf{Gl}_n F_\sigma \mid B\ \ r \text{ different eigenvalues}\}$ is dense in $\mathbf{Gl}_n F_\sigma$ we get $\mathbf{Gl}_n F_\sigma = \overline{S(\sigma)}$.
Now, it holds

$$\mathbf{Gl}_n \times^{P_{J(\sigma)}} F_\sigma = \{(A, U^\bullet) \in M_n(K) \times \mathbf{Gl}_n/P_{J(\sigma)} \mid \exists\, a_1, \dots, a_r \in K\colon (A - a_i E_n)(U^j) \subset U^{j-1}\ \forall \sum_{k=1}^{i} s_k < j \le \sum_{k=1}^{i+1} s_k,\ 1 \le i \le r-1\}$$

For $A \in S(\sigma)$ let $\chi_A = \prod_{i=1}^r (T - a_i)^{m_i}$, $JNF((A - a_i E_n)|_{\ker(A - a_i E_n)^{m_i}}) = \lambda^{(i)}, 1 \le i \le r$, then

$$\pi_\sigma^{-1}(A) = \bigsqcup_{\underline{b} \in I_\sigma \underline{a}} \pi^{-1}_{b_1, \dots, b_r; \sigma}(A)$$

consists of I_σ-points $U^\bullet_{A, \underline{b}} \in \mathbf{Gl}_n/P_{J(\sigma)}, \underline{b} \in I_\sigma \cdot \underline{a}$. Now, a morphism of algebraic varieties is generically smooth, therefore this one is generically etale. To see that it is etale over $S(\sigma)$ we show that $\pi_\sigma^{-1}(S(\sigma)) \to S(\sigma)$ is surjective on tangent spaces. Recall H, U, Φ from Lemma 50. Observe that for $\underline{a}, \underline{b} \in U$ it holds $U^\bullet_{q_\sigma(\underline{a}), \underline{a}} = U^\bullet_{q_\sigma(\underline{b}), \underline{b}}$. We define a morphism $\Psi\colon \mathbf{Gl}_n/H \times U \to \pi_\sigma^{-1}(S(\sigma)),\ (g, \underline{a}) \mapsto ({}^g q_\sigma(\underline{a}), g U^\bullet_{q_\sigma(\underline{a}), \underline{a}})$, then we have a commutative diagram

$$\begin{array}{ccc} & \mathbf{Gl}_n/H \times U & \\ {}^{\Psi}\swarrow & & \searrow^{\Phi} \\ \pi_\sigma^{-1}(S(\sigma)) & \xrightarrow[\pi_\sigma]{} & S(\sigma) \end{array}$$

As Φ is surjective on tangent spaces, therefore $\pi_\sigma|_{\pi_\sigma^{-1}(S(\sigma))}$ is surjective on tangent spaces, that implies the smoothness. It also follows that Ψ is an isomorphism. $\square$

Remark. The maps Ψ, Φ gives the following dimension identity, using
$\dim F_\sigma = \dim \mathfrak{u}_{J(\sigma)} + r$ gives

$$\begin{aligned} \dim S(\sigma) = \dim \mathbf{Gl}_n - \dim H + r = \dim \mathbf{Gl}_n - \dim P_{J(\sigma)} + \dim F_\sigma \\ = \dim \mathbf{Gl}_n - \dim L_{J(\sigma)} + r = n^2 + r - \sum_{i=1}^{r} \sum_{s=1}^{s_i} [(\lambda^{(i)})_s^t]^2 \end{aligned}$$

where $L_{J(\sigma)}$ is the Levi part of $P_{J(\sigma)}$. It also gives the (curious) dimension identity $\dim H = \dim L_{J(\sigma)}$, even though the two groups H and $L_{J(\sigma)}$ look quite different.

5.3.3 Springer maps for homogeneous decomposition classes of the Kronecker quiver

Let $n \in \mathbb{N}$, we fix $\underline{d} = (n, n)$ and $\mathbf{Gl}_{\underline{d}} = \mathbf{Gl}_n \times \mathbf{Gl}_n, B_{\underline{d}} = B_n \times B_n$ operating on $\mathrm{R_Q}(\underline{\mathrm{d}}) = \mathrm{M_n}(\mathrm{K}) \times \mathrm{M_n}(\mathrm{K})$ via $(g, h) \cdot (L, R) := (gLh^{-1}, gRh^{-1})$. There is also a $\mathbf{Gl}_2$-operation on $\mathrm{R_Q}(\underline{\mathrm{d}})$ given by

$$(L, R) \cdot \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) := (aL + bR, cL + dR),$$

for $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathbf{Gl}_2, (g, h) \in \mathbf{Gl}_{\underline{d}}$ it holds $[(g, h)(L, R)]A = (g, h)[(L, R)A]$. The $\mathbf{Gl}_2$-operation maps homogeneous decomposition classes to itself. We will introduce now an open covering which will allow us to reduce to the Jordan quiver considerations. For $A \in \mathbf{Gl}_2(K)$ we define

$$U_E := \mathbf{Gl}_n(K) \times M_n(K), \quad U_A := \{(L, R) \in \mathrm{R_Q}(\underline{\mathrm{d}}) \mid \mathrm{(L,R)A^{-1} \in U_E}\} = \mathrm{(U_E)A}.$$

Then, $\{U_A\}_{A \in \mathbf{Gl}_2}$ is an open $\mathbf{Gl}_{\underline{d}}$-invariant covering of the regular homogeneous locus $Reg_{\underline{d}} \subset \mathrm{R_Q}(\underline{\mathrm{d}})$. We have a maps

$$\begin{array}{ll} \Phi_E \colon U_E \rightleftarrows \mathbf{Gl}_n(K) \times M_n(K) \colon \Psi_E & \Phi_A \colon U_A \rightleftarrows \mathbf{Gl}_n(K) \times M_n(K) \colon \Psi_A \\ \Phi_E(L, R) := (L, L^{-1}R) & \Phi_A(L, R) := \Phi_E((L, R)A^{-1}) \\ \Psi_E(X, Y) := (X, XY) & \Psi_A(X, Y) := [\Psi_E(X, Y)]A \end{array}$$

Obviously Φ_A and Ψ_A are inverse isomorphisms of varieties. We consider the right hand side $\mathbf{Gl}_n(K) \times M_n(K)$ with the following $(\mathbf{Gl}_n)^2$-operation $(g, h) \star (X, Y) := (gXh^{-1}, hYh^{-1})$, orbits under this operation are of the form $\mathbf{Gl}_n \times \mathcal{O}_Y$ where $\mathcal{O}_Y \subset M_n(K)$ is a $\mathbf{Gl}_n$-orbit under the conjugation operation. On U_A we have restriction of the $\mathbf{Gl}_{\underline{d}}$-operation on

$\mathrm{R}_Q(\underline{d})$. Then, it holds

$$\Phi_E((g,h)\cdot(L,R)) = (gLh^{-1}, hL^{-1}Rh^{-1}) = (g,h)\star\Phi_E(L,R)$$
$$\Psi_E((g,h)\star(X,Y)) = (gXh^{-1}, gXYh^{-1}) = (g,h)\cdot\Psi_E(X,Y)$$

So, for $M=(L,R)\in U_A$ we have $\Phi_A(\mathcal{O}_M) = \mathbf{Gl}_n\times\mathcal{O}_{(rL+sR)^{-1}(tL+uR)}$ where $A^{-1}=\left(\begin{smallmatrix} r & t\\ s & u\end{smallmatrix}\right)$ and for any homogeneous decomposition class we have $\Phi_E(D(0,\sigma)\cap U_A) = \mathbf{Gl}_n\times S(\sigma)$.

Now for $\sigma=(\lambda^{(1)},\dots,\lambda^{(r)})$, recall that we had defined $J(\sigma) = ((\lambda^{(1)})^t,\dots,(\lambda^{(r)})^t)$, let $\mathfrak{p}_{J(\sigma)} = Lie P_{J(\sigma)})$, we will always write elements $L\in\mathfrak{p}_{J(\sigma)}$ as

$$L=\begin{pmatrix} L_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & L_r\end{pmatrix},\quad L_i\in\mathfrak{p}_{J(\lambda^{(i)})}.$$

For $[\underline{x}\colon\underline{y}] := ([x_1:y_1],\dots[x_r:y_r])\in(\mathbb{P}^1(K))^r$ with $[x_i\colon y_i]\neq[x_j\colon y_j]$ for all $i\neq j$ we set

$$F_{[\underline{x}\colon\underline{y}];\sigma} := \{(L,R)\in\mathfrak{p}_{J(\sigma)}\times\mathfrak{p}_{J(\sigma)}\mid x_iL_i-y_iR_i\in\mathfrak{u}_{J(\lambda^{(i)})}, 1\le i\le r\}$$

It holds $F_{[\underline{x}\colon\underline{y}];\sigma}$ is a $(P_{J(\sigma)})^2$-subrepresentation of $\mathrm{R}_Q(\underline{d})$.

Lemma 52. *Let $\underline{d}=(n,n)$. Let $\sigma=(\lambda^{(1)},\dots,\lambda^{(r)})$ be a Segre symbol with $\sum_{i=1}^r|\lambda(i)|=n$. For $[x_1:y_1],\dots[x_r:y_r]\in\mathbb{P}^1(K)$ pairwise different points set*

$$M=\bigoplus_{i=1}\bigoplus_{j=1}^{r_i} U_{[x_i:y_i]}[\lambda_j^{(i)}].$$

Then, $\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma} = \overline{\mathcal{O}_M}\subset\mathrm{R}_Q(\underline{d})$ and the map

$$\pi_{[\underline{x}\colon\underline{y}];\sigma}\colon\mathbf{Gl}_{\underline{d}}\times^{(P_{J(\sigma)})^2}F_{[\underline{x}\colon\underline{y}];\sigma}\to\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma}$$

is a resolution of singularities.

Proof: Obviously, it holds $\overline{\mathcal{O}_M}\subset\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma}$. There exists an $A\in\mathbf{Gl}_2$ such that $M\in U_A$, i.e. pick $A=\left(\begin{smallmatrix} a & c\\ b & d\end{smallmatrix}\right)$, $\det(A)=1$ such that $y_id-x_ic\neq 0, 1\le i\le r$ and set $a_i:=\frac{x_ia-y_ib}{x_ic-y_id}$. It is enough to see that $\mathcal{O}_M$ is dense in $U_A\cap\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma} = \mathbf{Gl}_{\underline{d}}(F_{[\underline{x}\colon\underline{y}];\sigma}\cap U_A)$. It holds $\Phi_A(F_{[\underline{x}\colon\underline{y}];\sigma}\cap U_A) = P_{J(\sigma)}\times F_{a_1,\dots,a_r;\sigma}, \Phi_A(\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma}) = \mathbf{Gl}_n\times(\mathbf{Gl}_nF_{a_1,\dots,a_r;\sigma})$ and $\Phi_A(\mathcal{O}_M) = \mathbf{Gl}_n\times\mathcal{O}_{(dL-bR)^{-1}(-cL+aR)}$ is dense in $\mathbf{Gl}_n\times(\mathbf{Gl}_nF_{a_1,\dots,a_r;\sigma})$. This implies that $\mathcal{O}_M$ is dense in $U_A\cap\mathbf{Gl}_{\underline{d}}F_{[\underline{x}\colon\underline{y}];\sigma}$.
To see the rest of the lemma we define an $\mathbf{Gl}_{\underline{d}}$-equivariant isomorphism $\widetilde{\Phi_A}$ such that the following diagram, where we set $\underline{a}=(a_1,\dots,a_r)$

$$\begin{array}{ccc} \pi^{-1}_{[\underline{x}\colon\underline{y}];\sigma}(U_A) = \mathbf{Gl}_{\underline{d}}\times^{(P_{J(\sigma)})^2}(F_{[\underline{x}\colon\underline{y}];\sigma}\cap U_A) & \xrightarrow{\widetilde{\Phi_A}} & \mathbf{Gl}_n\times[\mathbf{Gl}_n\times^{P_{J(\sigma)}}F_{\underline{a};\sigma}] \\ \downarrow{\scriptstyle \pi_{[\underline{x}\colon\underline{y}];\sigma}} & & \downarrow{\scriptstyle \mathrm{id}\times\pi_{\underline{a};\sigma}} \\ \mathbf{Gl}_{\underline{d}}(F_{[\underline{x}\colon\underline{y}];\sigma}\cap U_A) & \xrightarrow[\Phi_A]{} & \mathbf{Gl}_n\times\mathbf{Gl}_nF_{\underline{a};\sigma} \end{array}$$

commutes, then the claim follows from lemma 49.
It holds

$$\begin{aligned}\mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} (F_{[\underline{x}:\, \underline{y}];\sigma} \cap U_A) = \{(L,R),(U^\bullet,V^\bullet) \in U_A \times \mathbf{Gl}_{\underline{d}}/(P_{J(\sigma)})^2 \mid \\ L(V^j) \subset U^j, R(V^j) \subset U^j,\ \ (x_iL - y_iR)(V^j) \subset U^{j-1}, \\ \sum_{k=1}^{i} s_k < j \leq \sum_{k=1}^{i+1} s_k, 0 \leq i \leq r-1\} \\ \mathbf{Gl}_n \times^{P_{J(\sigma)}} F_{a_1,\ldots,a_r;\sigma} = \{(A,U^\bullet) \in M_n(K) \times \mathbf{Gl}_n/P_{J(\sigma)} \mid \\ (a_iE_n - A)(U^j) \subset U^{j-1}, \\ \sum_{k=1}^{i} s_k < j \leq \sum_{k=1}^{i+1} s_k, 0 \leq i \leq r-1\}.\end{aligned}$$

We define $\widetilde{\Phi_A}((L,R),(U^\bullet,V^\bullet)) := (\Phi_A(L,R),V^\bullet)$, the inverse is given by $\widetilde{\Psi_A}(X,Y,U^\bullet) := (\Psi_A(X,Y),U^\bullet,XU^\bullet)$. □

Now for homogeneous decomposition classes, the natural thing to look at is

$$\begin{aligned}F_{(\sigma)} &:= \{(L,R) \in (\mathfrak{p}_{J(\sigma)})^2 \mid \exists\, [\underline{x}:\, \underline{y}] \in (\mathbb{P}^1(K))^r \colon x_iL_i - y_iR_i \in \mathfrak{u}_{J(\lambda^{(i)})}, 1 \leq i \leq r\} \\ F^{reg}_{(\sigma)} &:= F_{(\sigma)} \cap Reg_{\underline{d}}.\end{aligned}$$

both are $(P_{J(\sigma)})^2$-invariant subsets of $\mathrm{R_Q}(\underline{\mathrm{d}})$, obviously

$$F_{(\sigma)} = \mathbb{A} \times F_{((\lambda^{(1)}))} \times \cdots F_{((\lambda^{(r)}))}, \quad F^{reg}_{(\sigma)} = \mathbb{A} \times F^{reg}_{((\lambda^{(1)}))} \times \cdots F^{reg}_{((\lambda^{(r)}))},$$

where $\mathbb{A} := \{(L,R) \in (\mathfrak{p}_{J(\sigma)})^2 \mid L_i = 0, R_i = 0, 1 \leq i \leq r\}$ and $F_{((\lambda^{(i)}))}$ is $F_{(\sigma)}$ for the Segre symbol $\sigma = (\lambda^{(i)})$. Observe, that $\mathbf{Gl}_2^r$ operates from the right on $(\mathfrak{p}_{J(\sigma)})^2$ by right multiplictaion on the r diagonal blocks. Now consider the following B_2^r-subrepresentation of $(\mathfrak{p}_{J(\sigma)})^2$

$$F'_{(\sigma)} := \mathbb{A} \times F_{[1:\,0];(\lambda^{(1)})} \times \cdots \times F_{[1:\,0];(\lambda^{(r)})} = \mathfrak{u}_{J(\sigma)} \times \mathfrak{p}_{J(\sigma)}.$$

Then it holds $F_{(\sigma)} = F'_\sigma \cdot \mathbf{Gl}_2^r$ is closed and irreducible subset of $(\mathfrak{p}_{J(\sigma)})^2$. But it is not smooth, therefore $\mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} F_{(\sigma)}$ is not smooth, to overcome this we can either restrict to a smooth subvariety or find an iterated fibre bundle which is smooth. We have the following results.

Lemma 53. *Let $\underline{d} = (n,n)$. Let $\sigma = (\lambda^{(1)},\ldots,\lambda^{(r)})$ be a Segre symbol with $\sum_{i=1}^r |\lambda(i)| = n$, $I_\sigma := \langle (i,i+1) \in S_r \mid \lambda^{(i)} = \lambda^{(i+1)} \rangle$.*

(1) Consider the Springer map associated to $(\mathbf{Gl}_2^r, B_2^r, (\mathfrak{p}_{J(\sigma)})^2, F'_{(\sigma)})$

$$\pi' \colon F'_{(\sigma)} \times^{B_2^r} \mathbf{Gl}_2^r \to F_{(\sigma)}$$

It holds π' is an isomorphism over $F^{reg}_{(\sigma)}$, in particular $F^{reg}_{(\sigma)}$ is smooth. Observe that π' is also $(P_{J(\sigma)})^2$-equivariant.

(2) *It holds* $\mathbf{Gl}_{\underline{d}}F_{(\sigma)} = \overline{D(0,\sigma)}$ *and the map* $\pi_{(\sigma)}\colon \mathbf{Gl}_{\underline{d}} \times^{P_J(\sigma)^2} F_{(\sigma)} \to \mathbf{Gl}_{\underline{d}}F_{(\sigma)}$ *is a* I_σ*-Galois covering over* $D(0,\sigma)$*. It restricts to*

$$\pi^{reg}_{(\sigma)}\colon \pi^{-1}_{(\sigma)}(Reg_{\underline{d}}) = \mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} F^{reg}_{(\sigma)} \to \mathbf{Gl}_{\underline{d}}F^{reg}_{(\sigma)} \subset Reg_{\underline{d}}$$

and $\mathbf{Gl}_{\underline{d}}F^{reg}_{(\sigma)} = (\mathbf{Gl}_{\underline{d}}F_{(\sigma)}) \cap Reg_{\underline{d}}$ *is the closure of* $D(0,\sigma)$ *in* $Reg_{\underline{d}}$.

(3) *We set* $\pi'_{(\sigma)} := \mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} \pi'\colon \mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} [F'_{(\sigma)} \times^{B_2^r} \mathbf{Gl}_2^r] \to \mathbf{Gl}_{\underline{d}} \times^{P_J(\sigma)^2} F_{(\sigma)}$*. It holds* $\mathbf{Gl}_{\underline{d}} \times^{(P_{J(\sigma)})^2} [F'_{(\sigma)} \times^{B_2^r} \mathbf{Gl}_2^r] = (\mathbf{Gl}_{\underline{d}} \times \mathbf{Gl}_2^r) \times^{(P_{J(\sigma)})^2 \times B_2^r} F'_{(\sigma)}$ *and the Springer map*

$$\Pi_{(\sigma)}\colon (\mathbf{Gl}_{\underline{d}} \times \mathbf{Gl}_2^r) \times^{(P_{J(\sigma)})^2 \times B_2^r} F'_{(\sigma)} \to \mathbf{Gl}_{\underline{d}}F'_\sigma \mathbf{Gl}_2 = \overline{D(0,\sigma)}$$

is the composition $\Pi_{(\sigma)} = \pi'_{(\sigma)} \circ \pi_{(\sigma)}$ *and therefore a* I_σ*-Galois covering over* $D(0,\sigma)$*.*

Proof:

(1) $F'_{(\sigma)} \times^{B_2^r} \mathbf{Gl}_2^r = \{(L,R), [\underline{x}\colon \underline{y}] \in (\mathfrak{p}_{J(\sigma)})^2 \times (\mathbb{P}^1)^r \mid x_iL_i - y_iR_i \in \mathfrak{u}_{J(\lambda(i))}, 1 \le i \le r\}$, $(L,R) \in F^{reg}_{(\sigma)}$ implies that (L_i, R_i) is equitubular, $1 \le i \le r$. The map from the equitubular modules to its one supporting tube is regular, we denote it by $(X,Y) \mapsto t(X,Y) \in \mathbb{P}^1(K)$. Then, we have a regular map $F^{reg}_{(\sigma)} \to F'_{(\sigma)} \times^{B_2^r} \mathbf{Gl}_2^r, (L,R) \mapsto (L,R), (t(L_1,R_1), \dots, t(L_r,R_r))$, this is the inverse to $(\pi')^{-1}(F^{reg}_{(\sigma)}) \xrightarrow{\pi'} F^{reg}_{(\sigma)}$.

(2) This follows from intersecting down with the charts $U_A, A \in \mathbf{Gl}_2$, very similar to the proof of the previous lemma.

(3) $(\mathbf{Gl}_{\underline{d}} \times \mathbf{Gl}_2^r) \times^{(P_{J(\sigma)})^2 \times B_2^r} F'_{(\sigma)} =$

$$\begin{aligned}\{ \big((L,R), (U^\bullet, V^\bullet), [\underline{x}\colon \underline{y}]\big) \in \mathrm{R_Q(\underline{d})} \times \mathbf{Gl}_{\underline{d}}/(\mathrm{P_{J(\sigma)}})^2 \times (\mathbb{P}^1)^{\mathrm{r}} \mid \\ L(V^j) \subset U^j, R(V^j) \subset U^j, \quad (x_iL - y_iR)(V^j) \subset U^{j-1}, \\ \sum_{k=1}^{i} s_k < j \le \sum_{k=1}^{i+1} s_k, 0 \le i \le r-1\}\end{aligned}$$

and $\Pi_{(\sigma)}$ is just the projection on the first factor $\big((L,R), (U^\bullet, V^\bullet), [\underline{x}\colon \underline{y}]\big) \mapsto (L,R)$, that obviously factorizes as $\big((L,R), (U^\bullet, V^\bullet), [\underline{x}\colon \underline{y}]\big) \mapsto ((L,R), (U^\bullet, V^\bullet)) \mapsto (L,R)$, which is precisely the claimed factorization.

□

We shortly review some known results on the singularities which occur in the orbit closures.

Classical Springer Theory: Let G be a connected reductive group, $\overline{\mathcal{O}} \subset \mathcal{N}$ the closure of a G-orbit in the nilpotent cone (in $\mathrm{Lie}\, G$).
The singularity is locally factorial, normal Cohen Macaulay, Gorenstein with rational singularities, Springer resolution is crepant, Grothendieck's slice conjecture holds (i.e. if you slice down the the Springer maps over a subregular orbit, you obtain the known crepant ADE-singularity resolution of the same type as the group.)

Quiver-graded Springer Theory: Zwara and Bobinski together with Zwara investigated the geometric properties of orbit closures in the representation space $\mathrm{R_Q}(\underline{\mathrm{d}})$: For arbitrary finite quivers, orbit closures are regular in codimension one (cp. [Zwa05b]).

- For Dynkin quiver of type **A** or **D** orbit closures are normal Cohen Macaulay and have rational singularities in type **A** and rational singularities in characteristic zero in type **D** (cp. [BZ02], [BZ01]). In type **E** only unibranch is proven yet (cp. [Zwa02b]). Orbit closures are regular in codimension two ([Zwa05a]).
  They are not locally factorial, as the following example shows
  $$Q = \mathbb{A}_2, M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$
  The orbit closure is Spec $\mathrm{K[a,b,c,d]/(ad-bc)}$. This is not a factorial ring. The factorization into irreducible elements is not unique, $\overline{ad} = -\overline{bc}$ are two factorizations. (There is a result of Mehta that for locally factorial rings it holds Gorenstein is equivalent to normal and Cohen-Macaulay.)

- For extended Dynkin quiver, the singularities of orbit closures in codimension two are either regular, Kleinian of type **A** or an affine cone over a rational normal curve. For the oriented cycle the singularities are regular or Kleinian of type **A**. For the Kronecker quiver Zwara gave an example (without restrictions on char K) of a representation $M = P \oplus I$ with P indecomposable projective and I indecomposable injective such that the orbit closure has worse singularities than normal (cp. [Zwa03]), the Springer map of the directed decomposition $P \oplus I$ is a resolution of singularities which is not crepant because the singularities in the orbit closure are not Gorenstein (as they are not normal).
  If Q is extended Dynkin and M indecomposable not in a tube of rank ≥ 2, then $\overline{\mathcal{O}_M}$ is normal Cohen Macaulay (cp. [BZ06]).

Open problems: Of course, we would like to find collapsings onto closures of arbitrary decomposition classes (for tame quivers) and use this to study their singularities and if they are unions of decomposition classes. This certainly needs different methods from what we know at the moment.
When is $\overline{\mathcal{O}_M}$ Gorenstein and the collapsing map crepant? Are there general conditions ensuring that?
In case there is a crepant resolution found:
The slice theorem does not make sense for Dynkin quiver, but is interesting for extended Dynkin quivers.
In general of course we would like to find explicit instances for the decomposition theorem, so how does the decomposition of $\pi_* \underline{\mathbb{C}}$ look like for the constructed maps? Can we say something on Steinberg varieties (this is a parabolic case), can we find an example of an $\mathbb{I}$-graded Springer theory?

Chapter 6

Quiver flag varieties of finite type

Summary. Our aim in this section is to investigate quiver flag varieties for Dynkin quivers and in particular those with finitely many orbits. We start with a locally trivial fibre bundle over them and review some geometric properties. For studying irreducible components, we use tangent space method, more precisely we detect the generically smooth irreducible components by looking at dimensions of tangent spaces. It is then applied in different situations, such as for orbits, Reineke strata (definition see later on) and in the last subsection for a simple version of canonical a decomposition for quiver flag varieties. We conjecture that Dynkin quiver flag varieties are generically smooth, so the tangent method would detect all irreducible components.

Notation and definitions:

K	an algebraically closed field,
Q	a finite quiver with vertices Q_0 and arrows Q_1,
KQ	its path algebra,
$\underline{d}$	an element in $\mathbb{N}_0^{Q_0}$,
$\underline{\mathbf{d}}$	$:= (\underline{d}^0 = 0, \underline{d}^1 \ldots, \underline{d}^\nu)$ with $\underline{d}^k \in \mathbb{N}_0^{Q_0}, \underline{d}_i^k \le \underline{d}_i^{k+1}$ for $i \in Q_0$.
$\mathrm{R_Q}(\underline{\mathrm{d}})$	the representation space, defined to be $\prod_{(i\to j)\in Q_1} \mathbb{A}^{d_j d_i}$,
$\mathbf{Gl}_{\underline{d}}$	the linear algebraic group (over K), defined as $\prod_{i\in Q_0} \mathbf{Gl}_{d_i}$, operating on $\mathrm{R_Q}(\underline{\mathrm{d}})$ as follows: For any K-algebra R, $M = (M_{(i\to j)})_{i\to j} \in \prod_{i\to j} \mathrm{Hom}_R(R^{d_i}, R^{d_j})$, $g = (g_i) \in \mathbf{Gl}_{\underline{d}}(R)$ we have $gM := (g_j M_{i\to j} g_i^{-1})_{i\to j}$
$\mathbb{A}_\nu$	the equioriented quiver $1 \to 2 \to \cdots \to \nu$,
Λ	$:= KQ \otimes_K K\mathbb{A}_{\nu+1}$

For any K-algebra Λ and Λ-modules M, N, we will use the following notation

$$(M,N)^i_\Lambda \quad := \mathrm{Ext}^i_\Lambda(M,N),$$
$$[M,N]^i_\Lambda \quad := \dim \mathrm{Ext}^i_\Lambda(M,N),$$

For $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ we define $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ on K-valued points as

$$\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K) := \{U = (0 \subset U^1 \subset \cdots \subset U^\nu = M) \mid \\ U^k \subset U^{k+1} \text{ inclusion of } KQ\text{-modules}, \ \underline{\dim}\, U^k = \underline{d}^k\};$$

in fact one can define a scheme $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ of finite type over K which has $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K)$ as K-valued points (cp. [Wol09]). Except for the tangent spaces we will only work with the K-valued points, i.e. in that case we study the underlying reduced subscheme $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}_{red}$. If $M = \{0\} \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ is the unique semisimple KQ-module, i.e. all linear maps are zero, we set

$$\mathrm{F}(\underline{\mathbf{d}}) := \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}.$$

6.0.4 A locally trivial fibre bundle

An $\mathbb{A}_\nu$-representation in KQ-modules is a sequence of KQ-module morphism

$$M_1 \xrightarrow{A_1} M_2 \to \cdots \xrightarrow{A_{\nu-1}} M_\nu.$$

Specifically $M_s = (M_{is})_{i\in Q_0} \xrightarrow{A_s = (A_{is})_{i\in Q_0}} M_{s+1} = (M_{i,s+1})_{i \in Q_0}$ and for $i \xrightarrow{\alpha} j \in Q_1$ we have $M_{\alpha,s}\colon M_{i,s} \to M_{j,s}$ and $M_{\alpha,s+1}A_{i,s} = A_{j,s}M_{\alpha,s}$, $1 \le s \le \nu - 1$. That is the same as a representation of the quiver $Q_\nu := (Q \times \mathbb{A}_\nu, I)$ with the relations given by commuting squares.

Fix $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ and a filtration $\underline{\mathbf{d}}$ of $\underline{d}$.

$$X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(K) := \{(M_{\alpha,s})_{\alpha\in Q_1, 1\le s \le \nu}, (A_{i,s})_{i\in Q_0, 1 \le s \le \nu-1} \in \mathrm{R}_{Q\times\mathbb{A}_\nu}(\underline{\mathbf{d}})(K) \mid \\ M_\nu = M, M_{\alpha,s+1}A_{i,s} = A_{j,s}M_{\alpha,s}, \mathrm{rk}(A_{i,\nu-1}A_{i,\nu-2}\cdots A_{i,s}) = \underline{d}_i^s\}.$$

The condition on the ranks in the definition ensures that all $A_{i,s}$ are injective. We define

$$\begin{aligned} \phi\colon X_{Q_\nu}\binom{M}{\underline{\mathbf{d}}}(K) &\to \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K) \\ (M_1 \xrightarrow{A_1} M_2 \to \cdots \xrightarrow{A_{\nu-1}} M_\nu) &\mapsto (0 = U_0 \subset U_1 \subset \cdots \subset U_\nu = M) \\ &\quad \text{where } U_s := \mathrm{Im}(A_{\nu-1}A_{\nu-2}\cdots A_s). \end{aligned}$$

Let $G(K) := \prod_{1\le s\le \nu-1} \mathbf{Gl}_{\underline{d}^s}(K)$, it operates on $X_{Q_\nu}\binom{M}{\underline{d}}(K)$ via $(g_s)_s \cdot N = N'$ if there is a commutative diagramm

$$\begin{array}{ccccccccc} N_1 & \longrightarrow & N_2 & \longrightarrow \cdots \longrightarrow & N_{\nu-1} & \longrightarrow & M \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_{\nu-1}} & & \downarrow{\scriptstyle \mathrm{id}_M} \\ N'_1 & \longrightarrow & N'_2 & \longrightarrow \cdots \longrightarrow & N'_{\nu-1} & \longrightarrow & M \end{array}$$

We observe that two points in $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(K)$ have the same image under ϕ if and only if

they are in the same $G(K)$-orbit.
The G-operation is free. The explicit definitions of the scheme structures of $X_{Q^{\otimes\nu+1}}\binom{M}{\underline{\mathbf{d}}}$ and $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ are given later (see subsection 6.4.2).

Lemma 54. *The previously defined map $\phi\colon X_{Q_\nu}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ is a principal G-bundle (of schemes).*

Proof: For $Q = A_1$, this is already known, see for example [Hub13], section 7.1 and 8. Therefore it is true for $M = \{0\}$. But we have a cartesian diagram where the vertical arrows are closed embeddings:

$$\begin{array}{ccc} X_{Q_\nu}\binom{M}{\underline{\mathbf{d}}} & \longrightarrow & X_{Q_\nu}\binom{\{0\}}{\underline{\mathbf{d}}} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi_0} \\ \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} & \longrightarrow & \mathrm{F}(\underline{\mathbf{d}}) \end{array}$$

As ϕ_0 is a principal G-bundle, ϕ is as well. □

Remark. The map ϕ induces a bijection of the irreducible components of $X_{Q_\nu}\binom{M}{\underline{\mathbf{d}}}$ and of $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$. This follows from the next lemma and the property that for a principal G-bundles images of G-invariant closed subsets are closed.

Lemma 55. *Let G be an irreducible group scheme. Let X be a noetherian scheme with G-operation and $f\colon X \to Y$ a dominant morphism with:*

1) *The fibres of f are G-orbits, i.e. for $x \in X$ we have $f^{-1}(f(x)) = G \cdot x$.*

2) *The images of closed G-invariant sets are closed.*

Then, there is a bijection

$$\{ \textit{ irreducible components of } X\} \to \{ \textit{ irreducible components of } Y\}$$
$$C \mapsto f(C)$$

with inverse $Z \mapsto f^{-1}(Z)$.

Proof: The irreducible components of X are G-invariant (for an irreducible component C, the closure of the image of the map $G \times C \to X$ is an G-invariant irreducible closed subset containing C, so it equals C). Let Z be an irreducible component of Y; as f is dominant there is an irreducible component C of X dominating Z. By assumption 2) it follows that $Z = f(C)$. Now, suppose C' is another irreducible component of X with $f(C') \subset Z$. Take $x' \in C'$, then there is an $x \in C$ with $f(x') = f(x)$. Therefore, $x' \in G \cdot x' = G \cdot x \subset C$, proving that $C' = C$. □

6.1 Categories of flags

6.1.1 What is a flag?

Definition 18. Let $\mathcal{C}$ be an abelian category with a forgetful functor to sets and $\nu \in \mathbb{N}$. A sequence of monomorphisms in $\mathcal{C}$

$$U_\bullet = (0 = U_0 \xrightarrow{i_0} U_1 \xrightarrow{j_1} \cdots \xrightarrow{i_{\nu-1}} U_\nu = M)$$

is called **flag** in $\mathcal{C}$ with **flagpole** M of length ν if the monomorphisms under the forgetful functor are the subset inclusions.
A sequence of epimorphisms in $\mathcal{C}$

$$V_\bullet = (M = V_0 \xrightarrow{j_0} V_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{\nu-1}} V_\nu = 0)$$

is called **dual flag** in $\mathcal{C}$ with **flagpole** M if the induced monomorphisms $M/V_i \to M/V_{i+1}$,$1 \leq i \leq \nu - 1$ under the forgetful functor are just subset inclusions.
We define a series of categories. For shortness we leave out the phrase "'*of length* ν"'.

- Let $(\mathbb{A}_{\nu+1}, \mathcal{C})$ be the category of functors from the small category $\mathbb{A}_{\nu+1}$ to $\mathcal{C}$, i.e. an abelian category whose objects we denote by

  $$(T, f) = (T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{\nu-1}} T_\nu)$$

  for objects $T_i, 0 \leq i \leq \nu$ and morphisms $f_i, 0 \leq i \leq \nu - 1$ in $\mathcal{C}$.

- Let $\mathbb{X}$ be the full subcategory of $(\mathbb{A}_{\nu+1}, \mathcal{C})$ whose objects are of the form (T, f) with all f_i are monomorphism. We call it **category of monos** in $\mathcal{C}$.
  Let $\mathbb{Y}$ be the full subcategory of $(\mathbb{A}_{\nu+1}, \mathcal{C})$ whose objects are of the form (T, f) with all f_i are epimorphism. We call it **category of epis** in $\mathcal{C}$.

- Let $\mathcal{X}$ be the full subcategory of $\mathbb{X}$ whose objects are flags of length ν, we call $\mathcal{X}$ the **category of flags** in $\mathcal{C}$. Let $\mathbb{Y}$ be the full subcategory of $\mathbb{Y}$ whose objects are dual flags of length ν. We call $\mathbb{Y}$ the **category of dual flags** in $\mathcal{C}$.

- Let M be an object in $\mathcal{C}$.
  We define $\mathbb{X}^M$ to be the full subcategory of $\mathbb{X}$ with objects (T, f) such that $T_\nu = M$, $\mathcal{X}^M$ be the full subcategory of $\mathbb{X}^M$ whose objects are in $\mathcal{X}$, and call them respectively **category of monos to** M, **category of flags in** M.
  (Analogously you could define the dual versions $\mathbb{Y}^M$, $\mathcal{Y}^M$.)

For M an object in $\mathcal{C}$, the object $M = (M = M = \cdots = M)$ will be considered as a final object of $\mathcal{X}^M$ and $\mathbb{X}^M$. There are equivalences of categories $\mathbb{X} \to \mathbb{Y}, \mathcal{X} \to \mathcal{Y}, \mathbb{X}^M \to \mathbb{Y}^M, \mathcal{X}^M \to \mathcal{Y}^M$ defined via $U_\bullet \mapsto \mathrm{coker}(U_\bullet \to M)$, where $M = U_\nu$.
Furthermore, there is the functor which project on the flagpole $fp\colon \mathbb{X} \to \mathcal{C}$, $(T, f) \mapsto T_\nu$.

From now on let $\mathcal{C}$ be the category $KQ - mod$ of finite-dimensional left modules over the ring KQ. The dimension vectors of objects in $\mathbb{X}$ will be denoted as a filtration $\underline{\mathbf{d}}$.

Remark. We see these categories in an obvious way as *categorifications* of varieties and regular maps in the following way (where we are sloppy with the length ν).

(1) There is a bijection between $\mathrm{Aut}(M)$-orbits on $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$, $\underline{\mathbf{d}}$ dimension filtration of $\underline{\dim}\, M$, and isomorphism classes in $\mathcal{X}^M$.

(2) There is a bijection between $\mathrm{Aut}(M)$-orbits on $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}$, $\underline{\mathbf{d}}$ dimension filtration of $\underline{\dim}\, M$, and isomorphism classes in $\mathbb{X}^M$.

(3) The map $\phi\colon X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ from the previous section corresponds to the inclusion $\mathcal{X}^M \to \mathbb{X}^M$ which is an equivalence of categories (up to the issue with starting flags with 0).

(4) We define

$$\mathrm{RF}(\underline{\mathbf{d}}) := \{(U,M) \in \prod_{i\in Q_0} \mathrm{Fl}(\underline{\mathbf{d}}_\mathrm{i}) \times \mathrm{R_Q}(\underline{\mathrm{d}}) \mid \ \mathrm{M_{(i\to j)}}(\mathrm{U_i^k}) \subset \mathrm{U_j^k},\ \text{for } \mathrm{i} \to \mathrm{j} \in \mathrm{Q_1}, 1 \leq \mathrm{k} \leq \nu\}.$$

There is a bijection between $\mathbf{Gl}_{\underline{d}}$-orbits in $\mathrm{RF}(\underline{\mathbf{d}})$, $\underline{d} \in \mathbb{N}_0^d$ and isomorphism classes of objects in $\mathcal{X}$.

(5) There is a bijection between $\mathbf{Gl}_{\underline{\mathbf{d}}}$-orbits on $\mathrm{R}_{Q\times \mathbb{A}_\nu, I}(\underline{\mathbf{d}})$ and isomorphism classes of $\mathbb{X}$, where I is the ideal such that $KQ \otimes K\mathbb{A}_{\nu+1} = K(Q \times \mathbb{A}_{\nu+1})/I$.

There are also versions of reflection functors for $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ (and $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}$), see [Wol09] and analogously for $\mathcal{X}^M$ (and $\mathbb{X}^M$).
To understand the categories $\mathcal{X}$ and $\mathcal{X}^M$ it is enough to study the categories $(\mathbb{A}_{\nu+1}, \mathcal{C}), \mathbb{X}, \mathbb{X}^M$. We identify $KQ - mod$ with the category of functors $(Q, K - vs)$, where Q is seen as a small category and $K - vs$ denotes the category of finite dimensional K-vector spaces. Then there is an equivalence of categories

$$(\mathbb{A}_{\nu+1}, KQ - mod) \to (KQ \otimes_K K\mathbb{A}_{\nu+1}) - mod.$$

In the next subsection we have a look at the representation type of the tensor algebra $KQ \otimes_K K\mathbb{A}_{\nu+1}$.

6.1.2 On the tensor product $KQ \otimes K\mathbb{A}_n$

Fix numberings $\mathbb{A}_n := (1 \to 2 \to \cdots \to n)$. Let A be a finite dimensional K-algebra. We define the K-algebra of upper triangle $n \times n$-matriceswith coefficients in A to be

$$T_n(A) := \begin{pmatrix} A & A & \cdots & A \\ 0 & A & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{pmatrix}$$

An isomorphism $T_n(K) \cong K\mathbb{A}_n$ induces an isomorphism $T_n(A) \cong A \otimes_K K\mathbb{A}_n$.The next lemma has essentially already been seen in the beginning of the last section, therefore we leave out the proof.

Lemma 56. *Let Q be a quiver bound by an ideal $I \subset KQ$. $T_n(KQ/I) \cong K\Delta/J$, where $\Delta = (\Delta_0, \Delta_1)$ is the following quiver:*

- $\Delta_0 := Q_0 \times \{1, \dots, n\}$,
- *For each $\alpha\colon i \to j \in Q_1, s \in \{1, \dots, n\}$ there is an arrow $\alpha^{(s)}\colon (i, s) \to (j, s) \in \Delta_1$ and for each $i \in Q_0, t \in \{1, \dots, n-1\}$ there is an arrow $\iota^{(i,t)}\colon (i, t) \to (i, t+1) \in \Delta_1$*

and where the generators of J are $\rho^{(s)}, \alpha^{(t+1)}\iota^{(i,t)} - \iota^{(j,t)}\alpha^{(t)}$ with $\rho \in I, s \in \{1, \dots, n\}, (\alpha\colon i \to j) \in Q_1, t \in \{1, \dots, n-1\}$.
In particular, if KQ/I is the incidence *algebra of a poset*[1] *$(\Sigma, \leq)$ then $T_n(KQ/I)$ is the incidence algebra of the poset $(\Sigma \times \{1, \dots, n\}, \preccurlyeq)$ with $(x, s) \preccurlyeq (y, t)$ iff $s \leq t$ and $x \leq y$.*

Notation. In the situation of the previous lemma, we define

$$(Q, I)^{\otimes n} := (\Delta, J), \quad Q^{\otimes n} := (Q, 0)^{\otimes n}.$$

Remark. As we are not the first having a look at this algebras, I give a short collection of some results from the literature about them.

(a) In [Ass06a] one can find the following properties and results. Assume that $A = KQ/I$ is the incidence algebra of a poset. Then

1. A is schurian [2], triangular [3] and semi-commutative [4],
2. one easily sees that $\pi_1(T_n(A)) \cong \pi_1(A)$ using [Ass06a], section 4.1,
3. If A is a tree algebra, then $T_n(A)$ contains no crowns [5]. This is not true for more general algebras, for example $K\mathbb{A}_2^{\otimes 3}$ contains a crown. By [Ass06a], section 7.2, the following are equivalent for the incidence algebra A of a poset:
   * A does not contain any crowns[6]

[1] This is the case iff Q has no oriented cycles and I is the ideal generated by all $v - w$ with (v, w) **contour** (i.e. $v, w \in Q_*$ with the same source and target). Examples are given by KQ with the underlying graph of Q is a tree.

[2] KQ/I **schurian** $:\iff$ $\forall\, v, w \in Q_*$ with same source and target there is $(\lambda, \mu) \in K^2 \setminus \{(0, 0)\}$ with $\lambda v + \mu w \in I$

[3] KQ/I **triangular** $:\iff$ Q has no oriented cycles.

[4] A schurian triangular algebra KQ/I is **semi-commutative** if, for every $v, w \in Q_*$ with same source and target, we have $v \in I$ if and only if $w \in I$.

[5] Let $\Sigma = (S_0, \leq)$ a finite poset and $A = K\Sigma$. A **crown** is a cyclic sequence $(x_1, \dots, x_{n+1} = x_1)$ in S_0 with

(i) The only $\leq$-relations in the sequence are given between neighbours and they are all comparable, i.e. it holds either $x_i \leq x_{i+1} \geq x_{i+2}$ or $x_i \geq x_{i+1} \leq x_{i+2}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.
(ii) In the associated quiver it holds for all $i \in \mathbb{Z}/n\mathbb{Z}$: If z is a point on a path with end points $\{x_i, x_{i+1}\}$ and on a path with endpoints $\{x_{i+1}, x_{i+2}\}$, then $z = x_{i+1}$.

[6] By [Dou98], section 3.3, this is for an incidence algebra of a poset equivalent to be called completely separated. This property we will need later on.

* A is strongly simply connected. [7]

Furthermore, if these properties are fulfilled the following holds:

* A admits a preprojective component, i.e. its Auslander-Reiten quiver contains no oriented cycles and has a preprojective component which is a component which is a union of τ-orbits of projectives. In this case there is a preprojective component such that the τ-orbit quiver is a tree, see [Bon84a].

(b) There is also an isomorphism of (non graded) K-algebras $T_n(KQ) \cong T((KQ)^{(\mathbb{A}_n)_0}, (KQ)^{(\mathbb{A}_n)_1})$, where the right hand side is a tensor algebra considered in [Wol09], Appendix B. There he has also shown that its global dimension is ≤ 2. So, in particular, $T_n(KQ)$ is quasi-hereditary (see definition 19 later).

Theorem 6.1.1. *Let Q be a connected quiver without relations, $Q \neq \mathbb{A}_1$, $n \in \mathbb{N}_{\geq 2}$ and K an algebraically closed field.*

(i) $T_n(KQ)$ is of finite representation type iff one of the following conditions hold.

1) $n = 2$ and Q is Dynkin with graph in $\{A_2, A_3, A_4\}$.

2) $n = 3$ and Q is Dynkin with graph A_2.

3) $n = 4$ and Q is Dynkin with graph A_2.

We have (obviously) $T_3(K\mathbb{A}_2) \cong T_2(K\mathbb{A}_3), T_4(K\mathbb{A}_2) \cong T_2(K\mathbb{A}_4)$.

(ii) $T_n(KQ)$ is tame of infinite representation type iff one of the following hold.

1. $n = 2$ and Q is Dynkin of type A_5 or D_4.

2. $n = 3$ and Q is Dynkin of type A_3.

Remark. One can even prove the following: If Q is a Dynkin quiver with graph A_2, A_3, A_4, then $T_2(KQ)$ is tilted of type D_4, E_6, E_8. According to [Hap87], theorem 5. 12, they are tilted of this type if and only if they are derived equivalent to an algebra of this type. If Q is equioriented, this derived equivalence is part of the ADE-chain folklore, see for the homepage [ADE08]. For the other orientations it is some work to go through the cases (A. Hubery suggests the following: First check that the associated cluster tilted algebras are mutation equivalent to path algebras of Dynkin quivers of these types. Then show that the Grothendieck group does not change when passing to the cluster tilted algebras and use one of the other equivalences in Happel's result loc. cit. again to get the result.).

Remark. Before we discuss the proof, we give a short recall that in certain cases the Tits form is dominating the representation type of an algebra:
We see an algebra $A = KQ/I$ as a category whose objects are the idempotents e_i of A, $i \in Q_0$, and morphisms from e_i to e_j are given by e_jAe_i. A (full) subcategory of A is then of the form $C = eAe$, where $e = \sum_{i \in J} e_i$ for some $J \subset Q_0$; it is called **convex subcategory**

[7] we call a schurian triangular algebra **strongly simply connected** if every full convex subcategory (see remark 6.1.2) is simply connected. This is justified by [Ass06a], section 6.1.

if J is path-closed, i.e. for any path in Q with endpoints in J every vertex on the way is in J.

(a) (see [HV83] and [Bon84b]) Let A be an algebra which admits a simply connected preprojective component. Then, the following are equivalent:

* A is representation-finite.
* A does not contain a convex subcategory which is n-Kronecker ($n \geq 2$) or tame concealed[8]
* q_A is weakly positive.

In this case, there is a bijection between the vertices of the Auslander-Reiten quiver of A and the positive roots of q_A. The second point can be checked explicitly with the list in [HV83].

(b) (see [Sko97], thm 4.1, corollary 4.2) Let A be a strongly simply connected algebra having no convex pg-critical[9] subcategory. Then, the following are equivalent:

* A is tame.
* A has a directed Auslander-Reiten quiver.
* q_A is weakly non-negative.

The quivers with relations of pg-critical algebras are listed in [NS97] explicitly.

(c) (see [Drä94], section 4.3) Let A be a completely separating algebra. Then, the following are equivalent:

* q_A is weakly indefinite.
* A has a convex hypercritical[10] subalgebra.

In this case, it follows that A is wild. The second point can be checked explicitly using the list in [Ung90].

Proof of 6.1.1:

ad (i) A direct consequence of the main theorem in [LS79] is the following:
$T_2(KQ)$ is of finite representation type iff 1) holds true. Therefore, we just need to determine the representation type for $n \in \{3, 4\}$ and Q Dynkin of type $\in \{A_3, A_4\}$. For $n = 3, Q$ of type A_3 one can find the quiver of a tame concealed algebra (cf. [HV83]) as a subquiver, the same argument works for $n = 3, Q$ of type A_4.

[8] An algebra B is called **tame concealed** if there exists a tame connected hereditary algebra and a tilting module T_A which is preprojective (or preinjective) such that $B = \text{End}_A(T)$.

[9] pg-critical= minimal wrt. convex subcategories such that it is not a polynomial-growth algebra, see definition in [Sko97].

[10] hypercritical = minimal wild wrt. convex subcategories

ad (ii) Looking at the list in [Ung90] one finds that in the following cases there is a hypercritical subcategory:

* $n \geq 2$ and Q contains a subquiver of type A_6 or D_5,
* $n \geq 3$ and Q contains a subquiver of type A_4 or D_4,
* $n \geq 4$ and Q contains a subquiver of type A_3.

Therefore, these cases are wild.
For $n = 2$ and Q Dynkin of type A_5 or D_4, and for $n = 3$ and Q Dynkin of type A_3, we can neither find pg-critical subcategories nor hypercritical subcategories inside, see [NS97] and [Ung90]. Therefore, using c) of the previous remark, we get that their Tits forms are not indefinite which implies it is weakly non-negative. Then by b), all of them have to be tame.

□

Remark. Let Λ be a basic connected finite-dimensional algebra of finite global dimension with Gabriel quiver (Q, I) and $\underline{d} \in \mathbb{N}_0^{Q_0}$. If $q_\Lambda(\underline{d}) \leq 0$, then there are infinitely many orbits in $\mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}})$ (where $\mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}}) \subset \mathrm{R}_{\mathrm{Q}}(\underline{\mathrm{d}})$ is the closed variety of representations fulfilling the relations in I). In [Bon83] (see also introduction of [PS99]) the following inequality is proven

$$q_\Lambda(\underline{d}) \geq \dim \mathbf{Gl}_{\underline{d}} - \dim \mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}}).$$

Then by assumption $\dim \mathbf{Gl}_{\underline{d}} \leq \dim \mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}})$, so for any module $M \in \mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}})(\mathrm{K})$ we have:

$$\dim \mathcal{O}_M \leq \dim \mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}}) - [\mathrm{M}, \mathrm{M}] < \dim \mathrm{R}_{(\mathrm{Q,I})}(\underline{\mathrm{d}}),$$

and as there are no dense orbits there have to be infinitely many.

6.1.3 Categories of monomorphisms

Let $\Lambda = KQ \otimes K\mathbb{A}_{\nu+1}$. Now, we have a look at the categories $\mathbb{X} = \mathbb{X}_{Q^{\otimes \nu+1}}$ and $\mathbb{Y} = \mathbb{Y}_{Q^{\otimes \nu+1}}$ as subcategories of Λ-mod. We consider, often without mentioning, the full embedding

$$\begin{aligned} KQ - mod &\to \Lambda - mod, \\ M &\mapsto (M = M = \cdots = M). \end{aligned}$$

Lemma 57. *The following conditions hold.*

(i) *$\mathbb{X}$ is closed under subobjects, $\mathbb{Y}$ is closed under images. In particular both are closed under direct summands.*

(ii) *$\mathbb{X}$ and $\mathbb{Y}$ are closed under extensions.*

(iii) *$\mathbb{X}$ and $\mathbb{Y}$ have the Krull-Schmidt property. Indecomposable objects in $\mathbb{X}$ and $\mathbb{Y}$ are also indecomposable in Λ-mod.*

(iv) $\mathbb{X}$ contains all projective Λ-modules, $\mathbb{Y}$ contains all injective Λ-modules.

(v) For all objects U, U' in $\mathbb{X}$ with flagpoles $U_\nu = M, U'_\nu = M'$, V, V' in $\mathbb{Y}$ with flagpoles $V_0 = N, V'_0 = N'$ it hold

(a) $(V,U) = 0$ and $(U,U') = (M,M'), (V,V') = (N,N')$.

(b) $(U,M')^1 = (M,M')^1_{KQ}$, $(N,V')^1 = (N,N')^1$.

(c) $(U,U')^2 = (V,V')^2 = (U,V)^2 = 0$.

(vi) $\mathbb{X}$ and $\mathbb{Y}$ are functorially finite so they have relative almost split sequences[11].

Proof: (i),(ii),(iii) are straight-forward to see.

(iv) Follows from the description of the projective and injective modules given in [ASS06b], III, 2.4 and 2.6, applied to the quiver described in the previuous section.

(v) Follows from projective and injective dimensions of M, M', N, N' being less or equal 1.

(vi) One can directly write down the left- and right-approximations.

□

6.1.4 Description as Δ-filtered modules over the quasi-hereditary algebra $\Lambda = KQ \otimes K\mathbb{A}_\nu$

The literature background for this paragraph is [Rin91]. We assume that Q is a quiver without loops. Again, by forgetting the zero we see $\mathbb{X}_{Q^{\otimes \nu+1}}$ as a full subcategory $KQ \otimes \mathbb{A}_\nu$-mod. For every $(i,s) \in Q_0 \times \{1, \dots, \nu\} = (Q^{\otimes \nu})_0$ we define a $Q^{\otimes \nu}$-representation $\theta_{i,s}$ via

$$(\theta_{i,s})_{j,t} = \begin{cases} K, & \text{if } i = j, s \leq t, \\ 0, & \text{otherwise.} \end{cases}$$

and the condition that all morphisms are the identity whenever possible. If i is a sink in Q, then all $\theta_{i,s}$ are projective Λ-modules. Denote by q_{ij} the number of $\alpha \in Q_1$ with $\alpha \colon i \to j$, it holds

$$[\theta_{i,s}, \theta_{j,t}]^1 = \begin{cases} q_{i,j}, & \text{if } s \leq t \\ 0, & \text{else} \end{cases}$$

(For example we can use [ASS06b], III, Lemma 2.12 to see this.)

$$[\theta_{i,s}, \theta_{j,t}] = \begin{cases} 1, & \text{if } i = j, s \leq t \\ 0, & \text{otherwise.} \end{cases}$$

[11] see [Rin91] for the definitions of functorially finite and relative almost split sequence.

(This is easy to see directly.)

Let $\theta = \{\theta_{i,s} \mid i \in Q_0, s \in \{1, \dots, \nu\}\}$. We denote by $\mathcal{F}(\theta)$ the full subcategeory of Λ-mod having a filtration in θ. Thus, M belongs to $\mathcal{F}(\theta)$ if and only if M has submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ with M_s/M_{s-1} is isomorphic to a module in θ.

Lemma 58. *Assume that Q is without oriented cycles, then*

$$\mathbb{X} = \mathcal{F}(\theta).$$

Proof: For a KQ-module M we call the support subquiver Q_M to be the maximal subquiver with $M_i \neq 0$ for all $i \in (Q_M)_0$. For a module $U = (U_1 \xrightarrow{i_1} \dots \xrightarrow{i_{\nu-1}} U_\nu)$ in $\mathbb{X}$ take a source i in the support subquiver for U_ν. Let $V_i \subset U_{i,\nu}$ be a codimension 1 subvector space. Then, we can define a subrepresentation $U' \in \mathbb{X}$ of U via

$$(U')_{j,s} := \begin{cases} U_{j,s}, & \text{if } j \neq i, \\ (i_{\nu-1} \cdots i_{s+1} i_s)^{-1}(V_i), & \text{if } j = i, \end{cases}$$

with morphisms such that U' is a subrepresentation of U. Let $s := \min\{k \in \{1, \dots, \nu\} \mid (i_{\nu-1} \cdots i_{k+1} i_k)^{-1}(V_i) \neq U_{i,k}\}$, then we have a short exact sequence

$$0 \to U' \to U \to \theta_{i,s} \to 0,$$

induction on the dimension vector gives then a filtration of U with subquotients in θ.
For the other inclusion observe that Λ-modules can be written as $M = (M_i \xrightarrow{f_\alpha} M_j)_{\alpha \in Q_1}$ where the M_i are $K\mathbb{A}_\nu$-modules and the f_α are $K\mathbb{A}_\nu$-linear maps. Then, M is in $\mathbb{X}$ if and only if M_i is a projective $K\mathbb{A}_\nu$-module for each $i \in Q_0$. Let now $M = (M_i \xrightarrow{f_\alpha} M_j)_{\alpha \in Q_1}, N = (N_i \xrightarrow{g_\alpha} N_j)_{\alpha \in Q_1}$ be in $\mathcal{F}(\theta)$ and let

$$0 \to N \to M \to \theta_{i,s}^a \to 0$$

be a short exact sequence. At the vertex i we get a short exact sequence of $K\mathbb{A}_\nu$-modules

$$0 \to N_i \to M_i \to P(s)^a \to 0.$$

As $P(s)^a$ is projective the sequence is split, this implies $M_i \cong N_i \oplus P(s)^a$. Fixing a numbering $\theta_1, \dots, \theta_n$ of the set θ such that $[\theta_j, \theta_i]^1 = 0$ for all $j \geq i$ ensures that we can find a filtration $0 = X_{n+1} \subset X_n \subset \cdots \subset X_1 = M$ such that $X_i/X_{i+1} \cong \theta_i^{m_i}$. Now, using this filtration we see with the previous argument that each M_i is projective, so M is in $\mathbb{X}$.
□

Definition 19. Lett A be an artin algebra with (representatives of the isomorphism classes of the) simple modules $S_1, \dots, S_n$ and their projective covers $P_1, \dots, P_n$. Then, one defines

$$\Delta(i) := P_i / \left(\sum_{j>i} \operatorname{Im}(P_j \to P_i) \right)$$

where the sum goes over all A-linear maps $P_j \to P_i$. The algebra A is called **quasi-hereditary** with respect to the ordering $S_1, \dots, S_n$ if the following two conditions hold:

(1) $A \in \mathcal{F}(\Delta)$, where $\Delta = \{\Delta(i) \mid 1 \leq i \leq n\}$

(2) For all $i \in \{1, \dots, n\}$: S_i occurs with multiplicity one in a composition series of $\Delta(i)$.

If A is quasi-hereditary wrt. $S_1, \dots, S_n$, we call $\Delta(i), 1 \leq i \leq n$ the **standard modules** for A.

Remark. If Q has no oriented cycles, the set Θ has a partial ordering such that $[\theta_i, \theta_j]^1 = 0$ whenever $i \geq j$, and for all elements $\theta, \theta' \in \Theta$, all homomorphism between θ and θ' are zero or invertible. Let us fix a total ordering refining this partial ordering, wlog. $\Theta = \{\theta_1, \dots, \theta_n\}$ and the total ordering is the natural one on $\{1, \dots, n\}$. By [DR92], theorem 2, p.14: There exists a quasi-hereditary algebra A such that $\mathcal{F}(\Theta) = \mathbb{X}$ is the category of Δ-filtered modules.

Remember that we have fixed a numbering $\theta_1, \dots \theta_n$ in the previous remark. This means that we have chosen a numbering of the vertices of the quiver associated to $KQ \otimes K\mathbb{A}_\nu$.

Lemma 59. *Let Q be without oriented cycles. The algebra $A = KQ \otimes K\mathbb{A}_\nu$ has the structure as quasi-hereditary algebra with respect to the previously fixed numbering of the vertices.*
The category $\mathbb{X}$ is the category of Δ-filtered modules with respect to this quasi-hereditary structure.

Proof: Set $\Lambda := KQ \otimes K\mathbb{A}_\nu$. It is enough to prove that all projective Λ-modules are in $\mathbb{X}$, because then one can take in the proof of theorem 2, [DR92] the projective covers of $P_i \to \theta_i$ for the $P_\theta(i)$. By the theorem we get an identification of $\mathbb{X}$ with Δ-filtered modules for $A^{op} = \operatorname{End}_A(A) = \operatorname{End}_A(\bigoplus_{i=1}^n P_i)$ via the functor

$$A - mod \to A^{op} - mod, \quad X \mapsto \operatorname{Hom}_A(A, -).$$

But then clearly, applying the functor with exchanged roles of A and A^{op} again implies the claim of the lemma.
One can use the explicit recipe given in [ASS06b], III, lemma 2.4, to see that the indecomposable projectives $P(i,s),\ i \in Q_0, s \in \{1, \dots, s\}$ fulfill: For each $\alpha\colon (j,t) \to (j,t+1)$ we consider the map

$$P(i,s)_\alpha \colon P(i,s)_{(j,t)} \to P(i,s)_{(j,t+1)}, \quad w + I \mapsto w\alpha + I$$

where I is the ideal given by the commuting squares and w runs through path from (i,s) to (j,t) in the quiver $Q^{\otimes\nu}$. It holds that $w\alpha \in I$ implies $w \in I$. In other words $P(i,s)_\alpha$ is a monomorphism. Therefore, $P(i,s)$ is in $\mathbb{X}$. □

6.1.5 When is $\mathbb{X}$ representation-finite?

We say $\mathbb{X}$ is representation-finite if it has up to isomorphism only finitely many indecomposable objects.

Theorem 6.1.2. *The category $\mathbb{X}$ is representation-finite iff the pair Q,ν is one of the following*

(i) Q is of type A_2, ν arbitrary, and the indecomposables of $\mathbb{X}$ are

$$(P(i) \to P(j)), (P(s) \to 0), (0 \to P(t)), \text{ with } i \le j \text{ and } i,j,s,t \in \{1,\dots,\nu\}$$

where $P(i)$ denotes the projective $K\mathbb{A}_\nu$-module corresponding to i.

(ii) Q is of type A_3, $\nu = 3$,

(iii) $Q = \mathbb{A}_3$, $\nu = 4$,

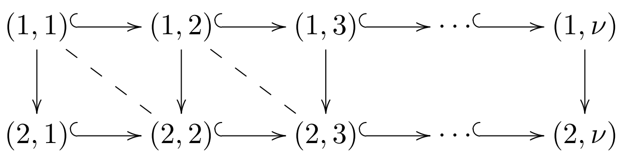
(iv) Q is of type A_4, $\nu = 2$.

(v) Q is of type A_5 equi-oriented or exactly one source or sink not in outer points or the middle , $\nu = 2$. [12]

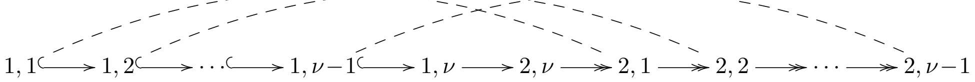
The answers for type A_2 and type A_3 have kindly been explained to me by W. Crawley-Boevey.

Sketch of proof: As $\mathbb{X}$ is equivalent to $\mathbb{Y}$ which is by reversing all arrows equivalent to $\mathbb{X}$ for the opposite quiver, we can ignore the opposite quivers.

- **Positive answers:** We write $\hookrightarrow$ (or $\twoheadrightarrow$) in a quiver when we mean the category of representations with this linear map injective (or surjective). There is an equivalence between the categories of representations of

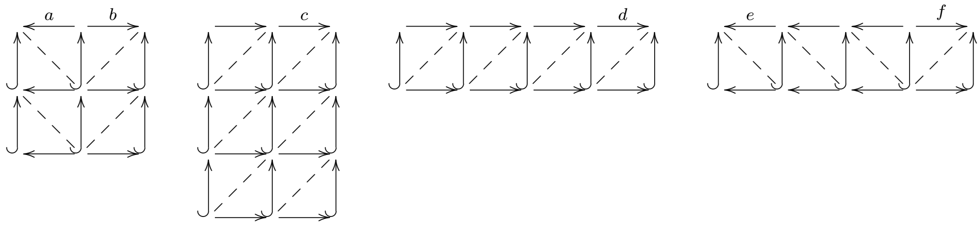


and this quiver, where the dotted arcs are zero-relations

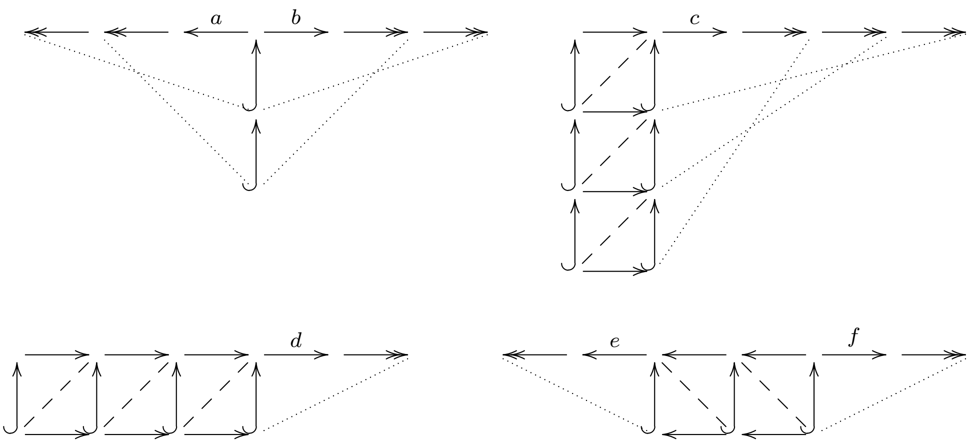


[12]In this case I was not able to find a dimension vector with Tits form is zero, I do not really have a proof that they belong to this list.

A representation of the first quiver is a linear map $f\colon X_\nu \to Y_\nu$ mapping a flag of subspaces $X_1 \subset \cdots \subset X_\nu$ to another flag of subspaces $Y_1 \subset \cdots \subset Y_\nu$. The functor to obtain a representation of the second quiver is to replace the second flag by its dual flag $Y \twoheadrightarrow Y/Y_1 \twoheadrightarrow \cdots \twoheadrightarrow Y/Y_{\nu-1}$, then the condition that $f(X_\bullet) \subset Y_\bullet$ translates into the zero-relations. As the category of representations of the last quiver is well-known to be representation-finite (i) follows.
With a similar argument we substitute the quivers

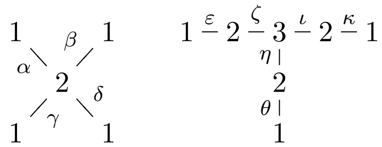


by the following (the dotted lines are the zero relations):



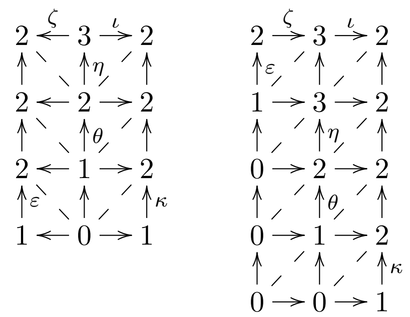
All algebras described by the last four quivers (consider the arrows $\twoheadrightarrow$ as usual arrows) are quotients of incidence algebras of posets not containing any crowns. By results in [Ass06a],[Drä94] these algebras admit a preprojective component whose τ-orbit quiver has as underlying graph a tree, which implies the preprojective component is simply connected and we can apply 6.1.2 (a).
The quivers do not contain a convex subcategory which is $m(\geq 2)$-Kronecker, nor one that can be found in the list [HV83], this gives a proof of finite representation type.

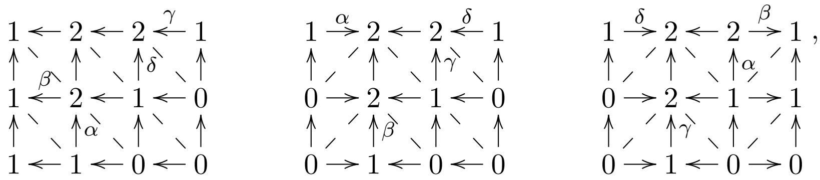
- **negative answers:** We give a list of dimension vectors written on the vertices of the quivers $Q^{\otimes \nu}$, which give negative answers for all quivers not in the list (i)-(v). We will choose infinite families of indecomposable $\tilde{D}_4$-representations and $\tilde{E}_6$-representations of the form

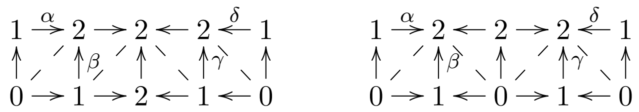


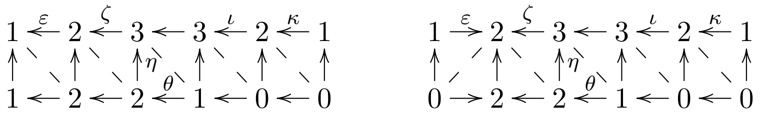
(the direction of the arrows are specified in the examples) subject to the condition

$\beta\alpha \neq 0,\ \varepsilon\zeta\eta\theta \neq 0$ if the composition is welldefined. It might help to use for the outer points of the quiver which are sinks the previous technique of *rolling out an arm*.





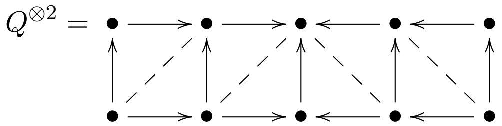




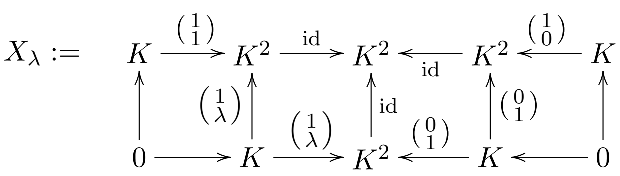
Example. An explicit example for the previously discussed indecomposables.

$$Q = \ 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5$$

and $\nu = 2$ (ignoring the zero representation). One can see $\mathbb{X}_{Q^{\otimes \nu+1}}$ as a subcategory of $(\mathbb{A}_\nu, \mathcal{C})$. It holds $KQ \otimes K\mathbb{A}_2 = KQ^{\otimes 2}$ with



Then, according to [HV83], case $\tilde{\mathbb{E}}_7$, the dimension vector $\underline{\mathbf{d}} = \left(\begin{smallmatrix}1&2&2&2&1\\0&1&2&1&0\end{smallmatrix}\right)$ is a root spanning the radical. There are infinitely many isomorphism classes of indecomposable $KQ^{\otimes 2}$-modules of dimension vector $\underline{\mathbf{d}}$.
For example, we can find the following family of pairwise non isomorphic indecomposable modules, for $\lambda \in K \setminus \{0\}$ define



Remark. For $Q = \mathbb{A}_2$ we saw that the associated category $\mathbb{X}$ is representation finite. One can also check directly that the Tits form is positive definite on this category.
For any $\nu \in \mathbb{N}$, and $\underline{\mathbf{d}} = \left(\begin{smallmatrix} d^0=0 & d^1 & \cdots & d^\nu \\ e^0=0 & e^1 & \cdots & e^\nu \end{smallmatrix}\right) \neq 0$ with $d^k \leq d^{k+1}, e^k \leq e^{k+1}$ for all $k \in \{0, \ldots, \nu-1\}$, we have

$$\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle > 0.$$

Furthermore, if X in $\mathbb{X}_{\mathbb{A}_2^{\otimes \nu+1}}$ is a module with $\langle \underline{\dim}\, X, \underline{\dim}\, X \rangle \leq 1$, then it holds that $d^\nu, e^\nu \in \{0,1\}$. Especially, all bricks fulfill this.
For the proof just consider.

$$\begin{aligned}
\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle &= \sum_{k=0}^{\nu} d^k d^k + \sum_{k=0}^{\nu} e^k e^k - \sum_{k=0}^{\nu-1} d^{k+1} d^k - \sum_{k=0}^{\nu-1} e^{k+1} e^k - \sum_{k=0}^{\nu} d^k e^k + \sum_{k=0}^{\nu-1} d^k e^{k+1} \\
&= \sum_{k=0}^{\nu-1} d^{k+1}(d^{k+1} - d^k) + \sum_{k=0}^{\nu-1} e^{k+1}(e^{k+1} - e^k) - \sum_{k=0}^{\nu-1} d^k (e^{k+1} - e^k) - d^\nu e^\nu \\
&= \sum_{k=0}^{\nu-1} d^{k+1}(d^{k+1} - d^k) + \sum_{k=0}^{\nu-1} e^{k+1}(e^{k+1} - e^k) - \sum_{k=0}^{\nu-1} (d^\nu - d^k)(e^{k+1} - e^k)
\end{aligned}$$

We set $x_k := d^{k+1} - d^k, y_k := e^{k+1} - e^k$, $0 \leq k \leq \nu - 1$, $A := (a_{ij})_{ij}$ with $a_{ij} = 1$ if $j \geq i$, $a_{ij} = 0$ if $j < i$, then it holds that

$$\begin{aligned}
\langle \underline{\mathbf{d}}, \underline{\mathbf{d}} \rangle &= \sum_k x_k \sum_{l \leq k} x_l + \sum_k y_k \sum_{l \leq k} y_l - \sum_k y_k \sum_{l \geq k} x_l \\
&= {}^t x {}^t A x + {}^t y {}^t A y - {}^t y A x \\
&= {}^t (x-y) A (x-y) + {}^t x A y
\end{aligned}$$

It holds $B = \frac{1}{2}(A + {}^t A)$ is positive definite. This is easy to see using the fact that main minors all have the same form as B and $\det B = \prod_{k=1}^{\nu} \frac{k+1}{2k} > 0$. So, if $x \neq y$ the first summand is positive; if $x = y$ the second summand is positive.

6.2 Tangent methods

Tangent spaces of the (not necessarily reduced) scheme $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ of finite type over K have been described by Stefan Wolf ([Wol09], Lemma 5.23). He showed that for $U \in \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}(K)$ the tangent space at U is

$$T_U \mathrm{Fl}_Q \binom{M}{\underline{\mathbf{d}}} = (U, M/U)_\Lambda$$

where M is considered as Λ-module via $M \xrightarrow{id} M \xrightarrow{id} \cdots \xrightarrow{id} M$.
Here it is important to let the flags start at 0 because then

$$M/U = \left(M \twoheadrightarrow M/U^1 \twoheadrightarrow M/U^2 \twoheadrightarrow \cdots \twoheadrightarrow M/U^\nu = 0\right).$$

Remark. We set $\mathrm{Gr}_\Lambda\binom{M}{\underline{\mathbf{d}}}(K) := \{U \mid U\ \Lambda-\text{submodule of } M, \underline{\dim}\, U = \underline{\mathbf{d}}\}$.
In his proof of [Wol09], Lemma 5.23, Stefan Wolf found an isomorphism (very similar to

the map ϕ below)

$$\mathrm{Gr}_\Lambda\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}, \quad (U_0 \xrightarrow{i_1} \cdots \xrightarrow{i_\nu} U_\nu) \mapsto (U_0 \subset U_1 \subset \cdots \subset U_\nu = M),$$

it is welldefined as the maps i_j are restrictions if id_M.

6.2.1 An example of a not generically reduced quiver flag variety

The quiver flag variety $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ is a closed subscheme of $\mathrm{F}(\underline{\mathbf{d}})$ which is just a product of $\#Q_0$ flag varieties. These have well-known affine charts given by matrices in column echolon form with 1 at pivot positions and the other nonzero entries are the coordinates of the affine space and row permutation of these, see example below. We can pull them back to $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ to get a family of affine schemes which glue together to $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$, We still call them charts even though they are not affine spaces in general.
We use this to give an example of a generically not reduced $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$.

Example. The six affine charts of the Grassmannian $Gr(2,4)$ are given by

$$A_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ X & Y \\ Z & U \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ R & S \\ 0 & 1 \\ Z & U \end{pmatrix}, A_3 := \begin{pmatrix} 1 & 0 \\ R & S \\ X & Y \\ 0 & 1 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} V & W \\ 1 & 0 \\ 0 & 1 \\ Z & U \end{pmatrix}, A_5 := \begin{pmatrix} V & W \\ 1 & 0 \\ X & Y \\ 0 & 1 \end{pmatrix}, A_6 = \begin{pmatrix} V & W \\ R & S \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let Q be the Jordan quiver. Let R be a K-algebra and $\underline{\mathbf{d}} := (0,2,4)$ we consider

$$R^2 \xrightarrow{\quad A_i \quad} R^4 \circlearrowleft M := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The charts U_{A_i} of the quiver flag variety are then given by the closed condition

$$M(\mathrm{Im}(A_i)) \subset \mathrm{Im}(A_i) \iff \mathrm{rk}(A_i, MA_i) \leq \mathrm{rk}(A_i) = 2$$

A short check shows that $U_{A_2} = U_{A_3} = U_{A_5} = U_{A_6} = \emptyset$.
For U_{A_1} we calculate

$$\mathrm{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & X & Y \\ X & Y & Z & U \\ Z & U & 0 & 0 \end{pmatrix} \leq 2 \iff \mathrm{rk}\begin{pmatrix} 1 & X & Y \\ Y & Z & U \\ U & 0 & 0 \end{pmatrix} \leq 1$$

and this holds if and only if $UX = 0, UY = 0, Z = YX, Y^2 = U$. So we find

$$U_{A_1} = \mathrm{Spec\ K[X,Y,Z,U]/(YX^2, Y^3, Z - YX, U - Y^2) = Spec\ K[X,Y]/(YX^2, Y^3)}.$$

Set $I_1 := (YX^2, Y^3)$, we see $\sqrt{I_1} = (Y)$, so U_{A_1} is a thickened affine line, in particular in it is generically not reduced.
For U_{A_4} we calculate

$$\mathrm{rk}\begin{pmatrix} V & W & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & Z & U \\ Z & U & 0 & 0 \end{pmatrix} \leq 2 \iff \mathrm{rk}\begin{pmatrix} V & W & 0 \\ -U & 1 & Z \\ Z & U & 0 \end{pmatrix} \leq 1$$

and this is equivalent to $ZW = 0, UZ = 0, V = -UW, Z = -U^2$. So we find

$$U_{A_4} = \text{Spec K[V, W, Z, U]}/(-\text{U}^2\text{W}, -\text{U}^3, \text{V} + \text{UW}, \text{Z} + \text{U}^2) = \text{Spec K[W, U]}/(\text{U}^2\text{W}, \text{U}^3)$$

We conclude $\text{Fl}_\text{Q}\binom{M}{\underline{\mathbf{d}}} = U_{A_1} \cup U_{A_4}$ is not generically reduced, we see it as a thickened $\mathbb{P}^1$.

6.2.2 Detecting irreducible components

Lemma 60. *Let X be a locally noetherian scheme, $Z \subset X$ be a locally closed, irreducible subset.*

a) The following are equivalent:

(i) $\overline{Z}$ is an irreducible component.

(ii) Z contains an open subset of X.

b) Let X be a scheme of finite type over an algebraically closed field. We give Z the reduced subscheme structure. Then the following are equivalent

(i) There is a closed point $x \in Z$ such that $\dim T_x X = \dim Z$.

(ii) $\overline{Z}$ is an irreducible component that is ***generically smooth in*** *X, which means by definition $\mathcal{O}_{X,z}$ is regular, where z is the generic point of Z.*

Proof: ad a):
Let $U \subset Z$ such that U is open in $\overline{Z}$. Let $\overline{Z} \subset Y \subset X$ such that Y is an irreducible component of X.
$(i) \Leftrightarrow (ii)$ Obviously, U open in $Y \iff \overline{U} = \overline{Z} = Y$.
Replacing U by $U' = U \cap (\bigcup_{T \text{ irred comp}} X \setminus T)$ gives an open in X, the other implication is clear.
ad b):
$(i) \Rightarrow (ii)$ Suppose $\overline{Z}$ is not an irreducible component, then $\overline{Z} \hookrightarrow\!\!\!\!\!\!\!\not\;\;\; Y$ with $\dim Z < \dim Y$. Then for all $x \in X$ $\dim Z < \dim T_x Y \leq \dim T_x X$.
We have shown by contraposition that condition (i) implies that Z is an irreducible component.
Also condition (i) ensures $\dim Z = \dim T_x Z$, so that Z is generically smooth. But that is not enough because we do not know that X is smooth in z yet (it could be that $\mathcal{O}_{X,z}$ is not reduced).
Since $i \colon Z \to X$ is an immersion, it induces an isomorphism on tangent spaces at x. As Z is smooth at x it follows that $Z \to X$ is etale at x and therefore X is smooth at x. As the locus of smooth points is open in X it follows that there exists an open neighbourhood $U \subset Z$ such that $i|_U$ is etale. Since it is also an immersion, it follows that it has to be an open immersion and therefore Z contains as an open subscheme an open subscheme of X and the condition for the local ring follows.
$(ii) \Rightarrow (i)$ Suppose $\overline{Z}$ is an irreducible component. Let $U \subset Z$ such that U is open in X

with all points in U are smooth in X and Z. So the induced subscheme structures from Z and from X coincide on U and for all $x \in U$ it holds that

$$\dim T_x X = \dim T_x U = \dim T_x Z = \dim Z.$$

Corollary 6.2.0.1. *Let $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$ with K algebraically closed, $C \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ a locally closed irreducible subset. Then the following are equivalent:*

(i) C is an irreducible component that is generically smooth in $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$.

(ii) $\dim C = [C, M/C]$ where $[C, M/C] := \min\{[U, M/V]_\Lambda \mid U, V \in C(K)\}$.

Proof: Follows from the tangent space calculation for quiver flag varieties and the previous lemma b).

6.3 Stratifications

Definition 20. Let X be a scheme. A **stratification** of X is a family $X_i \subset X, i \in I$ of locally closed subsets with $X = \bigcup_{i \in I} X_i$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$ in I. We say a stratification $(X_i)_{i \in I}$

- is finite if I is finite.
- has property $\mathcal{P} \in \{$smooth, affine, irreducible $\}$ if for all $i \in I$: X_i has property $\mathcal{P}$.
- fulfills the boundary condition if $\forall i \in I$ there is a $J_i \subset I$: $\overline{X_i} = \bigcup_{j \in J_i} X_j$

We recall the well-known fact.

Lemma 61. *If X is a scheme with a finite, irreducible (not necessarily disjoint) stratification of X, then every irreducible component is the closure of a stratum.*

Proof: As we have a finite stratification, a highest dimensional irreducible component C must be a closure of a stratum. Then look at the complement $U = X \setminus C$ with the stratification $(X_i \cap U)_{i \in I}$. If $X_i \cap U \neq \emptyset$, it is irreducible with $\overline{X_i \cap U} = \overline{X_i}$. So, we can repeat the initial argument. □

6.3.1 Stratification in orbits

We look at $H = (\mathrm{Aut}_{KQ}(M))$ operating on $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$. Recall that $H \subset End_{KQ}(M)$ is open in an affine space, therefore H is irreducible and smooth. The stratification in orbits is smooth, irreducible, and fulfills the boundary condition but unfortunately, unless the quiver is quite small, we can not expect it to be a finite stratification. For $U \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ set $\mathcal{O}_U := H \cdot U \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$. We first have a look at

Lemma 62. *Applying $(U,-)_\Lambda$ to the short exact sequence $0 \to U \to M \to M/U \to 0$ yields a 6-term exact sequence*

$$0 \to (U,U) \to (M,M) \xrightarrow{p} (U,M/U) \to (U,U)^1 \xrightarrow{i} (M,M)^1 \to (U,M/U)^1 \to 0.$$

Proof: For the vanishing of the higher Ext-groups and $(M,M)^1_\Lambda \cong (M,M)^1_{KQ}$ see [Wol09], proof of Thm 5.27. The isomorphism $(M,M) \cong (U,M)$ is clear, and for the map $(M,M)^1 = (M,M)^1_{KQ} \to (U,M)^1$ the map $X = (X_1 \to \cdots \to X_\nu) \mapsto X_\nu$ is an inverse (where $X_s \in (U^s, M)^1_{KQ}$). □

Corollary 6.3.0.2. *For a point $U \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K)$, the following are equivalent:*

(i) $\overline{\mathcal{O}_U}$ *is an irreducible component of* $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ *and U is a smooth point.*

(ii) p surjective ($\iff$ i injective).

(iii) $[M,M] - [U,U] = [U,M/U]$ ($\iff$ $[M,M]^1 - [U,U]^1 = [U,M/U]^1$).

In particular, if $[U,U]^1 = 0$, then U is smooth and $\overline{\mathcal{O}_U}$ is an irreducible component.

Proof: The inclusion $\mathrm{coker}((U,U) \to (M,M)) \to (U,M/U)$ can be identified with the inclusion $T_U\mathcal{O}_U \to T_U\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$. The rest follows from the lemma above and lemma 60. □

6.3.2 Reineke's stratification

We use this name because Reineke introduced the stratification in [Rei03].

We denote by $[N]$ the isomorphism class of a representation N. For a sequence of isomorphism classes of representations $N_* = ([N_0], \dots [N_{\nu-1}], N_\nu = M)$ Markus Reineke (see [Rei03]) considered the following subsets of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$

$$\mathcal{F}_{[N_*]} := \{U = (0 = U^0 \subset U^1 \subset \cdots \subset U^\nu = M) \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \mid U^s \in [N_s], 0 \le s \le \nu\}.$$

He showed that they are nonempty if and only if $\underline{\dim}\, N_s = \underline{d}^s$, $0 \le s \le \nu$ and there exist monomorphisms $N_k \to N_{k+1}$ for $k \in \{0, \dots, \nu-1\}$.

Furthermore, if they are nonempty he showed that they are locally closed, irreducible, smooth of dimension

$$\sum_{k=1}^{\nu}([N_{k-1}, N_k]_{KQ} - [N_{k-1}, N_{k-1}]_{KQ}),$$

and $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ is a disjoint union of these subsets. Obviously, if $U \in \mathcal{F}_{[N_*]}$, then it holds $\mathcal{F}_{[N_*]} = \mathcal{F}_{[U_*]}$. I will call them Reineke strata and the collection **Reineke stratification**. We reprove his result in the following remark:

Remark. Consider $\mathcal{U} := \prod_{k=1}^{\nu} \mathrm{Inj}(N_{k-1}, N_k) \subset X_{Q_\nu}\binom{M}{\underline{\mathbf{d}}}$ and $H := \prod_{k=1}^{\nu-1} \mathrm{Aut}_{KQ}(N_k)$. Then H operates freely on $\mathcal{U}$ (it follows for all orbits $\dim H \cdot x = \dim H$), and the restricted map

$$\phi|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{F}_{[N_*]}$$

is constant on H-orbits, surjective, and has the H-orbits as fibres. We also have a cartesian diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \prod_{k=1}^{\nu}\prod_{i\in Q_0}\mathrm{Inj}_K(K^{d_i^{k-1}},K^{d_i^k}) \\ \phi\downarrow & & \downarrow\phi_0 \\ \mathcal{F}_{[N_*]} & \longrightarrow & \mathrm{F}(\underline{\mathbf{d}}) \end{array}$$

Therefore, $\phi|_{\mathcal{U}}$ is a locally trivial fibration. It follows that it is a principal H-bundle. As $\mathcal{U}$ is smooth and irreducible, it follows $\mathcal{F}_{[N_*]}$ is smooth irreducible of dimension

$$\dim\mathcal{F}_{[N_*]} = \dim\mathcal{U} - \dim H.$$

Remark. Stefan Wolf constructed also the following exact sequence as a generalization of the standard sequence for KQ-modules (see [Wol09], Appendix B):
Let $U = (U^0 \to \cdots \to U^\nu), V = (V^0 \to \cdots \to V^\nu = M)$ be two Λ-modules.

$$0 \to (U,V) \to \prod_{k=0}^{\nu}(U^k,V^k)_{KQ} \xrightarrow{\varphi_{U,V}} \prod_{k=1}^{\nu}(U^{k-1},V^k)_{KQ}$$

$$\to (U,V)^1 \xrightarrow{\eta_{U,V}} \prod_{k=0}^{\nu}(U^k,V^k)^1_{KQ} \to \prod_{k=1}^{\nu}(U^{k-1},V^k)^1_{KQ} \to (U,V)^2 \to 0$$

If $U \in \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}(K)$ and $V \in \{U, M, M/U\}$, then $(U,V)^2 = 0$ (see [Wol09], proof of thm 5.27).
Let $U \in \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}(K)$. The following conditions are equivalent:

(i) $\mathcal{O}_U$ is dense in $\mathcal{F}_{[U_*]}$.

(ii) $\varphi_{U,U}$ is surjective (equivalently $\eta_{U,U}$ injective).

(iii) $[U,U] - \sum_{k=0}^{\nu}[U^k,U^k]_{KQ} + \sum_{k=1}^{\nu}[U^{k-1},U^k]_{KQ} = 0$
($\iff$ $[U,U]^1 - \sum_{k=0}^{\nu}[U^k,U^k]^1_{KQ} + \sum_{k=1}^{\nu}[U^{k-1},U^k]^1_{KQ} = 0$.)

Just compare dimensions of tangent spaces and use lemma 60.

Remark. Recall that for every $\underline{d} \in \mathbb{N}_0^{Q_0}$, $N, L \in \mathrm{R_Q(\underline{d})(K)}$ we write $L \leq N$ if $N \in \overline{\mathcal{O}_L}$ where $\mathcal{O}_L \subset \mathrm{R_Q}(\underline{\mathrm{d}})$ is the $\mathbf{Gl}_{\underline{d}}$-orbit of L. We call this the degeneration order, it is a partial order on $\mathrm{R_Q}(\underline{\mathrm{d}})$. Let $N_* = ([N_0], \ldots, [N_\nu] = [M])$ such that $\mathcal{F}_{[N_*]} \neq \emptyset$ with for all other L_* with $\mathcal{F}_{[L_*]} \neq \emptyset$ it holds for all $k \in \{0, \ldots, \nu\}$: Either $N_k \leq L_k$, or N_k and L_k are not comparable in the degeneration order.
Then if the Reineke stratification is finite, $\overline{\mathcal{F}_{[N_*]}}$ is an irreducible component of $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ and $\mathcal{F}_{[N_*]}$ is open in $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$.
(Proof: Assume $\mathcal{F}_{[N_*]} \subset \overline{\mathcal{F}_{[L_*]}}$, then $\mathcal{F}_{[N_*]} \cap \overline{\mathcal{F}_{[L_*]}} \neq \emptyset$. This implies for all $k \in \{0, \ldots, \nu\}$: $L_k \leq N_k$, so by assumption $L_k \cong N_k$ and $\mathcal{F}_{[N_*]} = \mathcal{F}_{[L_*]}$.
We also proved that $\mathcal{F}_{[N_*]}$ has empty intersection with closures of other strata, therefore it follows that $\mathcal{F}_{[N_*]}$ is open if the Reineke stratification is finite.)

Corollary 6.3.0.3. *The following conditions are equivalent.*

(i) $\overline{\mathcal{F}_{[N_*]}}$ *is an irreducible component of* $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ *that is generically smooth inside* $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$.

(ii) *There is* $U \in \mathcal{F}_{[N_*]}$ *such that* $\dim T_U \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} = \dim \mathcal{F}_{[N_*]}$, *i.e.*

$$[U, M/U] = \sum_{k=1}^{\nu}([U^{k-1}, U^k]_{KQ} - [U^{k-1}, U^{k-1}]_{KQ}).$$

(iii) *There is* $U \in \mathcal{F}_{[N_*]}$ *such that*

$$[U, M/U]^1 = \sum_{k=1}^{\nu}([U^{k-1}, U^k]^1_{KQ} - [U^{k-1}, U^{k-1}]^1_{KQ}).$$

If the Reineke stratification is finite, this detects all irreducible components that are generically smooth inside $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$.

Proof: By lemma 60, it remains to show *(ii)* $\iff$ *(iii)*. But using lemma 62 and remark 6.3.2 we obtain two formulas for $[U, U] - [U, U]^1$. The resulting equality easily shows the claim. □

6.4 A conjecture on generic reducedness of Dynkin quiver flag varieties

We introduce scheme structures defined by rank conditions on orbits and Reineke strata, following the work of Zwara in [Zwa02a].

6.4.1 Schemes defined by rank conditions

Let $M_{u\times v}$ be the $\mathbb{Z}$-scheme of $u \times v$-matrices with the operation of the following group $G = \mathbf{Gl}_u \times \mathbf{Gl}_v$ via

$$(g_1, g_2) \star x := g_1 \cdot x \cdot g_2^{-1}$$

for any $g = (g_1, g_2) \in G(S), x \in M_{u\times v}(S)$ and commutative ring S. Let $s \leq \min(u, v)$, we define Y_s to be the closed subscheme of $M_{u\times v}$ given by the condition that all determinants of $(s+1) \times (s+1)$-minors vanish. Let V_s be the open subscheme of Y_s given as the complement of Y_{s-1}. For any field K, it is easy to see that $V_s(K) = \{x \in M_{u\times v}(K) \mid rk(x) = s\} = G(K) \cdot \begin{pmatrix} E_s & 0 \\ 0 & 0 \end{pmatrix}$. Furthermore, it is known that

$$\begin{aligned} V_s(R) &:= \{x \in M_{u\times v}(R) \mid \mathrm{Im}(x) \subset R^u \text{ is a direct } R-\text{module summand of rank } s\} \\ &= \{x \in M_{u\times v}(R) \mid \ker(x) \subset R^v \text{ is a direct } R-\text{module summand of rank } (v-s)\} \end{aligned}$$

for any commutative ring R ([Wol09], Prop.5.4 for the first description, the second equality is easy to prove). We want to calculate its tangent spaces, therefore the following lemma is useful, it is due to Zwara .

Lemma 63. *([Zwa02a], Lemma 3.2, 3.3) Let K be a field and S a local commutative ring such there exist ring homomorphisms $\mathrm{id}_K \colon K \hookrightarrow S \twoheadrightarrow K$, set $\mathfrak{m} := \ker(S \twoheadrightarrow K)$.*

(i) For any $x \in M_{u\times v}(S)$ there is a $g \in G(S)$ such that

$$g \star x = \begin{pmatrix} E_t & 0 \\ 0 & z \end{pmatrix}$$

for some $t \leq \min(u, v)$ and some $z \in M_{(u-t)\times(v-t)}(\mathfrak{m})$.

(ii)

$$V_s(S) = G(S) \cdot \begin{pmatrix} E_s & 0 \\ 0 & 0 \end{pmatrix}$$

If $\dim_K S = h < \infty$, then $V_s(S) = \{x \in M_{u\times v}(S) \mid rk(\overline{x}) = s, rk(x) = sh\}$ where $M_{u\times v}(S) \to M_{u\times v}(K), x \mapsto \overline{x}$ is the map induced by $S \twoheadrightarrow K$, and $x \colon S^v \to S^u$ is considered as a K-linear map between finite dimensional vector spaces.

Recall that for a scheme X of finite type over a field K and $x \in X(K)$ the tangent space T_xX is defined to be the preimage of x under the canonical map

$$X(K[T]/(T^2)) \to X(K).$$

As a shortcut we will set $K[\varepsilon] := K[T]/(T^2), \varepsilon \mapsto T$. Using the previous lemma we can describe now the tangent space of V_s.

Corollary 6.4.0.4. *Let $x \in V_s(K)$ with $x = g \star e$ for $g = (g_1, g_2) \in G(K), e = \begin{pmatrix} E_s & 0 \\ 0 & 0 \end{pmatrix}$, then*

$$T_xV_s = \{(g_1 \cdot \begin{pmatrix} T & 0 \\ A & 0 \end{pmatrix}, g_2 \cdot \begin{pmatrix} -T & B \\ 0 & 0 \end{pmatrix}) \in M_{u\times u}(K) \times M_{v\times v}(K) \mid T \in M_{s\times s}(K)\}.$$

Its dimension is $s(u + v - s)$. Furthermore, seeing $x \colon K^v \to K^u$ as a linear map, we find an identification

$$T_xV_s \cong \{\varphi \in \mathrm{Hom}_K(K^v, K^u) \mid \varphi(\ker x) \subset \mathrm{Im}\, x\}$$

explicitly given by $\left(g_1 \left(\begin{smallmatrix} T & 0 \\ A & 0 \end{smallmatrix}\right), g_2 \left(\begin{smallmatrix} -T & B \\ 0 & 0 \end{smallmatrix}\right)\right) \mapsto g_1 \left(\begin{smallmatrix} T & B \\ A & 0 \end{smallmatrix}\right) g_2^{-1}$

Proof: Using the description given by the previous lemma (ii) it is not difficult to see that $T_xV_s = g \cdot T_eV_s$, so wlog $x = e$. Again by the previous lemma (ii)

$$V_s(K[\varepsilon]) = G(K[\varepsilon]) \cdot e \cong G(K[\varepsilon])/\,\mathrm{Stab}_{G(K[\varepsilon])}(e).$$

Now, we have

$$\mathrm{Stab}_G(e) := \{(\begin{pmatrix} Z & X \\ 0 & Y \end{pmatrix}, \begin{pmatrix} Z^{-1} & 0 \\ V & W \end{pmatrix}) \in \mathbf{Gl}_u \times \mathbf{Gl}_v \mid Z \in \mathbf{Gl}_s\}$$

is the closed subscheme whose K-valued and $K[\varepsilon]$-valued points are the stabilizers of e. We find a short exact sequence of pointed sets

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & r_S^{-1}(1) & \longrightarrow & \operatorname{Stab}_G(e)(K[\varepsilon]) & \xrightarrow{r_S} & \operatorname{Stab}_G(e)(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & r_G^{-1}(1) & \longrightarrow & G(K[\varepsilon]) & \xrightarrow{r_G} & G(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & r_V^{-1}(e) & \longrightarrow & V_s(K[\varepsilon]) & \xrightarrow{r_V} & V_s(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where it is easy to the check that the induced first column is exact, for example for $r_G^{-1}(1) \to r_V^{-1}(e)$ surjective just note that for $g \star e \in r_V^{-1}(e)$ with $g = (g_0 + \varepsilon g_1, h_0 + \varepsilon h_1)$ we have $g \star e = (1 + \varepsilon g_1 g_0^{-1}, 1 + \varepsilon h_0^{-1} h_1) \star e$. Now, $(1 + \varepsilon g_1 g_0^{-1}, 1 + \varepsilon h_0^{-1} h_1) \in r_G^{-1}(1)$ and maps to $g \star e$. Thus, by definition we have a short exact sequence

$$0 \to T_1 \operatorname{Stab}_G(e) \to T_1 G \to T_e V_s \to 0.$$

which gives $T_e V_s \cong \{(\left(\begin{smallmatrix} T & 0 \\ A & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -T & B \\ 0 & 0 \end{smallmatrix}\right)) \in M_{u \times u}(K) \times M_{v \times v}(K) \mid T \in M_{s \times s}(K)\}$.
The vector space isomorphism is easy to check. □

6.4.2 Quiver-related schemes defined by rank condition

Let Q be a finite quiver, K be a field. We assume the reader knows $\mathrm{R_Q}(\underline{\mathrm{d}})$. We start with two very basic constructions. Let N, L be two finite-dimensional KQ-modules. Let R be a commutative K-algebra. We define

1)
$$(\operatorname{Inj}_{KQ}(N, L))(R) := \{(f_i)_{i \in Q_0} \in \operatorname{Hom}_{RQ}(N \otimes R, L \otimes R) \mid f_i \text{ split monomorphism for all } i \in Q_0\}.$$

   This defines an open subscheme of the affine space defined by $R \mapsto \operatorname{Hom}_{RQ}(N \otimes R, L \otimes R)$.

2) Fix $t \in \mathbb{N}$ and choose a free representation $(KQ)^p \xrightarrow{\chi} (KQ)^q \to L \to 0$ with $\chi = (a_{ij}) \in M_{q \times p}(KQ)$. Assume that the underlying graph of Q has no oriented cycles. Every $N \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{R})$ can be considered as an R-algebra homomorphism

$$N \colon RQ \to \bigoplus_{(k,l) \in Q_0 \times Q_0} M_{d_l \times d_k}(R) = M_{d \times d}(R)$$

with $d = \sum_{i \in Q_0} d_i$ via

$$e_i \mapsto (\delta_{(k,l)(i,i)} \operatorname{id}_{R^{d_i}})_{(k,l) \in Q_0 \times Q_0}, \quad (\alpha \colon i \to j) \mapsto (\delta_{(k,l),(i,j)} N_\alpha)_{(k,l) \in Q_0 \times Q_0}.$$

Then we can define $f \colon \mathrm{R_Q(\underline{d})} \to \mathrm{M_{qd \times pd}}$ on R-valued points via

$$N \mapsto (N(a_{ij} \otimes_K 1_R))_{1 \leq i \leq q, 1 \leq j \leq p}$$

Then we define $\mathrm{R_Q(\underline{d})[L,t]}$ as the pullback in the following diagram

$$\begin{CD}
\mathrm{R_Q(\underline{d})[L,t]} @>>> V_{qd-t} \\
@VVV @VVV \\
\mathrm{R_Q(\underline{d})} @>f>> M_{qd \times pd}
\end{CD}$$

or explicitly on R-valued points

$$\begin{aligned}
\mathrm{R_Q(\underline{d})[L,t](R)} &= \{\mathrm{N \in R_Q(\underline{d})(R) \mid Im((N(a_{ij} \otimes_K 1_R)) \colon R^{pd} \to R^{qd})} \\
&\qquad \text{is a direct } R-\text{module summand of } R^{qd} \text{ of rk } qd - t\} \\
&= \{N \in \mathrm{R_Q(\underline{d})(R) \mid Hom_{RQ}(L \otimes R, N)} \\
&\qquad \text{is a direct } R-\text{module summand of } R^{pd} \text{ of rank } t\}
\end{aligned}$$

Zwara proved that for S an n-dimensional commutative local K-algebra with $\operatorname{id}_K \colon K \to S \to K$,

$$\begin{aligned}
\mathrm{R_Q(\underline{d})[L,t](S)} = \{&\mathrm{N \in R_Q(\underline{d})(S)} \mid \\
&\dim_K \operatorname{Hom}_{KQ}(L, N) = tn, \dim_K \operatorname{Hom}_{KQ}(L, N \otimes_S K) = t\},
\end{aligned}$$

and the functor $\mathrm{R_Q(\underline{d})[L,t]}$ is uniquely determined by the isomorphism class of L and the integer $t \geq 0$.

There is an obvious generalization of Zwara's scheme structure: Given $\mathbb{L} := (L_1, \ldots, L_r)$, a sequence of finite dimensional KQ-modules, $\underline{t} := (t_1, \ldots, t_r) \in \mathbb{N}_0^r$, and a choice of free resolution

$$(KQ)^{p_i} \to (KQ)^{q_i} \to L_i \to 0$$

we find

$$\begin{aligned}
\mathrm{R_Q(\underline{d})[\mathbb{L},\underline{t}](R)} := \{&\mathrm{N \in R_Q(\underline{d})(R) \mid Hom_{RQ}(L_i \otimes R, N)} \\
&\text{is a direct } R-\text{module summand of } R^{p_i d} \text{ of rank } t_i, 1 \leq i \leq r\}
\end{aligned}$$

by definition $\mathrm{R_Q(\underline{d})[\mathbb{L},\underline{t}] = R_Q(\underline{d})[L_1,t_1] \times_{R_Q(\underline{d})} \cdots \times_{R_Q(\underline{d})} R_Q(\underline{d})[L_r,t_r]}$, so it is a scheme.

Orbits defined by rank conditions

From now on, let Q be a connected Dynkin quiver and $L_1, \ldots, L_r$ be a complete set of isomorphism classes of indecomposable finite dimensional KQ-modules.

Let $N \in \mathrm{R_Q(\underline{d})(K)}$ be a module. We write

$$\underline{t}_N := (t_1, \ldots, t_r), \quad t_k := \dim \operatorname{Hom}(L_k, N),\ 1 \leq k \leq r$$

and define

$$\widetilde{\mathcal{O}_N} := \mathrm{R_Q(\underline{d})[\mathbb{L}, \underline{t}] = R_Q(\underline{d})[L_1, t_1] \times_{R_Q(\underline{d})} \cdots \times_{R_Q(\underline{d})} R_Q(\underline{d})[L_r, t_r]};$$

this is a subscheme of $\mathrm{R_Q(\underline{d})}$, the morphism $\widetilde{\mathcal{O}_N} \to \mathrm{R_Q(\underline{d})}$ is an immersion. It holds that $\widetilde{\mathcal{O}_N}(K) = \mathbf{Gl}_{\underline{d}}(K) \cdot N$. If $\widetilde{\mathcal{O}_N}$ is reduced, then it is equivalent to the orbit scheme $\mathcal{O}_N$, which is defined as the reduced subscheme with $\mathcal{O}_N(K) = \mathbf{Gl}_{\underline{d}}(K) \cdot N$. In general, we do not know when this is reduced. We call the scheme $\widetilde{\mathcal{O}_N}$ the **orbit by rank condition**.

For the reduced scheme structure, we remark the following lemma, which is a special case of [Hub13], Prop. 5.7.

Lemma 64. *Let Q be a quiver, $\underline{d} \in \mathbb{N}_0^{Q_0}$, $N \in \mathrm{R_Q(\underline{d})(K)}$, $G = \mathbf{Gl}_{\underline{d}}$ and we write $\mathcal{O} \subset \mathrm{R_Q(\underline{d})}$ for the $\mathbf{Gl}_{\underline{d}}$-orbit. Then it holds*

$$\mathcal{O}_N(K[\varepsilon]) = G(K[\varepsilon]) \cdot N$$

Proof: Let $p \colon \mathcal{O}_N(K[\varepsilon]) \to \mathcal{O}_N(K)$ be the map induced from $K[\varepsilon] \twoheadrightarrow K,\ \varepsilon \mapsto 0$. By definition it holds $p^{-1}(L) = T_L\mathcal{O}_N$ for $L \in \mathcal{O}_N(K)$. Recall that by Voigt's Lemma we know that for the standard exact sequence

$$0 \to \operatorname{End}_{KQ}(L) \to \prod_{i \in Q_0} M_{d_i}(k) \xrightarrow{\phi} \prod_{i \to j \in Q_1} M_{d_j \times d_i}(K) \to \operatorname{Ext}^1_{KQ}(L, L) \to 0$$

with ϕ is given by $(x_i)_i \mapsto (x_j L_{i \to j} - L_{i \to j} x_i)_{i \to j}$ it holds that $\operatorname{Im}(\phi) = T_L\mathcal{O}_N$. For the free $K[\varepsilon]$-module $K[\varepsilon]^n$, we fix a K-vector space basis $e_1, \cdots, e_n, \varepsilon e_1, \ldots, \varepsilon e_n$, that means that ε operates on $K[\varepsilon]^n$ by the nilpotent operator $\left(\begin{smallmatrix} 0 & 0 \\ E & 0 \end{smallmatrix}\right)$. Now a $K[\varepsilon]$-linear map $K[\varepsilon]^n \to K[\varepsilon]^m$ corresponds in the basis to a matrix $\left(\begin{smallmatrix} X & 0 \\ Y & X \end{smallmatrix}\right) \in M_{2m \times 2n}(K)$ with $X, Y \in M_{m \times n}(K)$. In particular, we have

$$G(K[\varepsilon]) = \prod_{i \in Q_0} \mathbf{Gl}_{d_i}(K[\varepsilon]) = \prod_{i \in Q_0} \{ \begin{pmatrix} T_i & 0 \\ x_i & T_i \end{pmatrix} \in \mathbf{Gl}_{2d_i}(K) \mid T_i \in \mathbf{Gl}_{d_i}(K), x_i \in M_{d_i}(K) \}$$

and we write

$$U_\varepsilon := \prod_{i \in Q_0} \{ \left(\begin{smallmatrix} E & 0 \\ x_i & E \end{smallmatrix}\right) \in \mathbf{Gl}_{2d_i}(K) \mid x_i \in M_{d_i}(K) \}.$$

We can see

$$T_L\mathcal{O}_N = \{ A := (\begin{pmatrix} L_{i \to j} & 0 \\ A_{i \to j} & L_{i \to j} \end{pmatrix})_{i \to j} \in \mathrm{R}_{2\underline{d}}(Q)(K) \mid A \cong N \oplus N \text{ as } KQ-\text{module} \}.$$

We set $L_\varepsilon := L \otimes_K K[\varepsilon] = (\begin{pmatrix} L_{i\to j} & 0 \\ 0 & L_{i\to j} \end{pmatrix})_{i\to j}$. We claim that as a consequence of Voigt's Lemma we directly get

$$T_L \mathcal{O}_N = U_\varepsilon \cdot L_\varepsilon$$

where the operation is the restriction of the operation of $\mathbf{Gl}_{2\underline{d}}(K)$ on $\mathrm{R}_{2\underline{d}}(Q)(K)$. To see this the diagram

$$\begin{array}{ccc} K^{2d_i} & \xrightarrow{\begin{pmatrix} L_{i\to j} & 0 \\ 0 & L_{i\to j} \end{pmatrix}} & K^{2d_j} \\ {\scriptstyle \begin{pmatrix} E & 0 \\ x_i & E \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} E & 0 \\ x_j & E \end{pmatrix}} \\ K^{2d_i} & \xrightarrow[\begin{pmatrix} L_{i\to j} & 0 \\ A_{i\to j} & L_{i\to j} \end{pmatrix}]{} & K^{2d_j} \end{array}$$

commutes if and only if $A_{i\to j} = x_j L_{i\to j} - L_{i\to j} x_i$ which is how we discribe all points in the tangent space by Voigt's Lemma.
Now, it holds by definition of the scheme structure on $\mathcal{O}_N$ that $G(K[\varepsilon]) \cdot N_\varepsilon \subset \mathcal{O}_N(K[\varepsilon])$. But we also know $\mathcal{O}_N(K[\varepsilon]) = \bigcup_{L \in \mathcal{O}_N(K)} p^{-1}(L)$. If $T \in p^{-1}(L)$, we fix an isomorphism $f \colon L \to N$ in $\mathbf{Gl}_{\underline{d}}(K)$, then $f \otimes 1 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} =: N_\varepsilon \to L_\varepsilon$ is in $G(K[\varepsilon])$. By the previous discussion it holds that there is a $g \in U_\varepsilon$ such that

$$T = g \cdot L_\varepsilon = g \cdot ((f \otimes 1) \cdot N_\varepsilon) \in G(K[\varepsilon]) \cdot N_\varepsilon.$$

$\square$

Reineke stratification defined by rank conditions

We consider $N_1, \cdots, N_\nu := M$ finite dimensional KQ-modules,
$\underline{\mathbf{d}} := (\underline{d}^1 := \underline{\dim}\, N_1, \ldots, \underline{d}^\nu := \dim N_\nu)$. For any KQ-module N we define $\underline{t}_{[N_*]} := (\underline{t}_{N_1}, \ldots, \underline{t}_{N_\nu})$.
Let R be a commutative K-algebra. Let (Δ, I) be the the quiver with the relations such that $KQ \otimes K\mathbb{A}_\nu = K\Delta/I$. We define the following

$$
\begin{aligned}
X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(R) &:= \{U = (0 \to U_1 \xrightarrow{f_1} U_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{\nu-2}} U_{\nu-1} \xrightarrow{f_{\nu-1}} M) \\
&\qquad \in \mathrm{R}_{(\Delta,I)}(\underline{\mathbf{d}})(R) \mid \ (f_k)_i \text{ split monom. } , 1 \leq k \leq \nu-1, i \in Q_0\} \\
\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(R) &:= \{U = (0 \to U_1 \subset U_2 \subset \cdots \subset U_{\nu-1} \subset M) \mid \\
&\qquad \text{flag of } KQ - mods, \underline{\dim}\, U = \underline{\mathbf{d}}\} \\
\widetilde{\mathrm{X}_{[N_*]}}(R) &:= \{U = (0 \to U_1 \xrightarrow{f_1} U_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{\nu-2}} U_{\nu-1} \xrightarrow{f_{\nu-1}} M) \in X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(R) \mid \\
&\qquad U_i \in \mathrm{R_Q}(\underline{\mathrm{d}})[\mathbb{L}, \mathrm{rk}_{\mathbb{L}}(\mathrm{N_i})](\mathrm{R}), 1 \leq \mathrm{i} \leq \nu - 1\} \\
\widetilde{\mathcal{F}_{[N_*]}}(R) &:= \{U = (0 \subset U_1 \subset \cdots \subset U_{\nu-1} \subset M) \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(R) \mid \\
&\qquad \mathrm{Hom}_{RQ}(L_j \otimes R, U_i) \text{ is a free direct } R-\text{module summand of } R^{p_j d_i} \\
&\qquad \text{of rank } \dim_K \mathrm{Hom}_{KQ}(L_j, N_i), 1 \leq j \leq r, 1 \leq i \leq \nu - 1\}
\end{aligned}
$$

Lemma 65. *All of these functors are represented by schemes. There is a cartesian commutative diagram*

$$
\begin{array}{ccc}
\widetilde{\mathrm{X}_{[N_*]}} & \longrightarrow & X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{F}_{[N_*]}} & \longrightarrow & \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}
\end{array}
$$

defined on R-valued points for a commutative K-algebra R by $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(R) \to \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(R)$

$$
(0 \to U_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{\nu-1}} M) \mapsto (0 \subset \mathrm{Im}(f_1 \cdots f_{\nu-1}) \subset \cdots \mathrm{Im}(f_{\nu-1}) \subset M)
$$

and the left vertical map is its restriction to $\widetilde{\mathrm{X}_{[N_*]}}(R) \to \widetilde{\mathcal{F}_{[N_*]}}(R)$. *The vertical morphisms are principal G-bundles with $G = \prod_{1 \leq s \leq \nu-1} \mathbf{Gl}_{\underline{d}^s}$, the vertical morphisms are immersions. We call* $\widetilde{\mathcal{F}_{[N_*]}}$ *a* ***Reineke stratum as rank scheme****.*

Proof: The principal G-bundle $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ is the one from the beginning of this chapter. That $\widetilde{\mathrm{X}_{[N_*]}}$ defines a scheme is because it is an iterated pullback of a rank scheme as defined before (left to the reader). Then, by [Hub13], Lemma 4.7 there exists the geometric quotient $\widetilde{\mathrm{X}_{[N_*]}}/G$. It follows that this has to be $\widetilde{\mathcal{F}_{[N_*]}}$. □

Remark. If $\widetilde{\mathcal{F}_{[N_*]}}$ is reduced, then it equals the smooth scheme $\mathcal{F}_{[N_*]}$ which we called a Reineke stratum.

Lemma 66. *Let Q be a quiver, $\underline{\mathbf{d}} = (0, \underline{d}^1, \ldots, \underline{d}^\nu) \in \mathbb{N}_0^{Q_0}$ and $0 \to N_1 \to \cdot \to N_\nu = M$ a Λ-module in $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}(K)$ such that $\widetilde{\mathcal{O}_{N_k}} = \mathcal{O}_{N_k}$ is reduced $1 \leq k \leq \nu$. Then, the Reineke stratum $\widetilde{\mathcal{F}_{[N_*]}} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ as rank scheme is smooth, i.e. $\widetilde{\mathcal{F}_{[N_*]}} = \mathcal{F}_{[N_*]}$.*

Proof: It is enough to show that $\widetilde{\mathrm{X}_{[N_*]}}$ is smooth, where $\widetilde{\mathrm{X}_{[N_*]}} \subset X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}$ is the pullback of $\widetilde{\mathcal{F}_{[N_*]}}$ by the principal $G := \prod_{1 \leq s \leq \nu-1} \mathbf{Gl}_{\underline{d}^s}$-bundle $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$.

We have a natural map

$$\phi\colon \prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times G \to \widetilde{\mathrm{X}_{[N_*]}}$$

which induces a morphism

$$\overline{\phi}\colon \prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times^H G \to \widetilde{\mathrm{X}_{[N_*]}}$$

with $H = \prod_{1\leq k\leq \nu-1} \mathrm{Aut}(N_k)$. It holds that $\overline{\phi}(K)$ is an isomorphism. Since $\prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times^H G$ is smooth, irreducible of the same dimension as $\widetilde{\mathrm{X}_{[N_*]}}$ it is enough to show that

$$\phi(K[\varepsilon])\colon \prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k)(K[\varepsilon]) \times G(K[\varepsilon]) \to \widetilde{\mathrm{X}_{[N_*]}}(K[\varepsilon])$$

is surjective, because it implies that $\overline{\phi}(K[\varepsilon])$ is surjective, and that implies that the tangent space dimensions on $\prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times^H G$ are greater or equal to the tangent space dimensions on $\widetilde{\mathrm{X}_{[N_*]}}$. For every $x \in \left(\prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times^H G\right)(K)$ that gives us

$$\dim \widetilde{\mathrm{X}_{[N_*]}} = \dim T_x \prod_{k=1}^{\nu} \mathrm{Inj}_{KQ}(N_{k-1}, N_k) \times^H G \geq \dim T_{\overline{\phi}(x)} \widetilde{\mathrm{X}_{[N_*]}} \geq \dim \widetilde{\mathrm{X}_{[N_*]}},$$

i.e. the tangent spaces have all $\dim \widetilde{\mathrm{X}_{[N_*]}}$ which implies that $\widetilde{\mathrm{X}_{[N_*]}}$ is smooth.
But that $\phi(K[\varepsilon])$ is onto follows directly from the lemma 64 because a point in $\widetilde{\mathrm{X}_{[N_*]}}$ is of the form

$$L := (0 \to M_1 \xrightarrow{f_1} M_2 \to \cdots \xrightarrow{f_{\nu-1}} M_\nu = M_\varepsilon)$$

with M_i is in $\mathcal{O}_{N_i}(K[\varepsilon])$ and f_i is a $KQ \otimes K[\varepsilon]$-linear map which is an injective K-vector space homomorphism. Then, by the previous lemma there exist $g_i \in \mathbf{Gl}_{\underline{d}^i}(K[\varepsilon])$ such that $M_i = g_i \cdot (N_i)_\varepsilon$. This implies

$$L = \phi((g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \ldots, g_{\nu-1} f_{\nu-2} g_{\nu-2}^{-1}, f_{\nu-1} g_{\nu-1}),\ (g_i)_{1\leq i\leq \nu-1}).$$

□

The conjecture

Conjecture. Every Dynkin quiver flag variety is generically reduced. This follows from (1) and (2).

(1) For every representation $M \in \mathrm{R_Q}(\underline{\mathrm{d}})(\mathrm{K})$, it holds that $\widetilde{\mathcal{O}_M} = \mathcal{O}_M$.

(2) For every $\widetilde{\mathcal{F}_{[N_*]}} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ such that $\overline{\mathcal{F}_{[N_*]}(K)}$ is an irreducible component of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K)$, there is an open subscheme

$U \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ such that $U(K) \subset \mathcal{F}_{[N_*]}(K)$ and for all $x \in U(K)$

$$T_x \widetilde{\mathcal{F}_{[N_*]}} = T_x \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}.$$

In fact, ad (2): Let $\mathcal{F}_{[L_*,-]\leq \underline{t}}$ the open subscheme such that $\mathcal{F}_{[L_*,-]=\underline{t}}(K) = \mathcal{F}_{[N_*]}(K)$ and assume $\overline{\mathcal{F}_{[N_*]}(K)}$ is an irreducible component $C(K)$ of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}(K)$. In [Hub13], Hubery describes a natural generic scheme structure on C coming from the primary ideal decomposition. Observe that $C(K) \cap \mathcal{F}_{[L_*,-]\leq \underline{t}}(K) = \mathcal{F}_{[N_*]}(K)$ is an open subset of $C(K)$. We conjecture that

$$\widetilde{\mathcal{F}_{[N_*]}} = C \times_{\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}} \mathcal{F}_{[L_*,-]\leq \underline{t}},$$

this would implies (2).

Corollary 6.4.0.5. *Let Q be a Dynkin quiver and $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ generically reduced. For all non-empty Reineke strata $\mathcal{F}_{[N_*]} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ we fix a $\Lambda = KQ \otimes K\mathbb{A}_{\nu+1}$-module $N := (0 \hookrightarrow N_1 \hookrightarrow N_2 \hookrightarrow \cdots \hookrightarrow N_\nu = M)$ in $\mathbb{X}$. Then the irreducible components of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ are*

$$\{\overline{\mathcal{F}_{[N_*]}} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \mid [N, M/N] = \dim \mathcal{F}_{[N_*]}\}.$$

6.4.3 Canonical decomposition

This is a review of a result due to Hubery saying that irreducible components of quiver flag varieties admit *canonical decompositions* analogously to Crawley-Boevey's and Schröer's article *Irreducible components of varieties of modules*, see [CBS02].

Notation of this subsection: K is an algebraically closed field. Let ${}_1\underline{\mathbf{d}}, \ldots, {}_t\underline{\mathbf{d}}$ be dimension filtrations of length ν, $M_1 \in \mathrm{R_Q}({}_1\underline{\mathrm{d}}^\nu)(\mathrm{K}), \ldots, \mathrm{M_t} \in \mathrm{R_Q}({}_t\underline{\mathrm{d}}^\nu)(\mathrm{K})$ be KQ-representations and $C_1 \subset \mathrm{Fl_Q}\binom{M_1}{{}_1\underline{\mathbf{d}}}, \ldots, C_t \subset \mathrm{Fl_Q}\binom{M_t}{{}_t\underline{\mathbf{d}}}$ locally closed irreducible subsets. We defined locally trivial fibrations $\varphi_i \colon X_{Q^{\otimes \nu+1}}\binom{M_i}{{}_i\underline{\mathbf{d}}} \to \mathrm{Fl_Q}\binom{M_i}{{}_i\underline{\mathbf{d}}}$, $1 \leq i \leq t$ in the first subsection of this chapter and we call $D_1 \subset X_{Q^{\otimes \nu+1}}\binom{M_1}{{}_1\underline{\mathbf{d}}}, \ldots, D_t \subset X_{Q^{\otimes \nu+1}}\binom{M_t}{{}_t\underline{\mathbf{d}}}$ the preimages of $C_1, \ldots C_t$. We set

$$M := \bigoplus_{i=1}^t M_i, \quad \underline{\mathbf{d}} := \sum_{i=1}^t {}_i\underline{\mathbf{d}},$$

the groups $H := \mathrm{Aut}(M)$, $\prod_{1\leq s\leq \nu-1} \mathbf{Gl}_{\underline{d}^s}$ operate on $X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}$ by Λ-module isomorphism,
the groups $H_i := \mathrm{Aut}(M_i)$, $\prod_{1\leq s\leq \nu-1} \mathbf{Gl}_{{}_i\underline{d}^s}$ operate on $X_{Q^{\otimes \nu+1}}\binom{M_i}{{}_i\underline{\mathbf{d}}}$, $1 \leq i \leq t$ by Λ-module isomorphism.
Furthermore, we consider the map

$$a\colon \mathrm{Aut}(M) \times C_1 \times \cdots C_t \to \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}},$$

defined via $\tilde{a}\colon \mathrm{Aut}(M) \times D_1 \times \cdots \times D_t \to X_{Q^{\otimes \nu+1}}\binom{M}{\underline{\mathbf{d}}}, (h, x_1, \ldots, x_t) \mapsto h \cdot (\bigoplus_{i=1}^t x_i)$, then check that $\varphi \circ \tilde{a}$ is constant on orbits under the group action of $\prod_{1\leq s\leq \nu-1} \mathbf{Gl}_{{}_1\underline{d}^s} \times \cdots \times$

$\prod_{1\le s\le \nu-1}\mathbf{Gl}_{{}_t\underline{d}^s}$. We set $\mathrm{Im}(a) := C_1 \oplus \cdots \oplus C_t$, then

$$\overline{C_1 \oplus \cdots \oplus C_t}$$

is an irreducible closed subset of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$.
Furthermore, we call the projection on the sink functor $\mathrm{fp}\colon \Lambda-\mathrm{mod} \to \mathrm{KQ}-\mathrm{mod}, (\mathrm{U}_0 \to \cdots \to \mathrm{U}_\nu) \mapsto \mathrm{U}_\nu$ and the regular mapping $fp := \pi\varphi\colon X_{Q\otimes \nu+1}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{R_Q}(\underline{\mathrm{d}})$ as a shortage for flagpole (even though Λ-modules are no flags).

Theorem 6.4.1. *Let $C \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ be an irreducible component, then*

$$C = \overline{C_1 \oplus \cdots \oplus C_t}$$

for some irreducible components $C_i \subset \mathrm{Fl_Q}\binom{M_i}{{}_i\underline{\mathbf{d}}}$, with the property that the general module in each $D_i = \varphi_i^{-1}(C_i)$ is indecomposable. Moreover, $C_1, \ldots, C_t$ are uniquely determined by this, up to reordering.

Lemma 67. *([Hub13], Lemma 6.5) Let $\underline{\mathbf{d}}$ and $\underline{\mathbf{e}}$ be two filtrations of length ν, $M \in \mathrm{R_Q}(\underline{\mathrm{d}}^\nu)(\mathrm{K}), \mathrm{N} \in \mathrm{R_Q}(\underline{\mathrm{e}}^\nu)(\mathrm{K})$. The functions*

$$\mathrm{Fl_Q}\begin{pmatrix} N \\ \underline{\mathbf{e}} \end{pmatrix} \times \mathrm{Fl_Q}\begin{pmatrix} M \\ \underline{\mathbf{d}} \end{pmatrix} \to \mathbb{Z}, \quad X_{Q\otimes\nu+1}\begin{pmatrix} N \\ \underline{\mathbf{e}} \end{pmatrix} \times X_{Q\otimes\nu+1}\begin{pmatrix} M \\ \underline{\mathbf{d}} \end{pmatrix} \to \mathbb{Z}$$

defined by $(U,V) \mapsto [U, M/V]^1$ are welldefined and upper semicontinuous. In particular, in the notation from above, the following conditions are equivalent:

1) for $i \neq j$: $[C_i, C_j] - [M_i, M_j] + [C_i, M_j/C_j] = 0$,

2) for $i \neq j$: $\exists\, U \in C_i, V \in C_j$ with $[U,V] - [M_i, M_j] + [U, M_j/V] = 0$,

3) for $i \neq j$: $\exists\, U \in C_i, V \in C_j$ with $\dim\ker((U,V)^1 \to (N,M)^1) = 0$.

Using the equivalence of 1) and 3) we can reformulate the main theorem.

Theorem 6.4.2. *(A. Hubery, [Hub13], Thm 6.7) If $C_i \subset \mathrm{Fl_Q}\binom{M_i}{{}_i\underline{\mathbf{d}}}, 1 \le i \le t$ are irreducible components, then $\overline{C_1 \oplus \cdots \oplus C_t} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ is an irreducible component if and only if for all $i \neq j$:*

$$[C_i, C_j] - [M_i, M_j] + [C_i, M_j/C_j] = 0.$$

To show that these are basically rewrites of results of Crawley-Boevey and Schröer we include the proofs of theorem 6.4.1 and lemma 67. A proof of theorem 6.4.2 following [CBS02] is also possible. Of course, you find more general proofs in [Hub13].

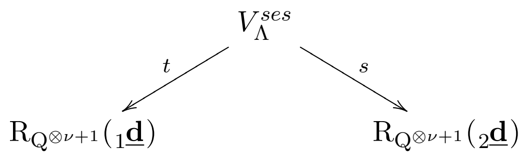
Proof of theorem 6.4.1: This is nearly a copy of [CBS02], theorem 1.1., but the situation is slightly different.
Let $D = \varphi^{-1}(C)$, $D \supset D^{ind}$ be the constructible subset of indecomposable $\underline{\mathbf{d}}$-dimensional Λ-modules. Every $\underline{\mathbf{d}}$-dimensional module $U \in D$ is isomorphic to a direct sum of indecomposables, so lies inside a set $S = D_1^{ind} \oplus \cdots \oplus D_t^{ind}$ for some irreducible components

$D_i \subset X_{Q^{\otimes\nu+1}}\binom{M_i}{{}_i\underline{\mathbf{d}}}$. Since all S are locally closed and there are only a finite number of possible S inside D covering D, there has to be one containing a dense open in D. Fixing this set S we get for it

$$\overline{S} = \overline{D_1^{ind} \oplus \cdots \oplus D_t^{ind}} = \overline{D_1 \oplus \cdots \oplus D_t} = D,$$

and therefore $C = \overline{C_1 \oplus \cdots \oplus C_t}$. The set $\tilde{a}^{-1}(S)$ has to be $\mathrm{Aut}(M) \times D_1^{ind} \times \cdots \times D_t^{ind}$ by Krull-Remark-Schmidt theorem. As S contains an open of D, it follows for $1 \leq i \leq t$ D_i^{ind} contains an open of D_i and C_i^{ind} contains an open of C_i. This finishes the proof of the existence statement.
The proof of the uniqueness statement is entirely the same as in [CBS02] mostly just using the Krull-Remark-Schmidt theorem. □

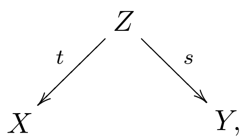
Proof of lemma 67: We use the symbol $\hookrightarrow\!\!\!\!\mapsto$ for closed immersions. In [CBS02], proof of thm 1.3(i), the authors construct for any algebra Λ and dimension vectors ${}_1\underline{\mathbf{d}}, {}_2\underline{\mathbf{d}}$ schemes $V_\Lambda^{ses} \hookrightarrow\!\!\!\!\mapsto \mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}({}_1\underline{\mathbf{d}} + {}_2\underline{\mathbf{d}}) \times \mathrm{V}^{\mathrm{ses}}({}_1\underline{\mathbf{d}}, {}_2\underline{\mathbf{d}})$. It comes together with two regular maps (for the existence, see loc.cit.):



Now, assume that ${}_1\underline{\mathbf{d}}$ is a filtration, ${}_1\underline{\mathbf{d}} + {}_2\underline{\mathbf{d}} = (\underline{d}, \ldots, \underline{d})$ and consider $\mathrm{R}_{\mathrm{Q}}(\underline{\mathrm{d}}) \hookrightarrow\!\!\!\!\mapsto \mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}((\underline{\mathrm{d}}, \ldots, \underline{\mathrm{d}}))$ via $M \mapsto (M = M = \cdots = M)$. We then can define a closed subscheme $Z \hookrightarrow\!\!\!\!\mapsto V_\Lambda^{ses}$ on K-valued points via

$$\begin{aligned} Z(K) = \{(m, \theta, \phi) \in V_\Lambda^{ses} \cap \mathrm{R}_{\mathrm{Q}}(\underline{\mathrm{d}}) \times \mathrm{V}^{\mathrm{ses}}({}_2\underline{\mathbf{d}}, {}_1\underline{\mathbf{d}}) \mid \\ (m, \theta, \phi) = \mathcal{E}\colon 0 \to U \to (M = \cdots = M) \to V \to 0 \text{ in } \Lambda - mod, \\ \mathrm{fp}(\mathcal{E}) = M \xrightarrow{id} M \to 0\} \end{aligned}$$

As before, if it is clear from the context, we will just write M instead of $(M = \cdots = M)$. Then, the mapping roof restricts to



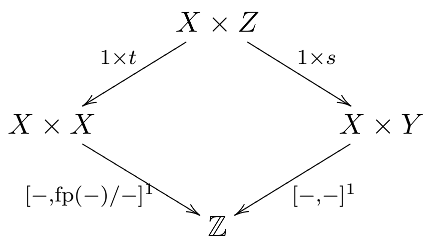
where $X(K) := \{(U_0 = 0 \hookrightarrow U_1 \hookrightarrow \cdots \hookrightarrow U_\nu) \in \mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}({}_1\underline{\mathbf{d}})\}$,
$Y(K) := \{(V_0 \twoheadrightarrow V_1 \twoheadrightarrow \cdots \twoheadrightarrow V_\nu = 0) \in \mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}({}_2\underline{\mathbf{d}})\}$. Let R, P be the $Q_0^{\otimes\nu+1}$-graded K-vector spaces underlying the points of $\mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}((\underline{\mathrm{d}}, \ldots, \underline{\mathrm{d}}))(\mathrm{K})$, $\mathrm{R}_{\mathrm{Q}^{\otimes\nu+1}}({}_2\underline{\mathbf{d}})(\mathrm{K})$ respectively, let G be the automorphism group of P and W be the variety whose K-rational points are given by the K-linear surjections $R \to P$. It has a transitive G-operation via $\theta \mapsto g\theta$, $g \in G, \theta \in W$. Then using the description from loc. cit., we get a closed embedding

$V_\Lambda^{ses} \hookrightarrow X \times W$, describing V_Λ^{ses} as $\{(U,\theta) \in X \times W \mid \theta(U) = 0\}$, we get a factorization of s in the following way

$$s\colon Z \hookrightarrow V_\Lambda^{ses} \hookrightarrow X \times W \xrightarrow{pr_1} X.$$

The first two closed embeddings are G-stable, therefore s is mapping closed G-stable subsets to closed subsets.

Now, have a look at the following commuting diagram:



with $[-, \mathrm{fp}(-)/-]^1\colon \mathrm{X} \times \mathrm{X} \to \mathbb{Z}, (\mathrm{U}, \mathrm{U}' = (0 = \mathrm{U}'_0 \hookrightarrow \cdots \hookrightarrow \mathrm{U}'_\nu = \mathrm{M}')) \mapsto [\mathrm{U}, \mathrm{M}'/\mathrm{U}']^1$ and $[,]^1\colon X \times Y \to \mathbb{Z}, (U,V) \mapsto [U,V]^1$, the second map is known to be upper semicontinuous by [CBS02]. Let $e \in \mathbb{Z}$, it follows that

$$(1 \times t)^{-1}\{(U,V) \in X \times Y \mid [U,V]^1 \geq e\} =$$
$$\{(U, 0 \to U' \to M \to V \to 0) \in X \times Z \mid [U,V]^1 \geq e\}$$

is a closed subset of $X \times Z$. Letting G operate just on the second factor of $X \times Z$, the subset is also G-stable. Now, as s maps G-stable closed subsets to closed subsets, it is easy to see that $1 \times s$ does the same, and it follows that

$$\{(U, U' = (0 = U'_0 \hookrightarrow \cdots \hookrightarrow U'_\nu = M')) \in X \times X \mid [U, M'/U']^1 \geq e\}$$

is a closed subset of $X \times X$. Then also the following map is upper-semicontinuous $\kappa\colon X \times X \to \mathbb{Z}, (U,U') \mapsto$

$$[U,U'] - [M,M'] + [U,M'/U'] = [U,U']^1 - [M,M']^1 + [U,M'/U']^1$$

which is the same as $\dim\ker((U,U')^1 \to (M,M')^1)$. So the equivalence is trivial as the intersection of the open locus, where $[,fp()/]^1$ is minimal with the open locus, where $[-,-]^1\colon X \times X \to \mathbb{Z}$ is minimal has to be the open locus where the map κ is minimal.

□

Corollary 6.4.2.1. *If $C_i \subset \mathrm{Fl}_Q\binom{M_i}{{}_i\underline{\mathbf{d}}}, 1 \leq i \leq t$ are irreducible components that are generically smooth and generically indecomposable Λ-modules inside $\mathrm{Fl}_Q\binom{M_i}{{}_i\underline{\mathbf{d}}}$, and $\overline{C_1 \oplus \cdots \oplus C_t} \subset \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ is an irreducible component, then it is also generically smooth in $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$.*

Proof: The irreducible component C is generically smooth if and only if $\dim C = [C, M/C]$. By studying fibres of the map a from the beginning one can prove

$$\dim C = \sum_{i=1}^{t} \dim C_i + \sum_{i \neq j} [M_i, M_j] - [C_i, C_j],$$

(this is analogue to [CBS02], page 3). Then using the generically smoothness of the C_i and the condition from theorem 6.4.2 the claim follows. □

We call an irreducible component of $C \subset \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ **an orbit closure** if it is of the form $C = \overline{C_1 \oplus \cdots \oplus C_t}$ with the property that the general module in each $D_i = \varphi_i^{-1}(C_i)$ is indecomposable Λ-module U with

$$[U, U] - [fp(U), fp(U)] + [U, fp(U)/U] = 0.$$

Since orbit closures are generically smooth, we see that all quiver flag varieties $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ with only finitely many $\mathrm{Aut}(M)$-orbits are generically smooths. For example, we see later that the quiver Grassmannians for quivers of type A_2, A_3 and A_4 have only finitely many orbits. An open question is whether all irreducible components $C \subset \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ with

$$[C, C] - [M, M] + [C, M/C] = 0$$

are orbit closures.

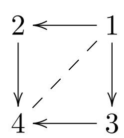
6.5 An example of a closure of a Reineke stratum which is not a union of Reineke strata

This is very detailed, please leave out parts of it.

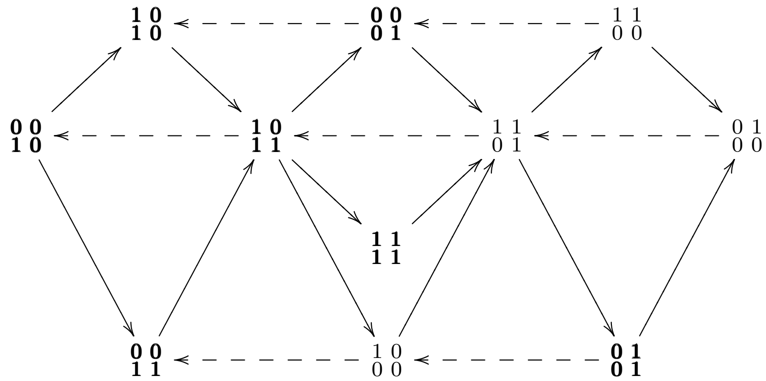
6.5.1 $\mathbb{A}_2$-Grassmannians

The quiver and the Auslander-Reiten quiver for $K\mathbb{A}_2 \otimes K\mathbb{A}_2$

Let (Q, I) be the quiver given by the following square



with I given by the relation $(1 \to 2 \to 4) = (1 \to 3 \to 4)$.
Its Auslander-Reiten quiver is given by



where the bold indecomposables are the indecomposables having monomorphisms at the verticals $\overset{2}{\underset{4}{\downarrow}}, \overset{1}{\underset{3}{\downarrow}}$. As $A := KQ/I$ is representation-finite, the dimension vector determines an indecomposable (right) A-module up to isomorphism, see [ASS06b]. In the following we identify the indecomposables with their dimension vectors.

Degenerations for modules with vertical monos

As $A := KQ/I$ is representation-finite, the degeneration order of A-modules is given by the Hom-order, i.e. for A-modules M, N it holds that

$$M \leq_{\deg} N :\Leftrightarrow [S, M] \leq [S, N] \quad \text{for all indecomposable } S,$$

see [Bon98].
As the category of (right) A-modules is Krull-Schmidt, every A-module M determines a multiplicity function

$$[M]\colon \{\text{vertices of the AR-quiver}\} \to \mathbb{N}_0, \;\; S \mapsto m_S^M.$$

From now on, we assume M and N to have monomorphisms at the vertical arrows $\overset{2}{\underset{4}{\downarrow}}, \overset{1}{\underset{3}{\downarrow}}$, i.e. the support of their multiplicity function is contained in the set of bold vertices of the Auslander Reiten quiver.
$[-,-] =?$ **and degeneration inequalities:** writing down a table $\begin{array}{c|c} & S' \\ \hline S & [S,S'] \end{array}$ with S an indecomposable and S' a bold indecomposable:

	$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$
$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	1	1	1	1	1	0	0
$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$	0	1	0	1	1	0	0
$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	0	0	1	1	1	1	1
$\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$	0	0	0	1	1	1	1
$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$	0	0	0	0	1	0	1
$\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$	0	0	0	0	0	1	1
$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	0	0	0	0	0	0	1
$\begin{smallmatrix}1&0\\0&0\end{smallmatrix}$	0	0	0	0	0	0	0
$\begin{smallmatrix}1&1\\0&1\end{smallmatrix}$	0	0	0	0	0	0	1
$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	0	0	0	0	0	0	0
$\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$	0	0	0	0	0	0	0

For $M \cong \bigoplus_{S \text{ indec}} m_S S, N \cong \bigoplus_{S \text{ indec}} n_S S$, $M \leq_{deg} N$ is equivalent to the following seven inequalities:

1) $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$: $m_{\begin{smallmatrix}0&0\\1&0\end{smallmatrix}} + m_{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} + m_{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}0&0\\1&0\end{smallmatrix}} + n_{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} + n_{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$

2) $\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$: $m_{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} + m_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} + n_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}}$

3) $\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$: $m_{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

4) $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$: $m_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

5) $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$: $m_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

6) $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$: $m_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + m_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \leq +n_{\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} + n_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

7) $\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$: $m_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \leq n_{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}$

Isomorphism classes of quiver flags

Let M as above a (Q, I)-represenation with vertical monomorphisms. We fix $\underline{d} := \underline{\dim}\, M := (d_1, \ldots, d_4)$ and $r := rk(M_4 \leftarrow M_3)$. We describe the isomorphism classes of (Q,I)-representations with vertical monomorphisms, dimension vector $\underline{d}$ and $rk(4 \leftarrow 3) = r$. By definition these are given by the solutions (in $\mathbb{N}_0^7$) of the following equations

1. $m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{0\,1}\\ \mathbf{0\,1}}} = d_1$
2. $m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,0}}} + m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} = d_2$
3. $m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{0\,0}\\ \mathbf{0\,1}}} + m_{\substack{\mathbf{0\,1}\\ \mathbf{0\,1}}} = d_3$
4. $m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,0}}} + m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,0}}} + m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} = d_4$
5. $m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,1}}} + m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} = r$

Its solutions in $\mathbb{N}_0^7$ are given by

$$\{ \begin{pmatrix} d_4-d_2-r+d_1 \\ d_2-d_1 \\ r-d_1 \\ 0 \\ d_1 \\ d_3-r \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \mid \lambda, \mu \in \mathbb{Z} \} \cap \mathbb{N}_0^7$$

In other words a solution is given by $(\lambda, \mu) \in \mathbb{Z}^2$ such that the following seven inequalities are fulfilled

(1) $m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,0}}} := d_4 - d_2 - r + d_1 - \lambda + \mu \geq 0$

(2) $m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,0}}} := d_2 - d_1 + \lambda - \mu \geq 0$

(3) $m_{\substack{\mathbf{0\,0}\\ \mathbf{1\,1}}} := r - d_1 + \lambda - \mu \geq 0$

(4) $m_{\substack{\mathbf{1\,0}\\ \mathbf{1\,1}}} := -\lambda \geq 0$

(5) $m_{\substack{\mathbf{1\,1}\\ \mathbf{1\,1}}} := d_1 + \mu \geq 0$

(6) $m_{\substack{\mathbf{0\,0}\\ \mathbf{0\,1}}} := d_3 - r + \mu \geq 0$

(7) $m_{\substack{\mathbf{0\,1}\\ \mathbf{0\,1}}} := -\mu \geq 0$

equivalent to

(4),(7) $(\lambda, \mu) \in (-\mathbb{N}_0) \times (-\mathbb{N}_0)$,

(5),(6) $\mu \geq \max(-d_1, r - d_3)$,

(2),(3) $\mu \leq \min(d_2 - d_1, r - d_1) + \lambda$,

(1) $\mu \geq -d_4 + d_2 + r - d_1 + \lambda$

Degenerations of quiver flags

We describe the degeneration order on isomorphism classes of (Q, I)-representations with vertical monomorphisms, dimension vector $\underline{d}$ and $rk(4 \leftarrow 3) = r$. The degeneration inequalities 1),2),3),5) are then redundant as the dimension vector is fixed. From the equation 4.,5. (i.e. the equality at d_4 and r) it follows that inequality 6) is an equality. Using this we can simplify 4) to $m_{\begin{smallmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} + m_{\begin{smallmatrix}\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} \leq n_{\begin{smallmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} + n_{\begin{smallmatrix}\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}}$. Again by the equality for the rank this is equivalent to $m_{\begin{smallmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} \geq n_{\begin{smallmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}}$. So, we conclude that for M, N with all the imposed properties, the following are equivalent

(a) $M \leq_{deg} N$,

(b) $m_{\begin{smallmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} \geq n_{\begin{smallmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}}$, $m_{\begin{smallmatrix}\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{1}\end{smallmatrix}} \leq n_{\begin{smallmatrix}\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{1}\end{smallmatrix}}$

(c) $\dim \mathrm{Im}(M_3 \to M_4) \cap \mathrm{Im}(M_2 \to M_4) \leq \dim \mathrm{Im}(N_3 \to N_4) \cap \mathrm{Im}(N_2 \to N_4)$,
$\dim \ker(M_1 \to M_2) \leq \dim \ker(N_1 \to N_2)$

The equivalence (b) $\Leftrightarrow$ (c) follows from $\dim \mathrm{Im}(M_3 \to M_4) \cap \mathrm{Im}(M_2 \to M_4) = m_{\begin{smallmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} + m_{\begin{smallmatrix}\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}}$ and $\dim \ker(M_1 \to M_2) = m_{\begin{smallmatrix}\mathbf{0}&\mathbf{1}\\\mathbf{0}&\mathbf{1}\end{smallmatrix}}$.
In other words, if $[M]$ corresponds to $(\lambda, \mu) \in \mathbb{Z}^2$, $[N]$ to $(\gamma, \delta) \in \mathbb{Z}^2$ as in the previous subsection 2.1, then (a) $\Leftrightarrow$ (b) says

$$M \leq_{deg} N \Leftrightarrow \quad \lambda - \mu \geq \gamma - \delta, \mu \geq \delta$$

In this case I also write $(\lambda, \mu) \leq_{deg} (\gamma, \delta)$ and set

$$\overline{(\lambda, \mu)} := \bigcup_{(\lambda,\mu) \leq_{deg} (\gamma,\delta)} (\gamma, \delta) = \{(\gamma, \delta) \mid (\gamma, \delta) \text{ an isom. class }, \lambda - \mu \geq \gamma - \delta, \mu \geq \delta\}.$$

Reineke's stratification

Again look at quiver representations with vertical monomorphisms, fixed dimension vector and rank at $(3 \to 4)$. For $s \in \{0, \ldots, \min(d_1, d_2, r)\}$ define the Reineke stratum

$$\mathcal{F}_s := \text{ union of isomorphism classes with } \mathrm{rk}(1 \to 2) = s.$$

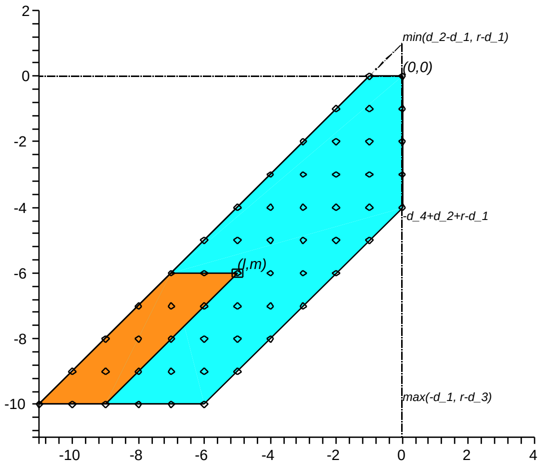
Observe $s = m_{\begin{smallmatrix}\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}\end{smallmatrix}} = d_1 + \mu_0$ for one $\mu_0 \in -\mathbb{N}_0$ and, therefore, the Reineke stratum $\mathcal{F}_s$ is the set Its solutions in $\mathbb{N}_0^7$ are given by

$$\{\begin{pmatrix} d_4-d_2-r+d_1 \\ d_2-d_1 \\ r-d_1 \\ 0 \\ d_1 \\ d_3-r \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu_0 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \mid \lambda, \mu \in \mathbb{Z}\} \cap \mathbb{N}_0^7$$

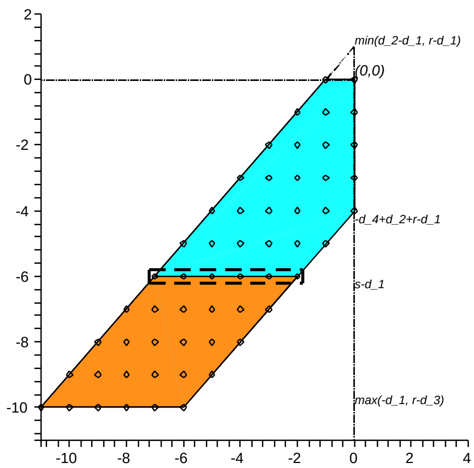
We set $\overline{\mathcal{F}_s} := \bigcup_{(\lambda,\mu) \in \mathcal{F}_s} \overline{(\lambda, \mu)}$ and look for the relative to inclusion maximal $\overline{\mathcal{F}_s}$, an easy way to see this is with a picture, see next subsection.

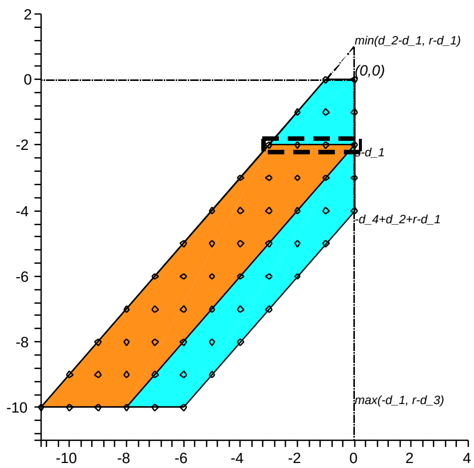
Visualization

The grey area is the set described in subsection 2.1 by the inequalities (1),...,(7). The darker grey area is $\overline{(l,m)}$.



The points in the dotted square are the orbits in the Reineke stratum to s, the darker grey area is its closure.





,

The left hand side is not maximal relative to inclusion of closures, the right hand side is maximal.

We conclude that the closures of Reineke strata maximal relative to inclusion are the ones with $s-d_1 \in \{\min(0, \min(d_2-d_1, r-d_1)), \min(0, \min(d_2-d_1, r-d_1))-1, \ldots, \min(0, -d_4+d_2+r-d_1)\}$. We see that the Reineke stratification does not fulfill the boundary condition.

There is only one irreducible component if and only if $-d_4+d_2+r-d_1 \geq 0$ or $\min(d_2-d_1, r-d_1) = -d_4+d_2+r-d_1$.

6.5.2 Open problems

It would be nice to have a better understanding of the AR-theory of the category $\mathbb{X}$ which might also leed to a better understanding of the decomposition graph for irreducible

components of quiver flag varieties. At the moment, $Q = A_2$ is the only understood example. The finite type investigation for the categories $\mathbb{X}$ has a similar result as for preprojective algebras, which is not surprising when using covering theory, is there more in this connection?
Also, the tame types could be investigated, the case of the Jordan quiver is contained in [BH00a]. In the spirit of [BHRR99] one can investigate a bijection between dense orbits in quiver flag varieties and tilting modules in $\mathbb{X}$. This might be related to the Richardson orbit lemma from the first section. It could be that the category $\mathbb{X}$ is already a special case of [BH00b], but I am not sure of this.

Chapter 7

A_n-equioriented quiver flag varieties

Summary. We study varieties of complete flags in quiver representations for the quiver A_n-equioriented. We refer to the classical case as the same constructions with the Jordan quiver.

* We stratify them by the isomorphism classes of the submodules, we call this Reineke stratification (see [Rei03]). To each stratum we associate a multi-tableau which we call root tableau.
* Then we refine Reineke stratification into a stratification parametrized by multi-tableau with relaxed rules which we call row root tableau. We prove that this gives an affine cell decomposition. As a corollary we can describe the Betti numbers of complete A_n-equioriented quiver flag varieties. In the classical case similar methods have been used by Lukas Fresse, [Fre09].
* We give some conjectural results which we did not prove for time reasons: The hook case can be analogously investigated to Fungs work, see [Fun03].

Remark. These methods can be generalized to uniserial algebras or categories but it is not obvious how they can be extended to other Dynkin quivers.

7.0.3 Notation and basic properties for A_n-equioriented representations

If not stated otherwise in this article, $Q := A_n$-equioriented with the following numbering $1 \to 2 \to \cdots \to n$. Let K be a field.

For $1 \leq i \leq n$ we write S_i for the simple left KQ-module supported in the vertex i. Recall that by Gabriel's theorem the set of (positive) roots can be identified with the set of dimension vectors of indecomposable KQ-modules which is given by $R^+ = \{\alpha_{ij} = \sum_{k=i}^{j} e_k \in \mathbb{N}_0^n \mid i \leq j\}$. We also write $\alpha_{ij} = (ij)$ and for a root α we denote by E_α (or E_{ij}) an indecomposable module with dimension vector α. By the Krull-Schmidt theorem, an isomorphism class of a finite dimensional left KQ-module M is determined by its multiplicities $m^M = (m^M_\alpha)_{\alpha \in R^+}$ i.e. $M \cong \bigoplus_{\alpha \in R^+} m^M_\alpha E_\alpha$.

We denote by $\operatorname{soc} E_{ij} = S_j \xrightarrow{s} E_{ij}$ the inculsion of the socle and by $\operatorname{rad} E_{ij} = E_{i+1,j} \xrightarrow{r} E_{ij}$ if $i < j$ and $\operatorname{rad} S_i = 0 \subset E_{ij}$ the inclusion of the radical.

Simple submodules

The following easy lemma has been observed by Reineke in [Rei03], Prop. 4.9 and Example on p.16.

Lemma 68. *Any short exact sequence $0 \to S_j \to M \to N \to 0$ is isomorphic to short exact sequence*

$$0 \to S_j \xrightarrow{\begin{pmatrix} 0 \\ s \end{pmatrix}} B \oplus E_{ij} \to B \oplus E_{i,j-1} \to 0$$

where s is the inclusion of the socle.

Let M be a representation of Q, by Krull-Schmidt theorem we find a direct sum decomposition into submodules

$$M = \bigoplus_{\alpha} [M_\alpha^1 \oplus \cdots \oplus M_\alpha^{m_\alpha}]$$

with $M_\alpha^t \cong E_\alpha, 1 \leq t \leq m_\alpha, \alpha \in R^+$, we call the projection on direct summands always pr.
Please note that we do not look at the isomorphism class of the module, so the numbering of the (isomorphic) direct summands is important information for us. We call this a **root blocked decomposition** (or **rb-decomposition**) because in difference to an arbitrary direct sum decomposition we require that our decomposition is a refinement of an isotypic decomposition.

Let $U \cong S_j$ be a simple submodule of M, then clearly $U \subset \bigoplus_{k=1}^{j} \bigoplus_{t=1}^{m_{(kj)}} M_{(kj)}^t \subset M$.
By lemma 68, we find (i, j) and a module B such that $M \cong B \oplus E_{ij}, M/U \cong N := B \oplus E_{i,j-1}$.
How do arbitrary simple submodules U of M with $M \cong B \oplus E_{ij}, M/U \cong N := B \oplus E_{i,j-1}$ look like?
The isomorphisms $M \cong B \oplus E_{ij}$ and $M/U \cong N := B \oplus E_{i,j-1}$ are equivalent to the following two conditions on U

a) $U \subset \bigoplus_{t=0}^{i-1} M_{(i-t,j)}^1 \oplus \cdots \oplus M_{(i-t,j)}^{m_{(i-t,j)}}$

b) the composition $U \subset \bigoplus_{t=0}^{i-1} [M_{(i-t,j)}^1 \oplus \cdots \oplus M_{(i-t,j)}^{m^M_{(i-t,j)}}] \xrightarrow{pr} M_{(i,j)}^1 \oplus \cdots \oplus M_{(i,j)}^{m_{(i,j)}}$ is not the zero map.

Therefore, any simple submodule U of M determines a unique $\alpha = (i, j)$ and $a \in \{1, \dots, m_\alpha\}$ such that $M \cong B \oplus E_{ij}, M/U \cong B \oplus E_{i,j-1}$ and $U \to M \xrightarrow{pr} M_\alpha^a$ is not zero and $U \to M \xrightarrow{pr} M_\alpha^t$, $t < a$ is zero. We write $(\alpha, a)[U] := (\alpha, a)$ and call it the **first**

relevant summand with respect to rb-decomposition M_*.
Furthermore, there an is an isomorphism determined by $U \subset M$

$$\phi_U \colon M/U \to M^1_{(1,1)} \oplus \cdots \oplus M^{a-1}_{(i,j)} \oplus \Big(M^a_{(i,j)}/(\operatorname{soc} M^a_{(ij)})\Big) \oplus M^{a+1}_{(ij)} \oplus \cdots \oplus M^{m_{(n,n)}}_{(n,n)}.$$

After a slight reordering this gives a rb-decomposition for M/U,

$$M/U = \bigoplus_{\alpha} \bigoplus_{t=1}^{m_\alpha^{M/U}} (M/U)^t_\alpha$$

with

$$\begin{aligned} (M/U)^t_{(i,j)} &= M^{t+1}_{(i,j)}, && \text{if } t \geq a \\ (M/U)^{m^{M/U}_{(i,j-1)}}_{(i,j-1)} &= M^a_{(i,j)}/(\operatorname{soc} M^a_{(ij)}), && \text{if } i \neq j \\ (M/U)^t_\alpha &= M^t_\alpha, && \text{for all other } \alpha, t. \end{aligned}$$

We call this **the from M induced rb-decomposition on the quotient** M/U. Observe, that the multiplicity function of M/U can be obtained from the multiplicity function of M and vice versa.

We can reformulate this to

Corollary 7.0.0.2. *Let M by a KQ-module. There is a function*

$$\{\mathit{Simples} \subset M\} \to R^+$$

mapping the simple U to the indecomposable (ij) such that $M \cong B \oplus E_{ij}, M/U \cong B \oplus E_{i,j-1}$. For a given rb-decomposition M_* of M, there is a function

$$\{\mathit{Simples} \subset M\} \to R^+ \times \mathbb{N}_0, \quad U \mapsto (\alpha, a)[U]$$

mapping a simple U to the first relevant summand with respect to the rb-decomposition M_* (see above).
This implies that we find for any simple submodule $U \subset M$ a function

$$\{\text{ rb-dec of } M\} \to \{\text{ rb-dec of } M/U\}.$$

defined by the from M induced rb-decomposition on M/U (see above).

7.1 Reineke strata and root tableaux

Now, let K be an algebraically closed field.

Definition 21. Let $Q = (Q_0, Q_1)$ be a quiver, M be a KQ-module of dimension $\underline{\dim}\, M =: \underline{d} \in \mathbb{N}_0^{Q_0}$ and $\mathbf{d} := (0 = \underline{d}^0, \underline{d}^1, \ldots, \underline{d}^r = \underline{d})$ with $\underline{d}^t \in \mathbb{N}_0^{Q_0}$ be a sequence with $\underline{d}^t_i \leq \underline{d}^{t+1}_i,\ 1 \leq$

$t \leq r-1, i \in Q_0$. We define

$$\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} := \{U = (0 \subset U^1 \subset \cdot \subset U^r = M) \mid U^t \ \ KQ\text{-module}, \underline{\dim}\, U^t = \underline{d}^t\}$$

where $\subset$ means submodule in the category of KQ-modules. This defines a projective K-variety, we call it the **quiver flag variety** associated to M and the dimension filtration $\underline{\mathbf{d}}$ and its (K-rational) points **quiver flags of dimension $\underline{\mathbf{d}}$**. Recall, for $i \in Q_0$, we have a simple module S_i with $\underline{\dim}\, S_i =: e_i$ is supported on the vertex i. If $\underline{d}^{t+1} - \underline{d}^t = e_{i_t}$ for an element $i_t \in Q_0$, $1 \leq t \leq r-1$, then we call the quiver flags of dimension $\underline{\mathbf{d}}$ **complete flags** and $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ a **variety of complete quiver flags**.

We also have the following stratifications of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$.

Definition 22. Let $M, \underline{\mathbf{d}}$ as before. Pick $N_* := (N_1, \dots, N_{r-1})$ a sequence of KQ-modules.

(1) We define

$$\mathcal{F}_{[N_*]} := \{U = (0 \subset U^1 \subset \cdots \subset U^r = M) \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \mid \\ U^t \cong N_t \text{ in } KQ\text{-mod},\ 1 \leq t \leq r-1\}.$$

If Q is a Dynkin quiver, this defines a stratification of $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ into locally closed irreducible smooth subsets see [Rei03]. We call this stratification **Reineke stratification**.

(2) We define

$$\mathcal{S}_{[N_*]} := \{U = (0 \subset U^1 \subset \cdots \subset U^\nu = M) \in \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \mid \\ M/U^t \cong N_t \text{ in } KQ\text{-mod},\ 1 \leq t \leq r-1\}.$$

If Q is Dynkin, this is a stratification into finitely many locally closed irreducible smooth subsets. We call this stratification **Spaltenstein stratification** because it has been studied for the Jordan quiver in [Spa76].

Remark. The two stratifications are mapped to each other under the following isomorphism

$$\hat{D}\colon \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Fl_{Q^{op}}}\binom{DM}{\underline{\mathbf{e}}}, \\ U \mapsto (\hat{D}(U))^t := \ker(DM \to DU^t) = D({}^{M}/{}_{U^t})$$

where $\underline{\mathbf{e}} = (\underline{e}^0, \underline{e}^1, \dots, \underline{e}^r)$, $\underline{e}^{r-t} = \underline{d}^r - \underline{d}^t$ and $D := \mathrm{Hom}_K(-, K)$, cp. [Wol09], definition 6.11, p.64. Therefore, it is enough to investigate one of the two stratifications.

We will investigate Spaltenstein strata for $Q := 1 \to 2 \to \cdots \to n$ and Reineke strata for $Q := 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$.

We associate to a Reineke stratum a combinatorial object called root tableau, this is the analogue of Spaltenstein's stratification of classical Springer fibres with respect to standard tableau, [Spa76].

Let us stick for a moment with Reineke strata and $Q = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ and leave the dual situation to the reader.

Definition 23. Let M be a finite dimensional representation of Q. A **root diagram** of type Q is a sequence of $Y = (Y_1, \ldots, Y_n)$ of possibly empty Young diagrams with the numbers of columns in Y_i is less or equal $n - i + 1$, $1 \le i \le n$.
Let $Y' = (Y'_1, \ldots, Y'_n), Y = (Y_1, \ldots, Y_n)$ be two root diagrams of type Q, the we write $Y' \subset Y$ if Y'_i is a Young subdiagram of Y_i, $1 \le i \le n$ (recall one Young diagram S' is a Young subdiagram of another S if for each row the length in S' is shorter than in S.)
A **root tableau** for Q is a sequence $T = (T_1 \subset T_2 \subset \cdots \subset T_r)$ with T_i root diagram of type Q, $1 \le i \le r$.
We visualize a root tableau of type Q via writing down the sequence of Young diagrams $(Y_1, \ldots, Y_n)$ of T_r.
Here, we renumber the columns as follows: The k-th column of Y_i is from now on in the $(i + k - 1)$th column of Y_i and we start with Y_1, write Y_2 under it, … , write Y_n under it (respecting the numbering of the columns), we end up with a skew diagram with n columns.
Then put 1 in all boxes lying inside the subdiagram T_1, put 2 inside all boxes inside T_2 not in T_1, …, put r in all boxes not inside T_{r-1}.

Definition 24. For a root diagram $Y = (Y_1, \ldots, Y_n)$ with Y_i a Young diagram with rows of lengths $\ell_1^{(i)} \ge \ell_2^{(i)} \ge \cdots \ge \ell_{r_i}^{(i)}$, $1 \le i \le n$, we associate a module

$$M_Y := \bigoplus_{i=1}^{n} \bigoplus_{s=1}^{r_i} E_{i, i+\ell_s^{(i)}-1}$$

and we set $\underline{\dim}\, Y := \underline{\dim}\, M_Y \in \mathbb{N}_0^n$.
For a root tableau $T = (T_1 \subset \ldots \subset T_r)$ we associate a Spaltenstein stratum

$$\mathcal{S}_T := \mathcal{S}_{[0, M_{T_1}, \ldots, M_{T_r} =: M]} \subset \mathrm{Fl_Q}\binom{M_{T_r}}{\underline{\mathbf{d}}_T}$$

and a Reineke stratum

$$\mathcal{F}_T := \mathcal{F}_{[M_{T_1}, \ldots, M_{T_r}]} \subset \mathrm{Fl_Q}\binom{M_{T_r}}{\underline{\mathbf{d}}_T}$$

with $\underline{\mathbf{d}}_T := (0, \underline{\dim}\, M_{T_1}, \underline{\dim}\, M_{T_2}, \ldots, \underline{\dim}\, M_{T_r})$.

By the previous considerations we get.

Lemma 69. *(1) There is a bijection between Spaltenstein strata in* $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ *and root tableau* $T = (T_1, \ldots, T_r)$ *with* $M_{T_r} \cong M$ *and* $\underline{\mathbf{d}}_T = \underline{\mathbf{d}}$.

(2) There is a bijection between Reineke strata in $\mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ *and root tableau* $T = (T_1, \ldots, T_r)$ *with* $M_{T_r} \cong M$ *and* $\underline{\mathbf{d}}_T = \underline{\mathbf{d}}$.

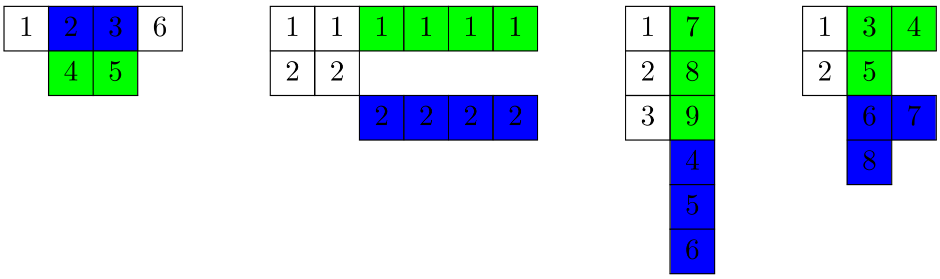
Proof: Ad (2), we map $T \mapsto \mathcal{F}_T$. For $N_1, \ldots N_{r-1}, N_r = M$ the Reineke stratum is nonempty if and only there exist monomorphisms $N_k \to N_{k+1}$, $1 \leq k \leq r-1$, cp. [Rei03] . Let T_i be the root diagram corresponding to the isomorphism class of N_i, since there are these monomorphisms we get a sequence $T = (T_1 \subset T_2 \subset \cdots \subset T_r)$, i.e. we obtain a root tableau.

Ad (1), follows from the bijection described in the first remark in this section. □

7.1.1 Swapping numbered boxes in row root tableaux

Definition 25. Let $Q = \mathbb{A}_n$ and let $T = (T_1, \ldots, T_r)$ be a root tableu for Q. We call a connected subset of numbered boxes in one Young tableau T_i of T a **brick**. We call it row brick if b lies in one row, column brick if b lies in one column. Let c be another brick obtained by translating b some rows up or down. We say that (b, c) is **admissible for T** if the skew tableau obtained by swapping b and c is again a root tableau. In case b, c are admissible for T, then we write $\delta_{bc}T$ for the root tableau obtained from T by swapping the boxes b and c. In this case we say (b, c) **decomposable** if there are admissible pairs (b', c') for T and (b'', c'') for $\delta_{b',c'}T$ such that $\delta_{b,c}T = \delta_{b''c''}\delta_{b'c'}T$, we say (b, c) is **indecomposable** if it is not decomposable.

Example. If the bricks in admissible pair is just a single boxes, then it is indecomposable. Here are further examples of indecomposable admissible pairs, b is the blue (=dark grey) brick and c is the green (=light grey) brick. They can have arbitrary shape.



An admissible pair (b, c) for T is indecomposable if and only if the maximal number within one of the bricks b, c is strictly smaller than the minimal number within the other brick.

Remark. Observe, that there might be boxes b which might not be part of any admissible pair. We call this the fixed boxes of T. When looking at the boxes fixed in all T with the same $\underline{\mathbf{d}}_T = \underline{\mathbf{d}}$ (i.e. in all columns are the same numbers just permuted in order) we call them the **fixed** boxes. Then, the boxes in the first column of (any) T will be fixed. If b, c are admissible for T and its shape is a root diagram $Y = (Y_1, \ldots, Y_n)$, then b and c belong to different Y_i.

7.1.2 Dimension of root tableau

Let $Q = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$. For a Reineke stratum defined by $0 \subset N_1 \subset \cdots \subset N_r = M$ we define

$$a_k := \dim \operatorname{Hom}(N_{k-1}, N_k) - \dim \operatorname{End}(N_{k-1}), 1 \leq k \leq r.$$

In [Rei03], thm 4.2 Reineke proved $\dim \mathcal{F}_{[N_*]} = \sum_{k=1}^{r} a_k$. In particular, $a_k = \dim \mathcal{F}_{[0,N_{k-1},N_k]}$ where $\mathcal{F}_{[0,N_{k-1},N_k]} \subset \mathrm{Gr}\binom{N_k}{\underline{\dim}\, N_{k-1}}$ is the Reineke stratum in the quiver Grassmannian.
We want to read this number from the root tableau. Even though it is enough to understand the Grassmannian case, we start with understanding the case of complete flags and then afterwards we look at the Grassmannian case. Recall that for Reineke strata in complete flags there is for every $k \in \{1, \ldots, r\}$ an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{k-1} & \longrightarrow & N_k & \longrightarrow & E_{j_k} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B \oplus E_{i_k,j_k-1} & \longrightarrow & B \oplus E_{i_k,j_k} & \longrightarrow & E_{j_k} & \longrightarrow & 0 \end{array}$$

for a module B and some $i_k < j_k$ in $\{1, \ldots, n\}$. It follows

$$\begin{aligned} a_k &= \dim \mathrm{Hom}(B \oplus E_{i_k,j_k-1}, B \oplus E_{i_k j_k}) - \dim \mathrm{Hom}(B \oplus E_{i_k,j_k-1}, B \oplus E_{i_k,j_k-1}) \\ &= \dim \mathrm{Hom}(B, E_{i_k j_k}) - \dim \mathrm{Hom}(B, E_{i_k,j_k-1}) \end{aligned}$$

Recall, that $\dim \mathrm{Hom}(E_{ab}, E_{ij}) = 1$ if and only if $a \leq i \leq b \leq j$ implying that for $B = E_{ab}$ we have $a_k \in \{0, 1\}$ and

$$a_k = 1 \iff a \leq i_k \text{ and } b = j_k.$$

Now, observe for $B = \bigoplus_{a \leq b} b_{ab} E_{ab}$ we have $b_{ab} = n_{ab}^{(k-1)}$ for all $(a, b) \neq (i_k, j_k - 1)$, therefore we proved the following

Lemma 70. *Let $N_k = \bigoplus_{a \leq b} n_{ab}^{(k)} E_{ab}$, $1 \leq k \leq r$ be defining a Reineke stratum of complete flags, suppose $N_{k-1} \to N_k$ is isomorphic to $\mathrm{id} \oplus (E_{i_k,j_k-1} \to E_{i_k,j_k})$. Then, it holds*

$$a_k = \sum_{a \leq i_k} n_{a j_k}^{(k-1)}.$$

In the case of partial flags, we know for all $k \in \{1, \ldots, r\}$ by [Lus91], Lemma 1.8 that there exists a complete flag of submodules in N_k/N_{k-1} this means that every Reineke stratum has a refinement to a Reineke stratum in a complete flag. Using our knowledge on complete flags this implies that there exists an injection $N \to M$ if and only if there exists direct sum decompostions $N = B \oplus S, M = B \oplus S^{+x} \oplus R$ such that $S = \bigoplus_{i \leq j} s_{ij} E_{ij}$ and $S^{+x} = \bigoplus_{i \leq j} \bigoplus_{s=1}^{s_{ij}} E_{i,j+x_s^{(ij)}}$ for certain integers $1 \leq x_1^{(ij)} \leq x_2^{(ij)} \leq \cdots \leq x_{s_{ij}}^{(ij)}$. The $B = \bigoplus_{a \leq b} b_{ab} E_{ab}$ is the largest common direct summand of N and M. There might be more than one choice of S^{+x} and R, but the root tableau gives us unique choices (up to isomorphism), see later. We get that the number

$$\begin{aligned}
a_{NM} &:= \dim \operatorname{Hom}(N, M) - \dim \operatorname{Hom}(N, N) \\
&= \dim \operatorname{Hom}(B \oplus S, B \oplus S^{+x} \oplus R) - \dim \operatorname{Hom}(B \oplus S, B \oplus S) \\
&= [\dim \operatorname{Hom}(B, S^{+x}) - \dim \operatorname{Hom}(B, S)] + \dim \operatorname{Hom}(N, R) \\
&= [\sum_{i \le j} \sum_{s=1}^{s_{ij}} (\dim \operatorname{Hom}(B, E_{i,j+x_s^{(ij)}}) - \dim \operatorname{Hom}(B, E_{ij}))] + \dim \operatorname{Hom}(N, R)
\end{aligned}$$

We have $\dim \operatorname{Hom}(E_{ab}, E_{i,j+x_s^{(ij)}}) - \dim \operatorname{Hom}(E_{ab}, E_{ij}) = 1$ if and only if $a \le i$ and $b \in \{j+1, \dots, j+x_s^{(ij)}\}$. So, we get

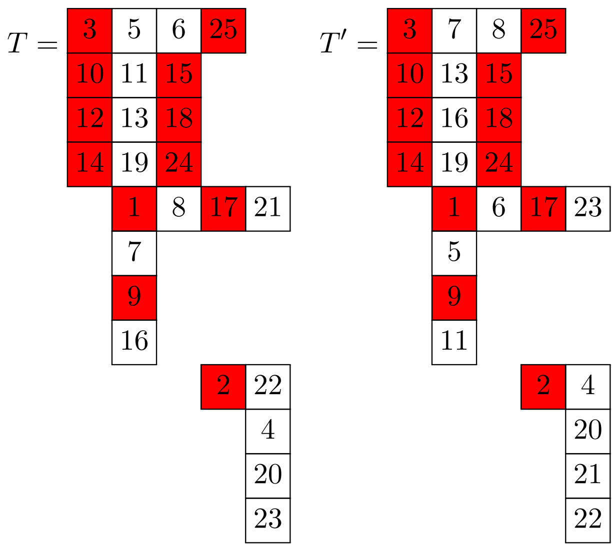
$$a_{NM} = [\sum_{i \le j} \sum_{s=1}^{s_{ij}} \sum_{a \le i} (b_{a,j+1} + \cdots + b_{a,j+x_s^{(ij)}})] + \dim \operatorname{Hom}(N, R)$$

Now, in terms of root tableau, N injects to M if and only if the root diagram of N is a subdiagram of M, let‘s fill the boxes of the root diagram for N with 1 and the remaining boxes of the root diagram of M we fill with 2. The largest common direct summand B is given by the the rows completely filled with 1, R is given by the rows completely filled wit 2, S^{+x} is given by the rows containing 1 and 2, S is given by the subdiagram of S^{+x} just filled with 1.
Then, the previous formula gives an easy recipe to calculate a_{NM}. The reader is encouraged to try an example on its own.

Example. Example for A_5-equioriented $\longleftarrow \; \longleftarrow \; \longleftarrow \; \longleftarrow$,

$$M = E_{14} \oplus 3E_{13} \oplus E_{25} \oplus 3E_2 \oplus E_{45} \oplus 3E_5,$$

we give two root tableau defining Reineke strata in the same flag variety, i.e. they have in each column the same numbers written (up to permutation). I take complete flags, so we can identify boxes with their numbers within. The fixed boxes are colored red (=grey) for the convenience of the reader.



We can try to use this now to calculate the dimension change when we do an admissible swap within on root tableau-

How do we calculate the maximal dimension of a Reineke stratum? Actually, I do not know, but I present my thoughts here.

We define for any root diagram and dimension filtration $\underline{\mathbf{d}}$, s.t. there exists at least one root tableau T with $\underline{\mathbf{d}}_T = \underline{\mathbf{d}}$, a root diagram T_{max} with $\underline{\mathbf{d}}_{T_{max}} = \underline{\mathbf{d}}$ in the following way. Go through the columns from left to right. Go downwards through each column filling in not fixed boxes always the smallest of all possible choices from the still free numbers for that column (possible choice means that there exists a root tableau containing the filling until that point).

We call it the **maximal** root tableau associated to a root diagram and a dimension filtration $\underline{\mathbf{d}}$. In particular we can associated to a quiver flag variety $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ a maximal root tableau T_{max}.

Then, $\mathcal{F}_{T_{max}}$ is often the highest dimensional Reineke stratum but not always. Also we have no control of the dimension of a Reinke stratum after wapping an admissible pair. Here is an example where it is not the highest dimensional stratum.

Example.

$$T_{max} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \hline & 5 \\ \cline{2-2} & 6 \\ \cline{2-2} \end{array} \qquad T = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \hline & 2 \\ \cline{2-2} & 6 \\ \cline{2-2} \end{array} \qquad T' = \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \hline & 2 \\ \cline{2-2} & 5 \\ \cline{2-2} \end{array}$$

Then, using the formular we calculate

(1) For T_{max} it holds $a_1 = a_2 = a_3 = 0$, $a_4 = a_5 = 1$, $a_6 = 2$, therefore $\dim \mathcal{F}_{T_{max}} = 4$.

(2) For T it holds $a_1 = a_2 = 0$, $a_3 = 1$, $a_4 = 2$, $a_5 = 0$, $a_6 = 2$, therefore $\dim \mathcal{F}_T = 5$.

(3) For T' it holds $a_1 = a_2 = 0$, $a_3 = 1$, $a_4 = 2$, $a_5 = 1$, $a_6 = 0$, therefore $\dim \mathcal{F}_{T'} = 4$.

These are the only Reineke strata in that quiver flag variety, it follows that its maximal dimension is 5.

7.2 rb-strata and row root tableaux

This implies, if we have a point in a complete quiver flag variety $(0 \subset U^1 \subset \cdots \subset U^r = M) \in \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$, then a rb-decomposition on M induces an rb-decomposition on M/U^k, $1 \leq k \leq r-1$.

Definition 26. Fix a direct sum decomposition of M indexed as before. Fix a complete dimension filtration $\underline{\mathbf{d}}$ of $\underline{\dim} M$. For a sequence $\underline{\alpha, a} := (\alpha_1, a_1), (\alpha_2, a_2), \ldots (\alpha_r, a_r)$ with

α_t roots, $1 \leq t \leq r$ and $a_t \in \mathbb{N}_0$ we define the set

$$\mathcal{S}_{\underline{\alpha,a}} := \{(0 \subset U^1 \subset \cdots \subset U^r = M) \in \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} \mid (\alpha_k, a_k) = (\alpha, a)[U^k/U^{k-1}]$$

$$\text{wrt to the induced rb-dec on } M/U^{k-1}$$

and call it the **rb-stratum** for $\underline{\alpha, a}$.

We see later that this defines a locally closed subset of $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$, see corollary 7.3.0.3. The rb-stratification is not $\mathrm{Aut}(M)$-invariant. Choosing another direct sum decomposition gives rise to a stratification which can be obtained by applying an element of $\mathrm{Aut}(M)$.

Definition 27. Let Y be a root diagram (of type A_n-equioriented) with r boxes. We call a filling τ of the boxes in Y by $1, \dots, s$ $(s \leq r)$ **row root tableau** of shape Y if the numbering is weakly increasing in the rows.
Each row root tableau τ has an associated root tableau $T(\tau) = (T_1, \dots, T_s)$, where T_i is the root diagram which you get when looking at the boxes filled by $1, \dots, i$ and permute the rows.

rb-stratification is finer than Spaltenstein stratification because it is not only fixing the isomorphism type of the module where the simple is mapping to but the module itself. So, if there are two isomorphic modules, it is saying into which one the simple is mapping. This is reflected in the row root tableau, where filling in the numbers sees every indecomposable submodule of M as a row of boxes and in each step you fill in the number in the row corresponding to the root where the simple is mapping to. In other words.

Lemma 71. *Let M be a representation of $Q = A_n$-equioriented, $\underline{\mathbf{d}}$ be a complete dimension filtration. Let $Y = Y_M$ be the root diagram of M and T be a root tableau of shape Y and dimension filtration $\underline{\mathbf{d}}_T = \underline{\mathbf{d}}$.*
Then, there is a bijection between rb-strata $\mathcal{S}_{(\underline{a})}$ inside the Spaltenstein stratum corresponding to T and row root tableau τ with $T(\tau) = T$.

Open questions: Are closures of rb–strata unions rb-strata? Is there a good formula for its dimension and or codimension in the Spaltenstein stratum? What is the right definition of the inversion number and can we give a formula for it as Fresse did?

Our main aim in the next section is to show that rb-strata are affine spaces. We first find a special point within each rb-stratum, a so called split module, see next subsection.

7.2.1 Split Modules

For a moment allow Q to be an arbitrary finite quiver. Let $\Lambda = KQ \otimes_K K\mathbb{A}_n$. Let $\mathbb{X} \subset \Lambda$ mod be the subcategory of $\mathbb{A}_n$-monomorphisms.

Definition 28. A Λ-module $0 \subset U_1 \subset \cdots \subset U_n = M$ in $\mathbb{X}$ is called **split** with respect to a direct sum decomposition $M = M_1 \oplus \cdots \oplus M_r$ in indecomposable submodules if

$$U_s = (U_s \cap M_1) \oplus \cdots \oplus (U_s \cap M_r), \quad 1 \leq s \leq n-1.$$

We call a Λ-module in $\mathbb{X}$ **split** if there exists a direct sum decomposition of its flagpole such that it is split with respect to this direct sum decomposition. Fix an additive family of modules with for each isomorphism type precisely one module. The full subcategory (of $\mathbb{X}$) of split modules with respect to this family defines an additive subcategory $S\mathbb{X}$ of $\mathbb{X}$.

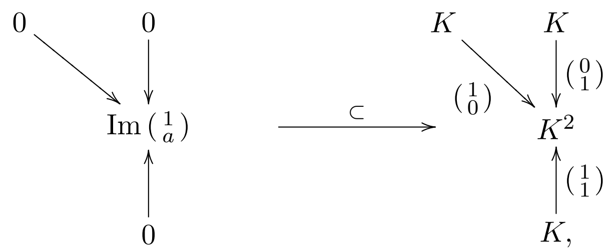
Remark. Let $U := (0 \subset U_1 \subset \cdots \subset U_n = M)$ be a split module with respect to $M = M_1 \oplus \cdots \oplus M_r$, we set $(U \cap M_i) := 0 \subset (U_1 \cap M_i) \subset \cdots \subset (U_n \cap M_i) = M_i$, then $U = (U \cap M_1) \oplus \cdots \oplus (U \cap M_r)$ is a direct sum decomposition into indecomposable Λ-submodules.

We recall some notions from [Rei03], section 4. A quiver is called **co-special**, if for all indecomposable E_α and all simples S_i, we have $\dim \operatorname{Hom}(S_i, E_\alpha) \leq 1$. We call a vertex $i \in Q_0$ thick if there exists a root α such that $\dim_i \alpha \geq 2$. Then, Reineke proved that a quiver is co-special if and only if no thick vertex is a sink ([Rei03], Prop. 4.8).

Lemma 72. *If Q is co-special, then $S\mathbb{X}$ is representation-finite. In particular, this holds for Q Dynkin of type A.*

Proof: We need to see that the number of Λ-modules with flagpole equal an indecomposable M is finite up to $\operatorname{Aut}(M)$-isomorphism. If M has a dimension vector in $\{0,1\}^{Q_0}$ it has only a finite number of submodules, so there is nothing to prove in this case.
In general, let $0 \subset U_1 \subset \cdots \subset U_n = M$. If $\dim \operatorname{Hom}(U'_j, M) = 1$ for all indecomposable direct summands U'_j of U_j, then there exists a unique submodule of isomorphism type U_j inside M. But as Q is co-special this is automatically fulfilled. $\square$

Example. Also, for Q arbitrary Dynkin and K a finite field the category $S\mathbb{X}$ is representation finite, for trivial reason. But if K is infinte, $S\mathbb{X}$ is representation infinite for Q (not co-special) Dynkin quiver of type D_4. Consider the following family of indecomposable Λ-modules



$a \neq 0, 1$ in K. They are pairwise non-isomorphic.

From now on we investigate the case $Q = \mathbb{A}_n$-equioriented more closely. Recall that $r+1$ is the length of the sequence of the flag. Let $i \leq j$ in $\{1, \ldots, n\}$. We define a Λ-modules as follows

(1) For $Q = 1 \leftarrow \cdots \leftarrow n$ let $\lambda = (\lambda_0, \lambda_{ii}, \lambda_{i,i+1}, \ldots, \lambda_{i,j})$ be a sequence $\lambda_0, \lambda_{i,i+k} \in \mathbb{N}_{\geq 1}, \lambda_0 + \sum_{k=0}^{j-i} \lambda_{i,i+k} = r+1$ we define

$$E_{ij}(\lambda) := \left(\underbrace{0 = \cdots = 0}_{\lambda_0 - \text{times } 0} \subset \underbrace{S_i = \cdots = S_i}_{\lambda_{ii} - \text{times } S_i} \subset E_{i,i+1} = \cdots = E_{i,j-1} \subset \underbrace{E_{ij} = \cdots E_{ij}}_{\lambda_{ij} - \text{times } E_{ij}} \right).$$

(2) For $Q = 1 \rightarrow \cdots \rightarrow n$ let $\lambda = (\lambda_0, \lambda_{jj}, \lambda_{j-1,j}, \ldots, \lambda_{i,j})$ be a sequence $\lambda_0, \lambda_{j-k,j} \in \mathbb{N}_{\geq 1}, \lambda_0 + \sum_{k=0}^{j-i} \lambda_{j-k,j} = r+1$ we define

$$E_{ij}(\lambda) := \left(\underbrace{0 = \cdots = 0}_{\lambda_0 - \text{times } 0} \subset \underbrace{S_j = \cdots = S_i}_{\lambda_{jj} - \text{times } S_j} \subset E_{j-1,j} = \cdots = E_{i+1,j} \subset \underbrace{E_{ij} = \cdots E_{ij}}_{\lambda_{ij} - \text{times } E_{ij}} \right).$$

As always we write $S_i(\lambda)$ instead of $E_{ii}(\lambda)$. Obviously, the $E_{ij}(\lambda)$ are indecomposable Λ-modules.

Lemma 73. *The category $S\mathbb{X}$ is Krull-Schmidt. The isomorphism classes of indecomposable objects in $S\mathbb{X}$ are given by the $E_{ij}(\lambda)$ defined above. It holds $\operatorname{End}_\Lambda(E_{ij}(\lambda), E_{k\ell}(\mu)) = K$.*

Proof: It is Krull-Schmidt because it is closed under taking direct summands. Clearly, the $E_{ij}(\lambda)$ are the indecomposable objects. □

Lemma 74. *Any rb-stratum containes precisely one split module. There is a bijection between isomorphism classes of split modules and row root tableau.*

Proof: Let $Q = 1 \leftarrow \cdots \leftarrow n$. Let $U = (0 \subset U^1 \subset \cdots \subset U^\nu = M)$ be a split module. Recall that U^t, U^{t+1} have an induced direct sum decomposition which is respected be the inclusion. Then, U^t is the image of a KQ-linear map $\bigoplus_k \phi_k$ where ϕ_k is on all but one summand the identity and on the last one it is a nonzero $s_{i,j} \colon E_{i,j-1} \to E_{ij}, i < j$ or $s_i \colon 0 \to S_i$ for some $i, j \in Q_0$. The data on which direct summand the non-identity map occurs is given by the row root tableau because the direct summands correspond to the rows in the tableau. We illustrate this with an example. □

Example. Let us write down the split module corresponding to

2	5	
1		
3		
	6	7
	4	

$Q = 1 \leftarrow 2 \leftarrow 3$ and $r + 1 = 8$ is the length of the flag. It is a Λ-module of the form

$$V = (0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_6} V_7 =: M)$$

with

$$V_1 = S_1,\ \ V_2 = S_1^2,\ \ V_3 = S_1^3,\ \ V_4 = S_1^3 \oplus S_2,$$
$$V_5 = E_{12} \oplus S_1^2 \oplus S_2,\ \ V_6 = E_{12} \oplus S_1^2 \oplus S_2^2,\ \ V_7 = E_{12} \oplus S_1^2 \oplus S_2 \oplus E_{23}.$$

and

$$f_1 = s_1 \oplus \mathrm{id}_{S_1},\ \ f_2 = \mathrm{id}_{S_1^2} \oplus s_1,\ \ f_3 = \mathrm{id}_{S_1^3} \oplus s_2$$
$$f_4 = s_{12} \oplus \mathrm{id}_{S_1^2} \oplus \mathrm{id}_{S_2},\ \ f_5 = \mathrm{id}_{E_{12}} \oplus \mathrm{id}_{S_1^2} \oplus s_2 \oplus \mathrm{id}_{S_2},\ \ f_6 = \mathrm{id}_{E_{12}} \oplus \mathrm{id}_{S_1^2} \oplus s_{23} \oplus \mathrm{id}_{S_2}.$$

It holds

$$V = E_{12}(2,3,3) \oplus S_1(1,7) \oplus S_1(3,5) \oplus S_2(4,4) \oplus E_{23}(6,1,1)$$

Example. Assume we have a semisimple KQ-representation M . We fix a root blocked decomposition

$$M = \underbrace{S_1 \oplus \cdots \oplus S_1}_{m_1-\text{times}} \oplus \cdots \oplus \underbrace{S_n \oplus \cdots \oplus S_n}_{m_n-\text{times}}$$

by choosing a basis $e_1^{(1)}, \dots, e_{m_1}^{(1)}, e_1^{(2)}, \dots, e_{m_n}^{(n)}$ such that the k-th summand in the S_i-block is $Ke_k^{(i)}$. We let a torus $T = (K^*)^{m_1} \times \cdots \times (K^*)^{m_n}$ act by rescaling the basis vectors. Every complete quiver flag variety $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ (for an arbitrary dimension filtration $\underline{\mathbf{d}}$) is isomorphic to $\mathbf{Gl}_{m_1}/B_1 \times \cdots \times \mathbf{Gl}_{m_n}/B_n$ for the upper triangular matrices $B_i \subset \mathbf{Gl}_{m_i}, 1 \leq i \leq n$. This can be seen as follows: The dimension filtration corresponds to a word in the vertices $\underline{i} = (i_1, \dots, i_r), i_j \in Q_0$. This induces a permutation of the basis vectors as above (for example $\underline{i} = (1,2,1,1)$ then the reordering is $e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_3^{(1)}$). This gives an element $\sigma \in S_N$ with $N = \sum_{k=1}^n m_k$. If we see the quiver flag naturally embedded into $\mathbf{Gl}_N/B$ with B upper triangular, then conjugation with π^{-1} gives the identification with $\mathbf{Gl}_{m_1}/B_1 \times \cdots \times \mathbf{Gl}_{m_n}/B_n$.
The described torus is (also after the permutation) the diagonal torus. We leave it to the reader to see that the T-fixed points on them are precisely the split modules and the rb-strata are the products of Schubert cells.

7.3 rb-stratification as affine cell decomposition

Spaltenstein's fibration

In [Spa76], Spaltenstein's main tool to proof the equi-dimensionality of the classical Springer fibres is that the *taking Spaltenstein strata* map is locally trivial when restricted to certain Schubert cells. We show here that we have the analogue in the case of an equioriented A_n-quiver, even though Spaltenstein strata are not equi-dimensional. As a corollary we obtain that the quiver flag varieties admit a affine cell decomposition. As we made the choice to work with Reineke strata instead of Spaltenstein strata, we stick with it and remark here that with applying the isomorphism $\hat{D}$ one can rewrite everything in this section in terms of Spaltenstein strata; instead of characteristic flags you would define co-characteristic flags, in the examples all kernels and images (or socles and radicals) would be swapped, instead projecting onto the Grassmannian of hyperplanes you would project onto the projective space. We write down both versions because in the literature both are common.

Short recall on relative position Let $\mathrm{Fl}(\underline{\mathbf{d}}) := \prod_{\mathrm{i}\in\mathrm{Q}_0}\mathrm{Fl}(\underline{\mathrm{d}}_\mathrm{i}), \mathrm{Fl}(\underline{\mathbf{e}}) := \prod_{\mathrm{i}\in\mathrm{Q}_0}\mathrm{Fl}(\underline{\mathrm{e}}_\mathrm{i})$ with $\underline{d}_i = (0 = d_i^0 \leq d_i^1 \leq \cdots \leq d_i^\nu), \underline{e}_i = (0 = e_i^0 \leq e_i^1 \leq \cdots \leq e_i^\mu)$ with $d_i^\nu = e_i^\mu$ for all $i \in Q_0$.
Then we define the relative position map to be

$$\mathrm{rp}\colon \mathrm{Fl}_\mathrm{Q}\binom{\mathrm{M}}{\underline{\mathbf{d}}} \times \mathrm{Fl}_\mathrm{Q}\binom{\mathrm{M}}{\underline{\mathbf{e}}} \to \prod_{i\in Q_0}\mathrm{Mat}\,\big((\nu+1)\times(\mu+1),\mathbb{N}_0\big)$$
$$(U_i^\bullet, V_j^\bullet)_{i,j\in Q_0} \mapsto \big((\dim U_i^k \cap V_i^l)_{0\leq k\leq\nu, 0\leq l\leq\mu}\big)_{i\in Q_0}$$

Given $w \in \prod_{i\in Q_0}\mathrm{Mat}\,\big((\nu+1)\times(\mu+1),\mathbb{N}_0\big), \mathrm{V} \in \mathrm{Fl}_\mathrm{Q}\binom{\mathrm{M}}{\underline{\mathbf{e}}}$ we call

$$\mathrm{Fl}(\underline{\mathbf{d}})_{\mathrm{V,w}} := \{\mathrm{U} \in \mathrm{Fl}_\mathrm{Q}\binom{\mathrm{M}}{\underline{\mathbf{d}}} \mid \mathrm{rp}(\mathrm{U},\mathrm{V}) = \mathrm{w}\}$$

generalized Schubert cell. This is an orbit under the diagonal $\mathbf{Gl}_{\underline{d}} := \prod_{i\in Q_0}\mathbf{Gl}_{d_i^\nu}$-operation. For another point in $V' \in \mathrm{Fl}_\mathrm{Q}\binom{M}{\underline{\mathbf{e}}}$ we get an isomorphic Schubert cell.
Let $M \in \mathrm{R}_\mathrm{Q}(\underline{\mathrm{d}})$ be a representation, then we set

$$\begin{aligned}\mathrm{Fl}_\mathrm{Q}\binom{M}{\underline{\mathbf{d}}}_{V,w} &:= \mathrm{Fl}_\mathrm{Q}\binom{M}{\underline{\mathbf{d}}} \cap \mathrm{Fl}(\underline{\mathbf{d}})_{\mathrm{V,w}} \\ &= \{U \in \mathrm{Fl}(\underline{\mathbf{d}}) \mid \forall(\mathrm{i}\to\mathrm{j}) \in \mathrm{Q}_1, \mathrm{k} \in \{0,\ldots,\nu\}\colon \\ &\quad M_{i\to j}(U_i^k) \subset U_j^k, \mathrm{rp}(\mathrm{U},\mathrm{V}) = \mathrm{w}\}.\end{aligned}$$

For a fixed V this gives the **stratification by relative position (with V)** in $\mathrm{Fl}_\mathrm{Q}\binom{M}{\underline{\mathbf{d}}}$. We leave out the index V if it clear which is meant.

Definition 29. Let Q be any quiver and M be a representation of Q.

(1) We call a flag F_M of Q_0-graded vector spaces inside the underlying Q_0-graded vector space $\underline{M}$ of M **characteristic flag for** $(M, \underline{\mathbf{d}} = (\underline{d}^1, \ldots, \underline{d}^\nu))$ if for all $N, N' \in \mathrm{Gr}_\mathrm{Q}\binom{M}{\underline{\mathbf{d}}^i}$: $\mathrm{ED}(\underline{\mathrm{N}} \subset \underline{\mathrm{M}}, \mathrm{F_M}) = \mathrm{ED}(\underline{\mathrm{N}}' \subset \underline{\mathrm{M}}, \mathrm{F_M})$ implies N, N' define the same Reineke strata (i.e. $N \cong N'$ as KQ-modules).
We call it **characteristic flag for** M if it is a characteristic flag for all filtration $\underline{\mathbf{d}}$. We call it **characteristic flag for cosimples in** M if it is a characteristic flag for all $\underline{d}$ with $\underline{d} + e_i = \underline{\dim}\, M$ for some $i \in Q_0$.

(2) We call F_M **co-characteristic flag for** $(M, \underline{\mathbf{d}} = (\underline{d}^1, \ldots, \underline{d}^\nu))$ if for all $N, N' \in \mathrm{Gr}_\mathrm{Q}\binom{M}{\underline{\mathbf{d}}^i}$: $\mathrm{ED}(\underline{\mathrm{N}} \subset \underline{\mathrm{M}}, \mathrm{F_M}) = \mathrm{ED}(\underline{\mathrm{N}}' \subset \underline{\mathrm{M}}, \mathrm{F_M})$ implies N, N' define the same Spaltenstein strata (i.e. $M/N \cong M/N'$ as KQ-modules). We call it **co-characteristic flag for** M if it is a co-characteristic flag for all filtration $\underline{\mathbf{d}}$. We call it **co-characteristic flag for simples in** M if it is a co-characteristic flag for all e_i for some $i \in Q_0$.

We start with some preparation for the main result.

Remark. * F_M is a characteristic flag for $(M, \underline{\mathbf{d}} = (\underline{d}^1, \ldots, \underline{d}^\nu))$ if and only if F_M is a co-characteristic flag for $(D(M), \underline{\mathbf{e}} = (\underline{e}^0, \underline{e}^1, \ldots, \underline{e}^\nu))$, $\underline{e}^{\nu-i} = \underline{d}^\nu - \underline{d}^i$.
* Any refinement of a (co-)characteristic flag for $(M, \underline{\mathbf{d}})$ is a again a (co-)characteristic flag for $(M, \underline{\mathbf{d}})$.
* A flag is characteristic for $(M, \underline{\mathbf{d}} = (\underline{d}^1, \ldots, \underline{d}^\nu))$ if the associated stratification by relative position is finer than the Reineke stratification in $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$. It is co-characteristic for $(M, \underline{\mathbf{d}} = (\underline{d}^1, \ldots, \underline{d}^\nu))$ if the associated stratification by relative position is finer than Spaltenstein stratification in $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$.

(1) Assume F_M is characteristic flag for $M, \underline{d}^{\nu-1}$ in a dimension filtration $\underline{\mathbf{d}} = \underline{d}^0, \ldots, \underline{d}^\nu = \underline{\dim}\, M)$. Let $\underline{\mathbf{e}} := (\underline{d}^0, \ldots, \underline{d}^{\nu-1})$. Let us denote by $p\colon \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}$ the *forgetting all other than the* $(\nu-1)$*-th subspace* map. For every $U, U' \in \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}_w$ seen as submodules of M it holds $U \cong U'$ as KQ-modules, so fix one submodule N in this isomorphism class. Then, the fibres $p^{-1}(U)$ over a relative position stratum $\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}_w$ are isomorphic to $\mathrm{Fl}_{\mathrm{Q}}\binom{N}{\underline{\mathbf{e}}}$. Therefore, we get a commutative diagram

$$\begin{array}{ccccc} \mathrm{Fl}_{\mathrm{Q}}\binom{N}{\underline{\mathbf{e}}} & \longrightarrow & p^{-1}\big(\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}_w\big) & \xrightarrow{p_w} & \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}_w \\ & \searrow^{(rs,[N])} & \downarrow{\scriptstyle rs} & & \\ & & \mathcal{R} & & \end{array}$$

where $rs\colon \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} \to \mathcal{R} := \{\text{Reineke strata in } \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}\}, (0 \subset U^1 \subset \cdots \subset U^\nu = M) \mapsto ([U^1], \ldots, [U^{\nu-1}])$ and p_w is the restriction of p.

(2) Assume F_M is co-characteristic flag for $M, \underline{d}^1$ in a dimension filtration $\underline{\mathbf{d}} = \underline{d}^0, \ldots, \underline{d}^\nu = \underline{\dim}\, M)$. Let $\underline{\mathbf{e}} := (\underline{d}^1 - \underline{d}^1, \underline{d}^2 - \underline{d}^1, \ldots, \underline{d}^\nu - \underline{d}^1)$. Let us denote by $p\colon \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^1}$ the *forgetting all other than the* 1*-st subspace* map. For every $U, U' \in \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^1}_w$ seen as submodules of M it holds $M/U \cong M/U'$ as KQ-modules, so fix one quotient module N in this isomorphism class. Then, the fibres $p^{-1}(U)$ over a relative position stratum $\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^{\nu-1}}_w$ are isomorphic to $\mathrm{Fl}_{\mathrm{Q}}\binom{N}{\underline{\mathbf{e}}}$. Therefore, we get a commutative diagram

$$\begin{array}{ccccc} \mathrm{Fl}_{\mathrm{Q}}\binom{N}{\underline{\mathbf{e}}} & \longrightarrow & p^{-1}\big(\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^1}_w\big) & \xrightarrow{p_w} & \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^1}_w \\ & \searrow^{([N],sp)} & \downarrow{\scriptstyle sp} & & \\ & & \mathcal{S} & & \end{array}$$

where $sp\colon \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}} \to \mathcal{S} := \{\text{Spaltenstein strata in } \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}\}, (0 \subset U^1 \subset \cdots \subset U^\nu = M) \mapsto ([M/U^1], \ldots, [M/U^{\nu-1}])$ and p_w is the restriction of p.

The morphism p is $\mathrm{Aut}(M)$-equivariant but the relative position stratum $\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}^1}_w$ is only $\mathrm{Aut}(M)$-invariant if the flag F_M is M-invariant, in that case p_w is also $\mathrm{Aut}(M)$-equivariant.

Our main result is that for some choices of $Q, \underline{\mathbf{d}}, F_M$ we get that p_w is isomorphic to the projection map $\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w \times \mathrm{Fl_Q}\binom{N}{\underline{\mathbf{e}}} \to \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w$.

Example. Let Q be $1 \to 2 \to \cdots \to n$. Let $M = (K^{d^1} \xrightarrow{A_1} K^{d^2} \to \cdots \to K^{d^{n-1}} \xrightarrow{A_{n-1}} K^{d^n})$.

(i) *a characteristic flag*:
is given by the underlying Q_0-graded flag of the following flag of submodules

$$0 \subset \mathrm{soc}^1(M) \subset \mathrm{soc}^2(M) \subset \cdots \subset \mathrm{soc}^n(M) = M$$

where

$$\mathrm{soc}^i(M) = \big(\ker(A_i A_{i-1} \cdots A_1) \to \ker(A_{i+1} A_i \cdots A_2) \to \cdots \to \ker(A_{n-1} \cdots A_{n-i}) \to \ker(A_{n-1} \cdots A_{n-i+1}) \to \cdots \to \ker(A_{n-1}) \to K^{d^n}\big).$$

Here we even get that the relative position stratification equals the Reineke stratification.

(ii) *a co-characteristic flag:*
is given by the underlying Q_0-graded flag of the following flag of submodules

$$0 \subset \mathrm{rad}^n(M) \subset \mathrm{rad}^{n-1}(M) \subset \cdots \subset \mathrm{rad}^1(M) \subset M$$

where

$$\mathrm{rad}^i(M) = \big(0 \to \cdots \to 0 \to \mathrm{Im}(A_i \cdots A_1) \to \mathrm{Im}(A_{i+1} \cdots A_2) \to \cdots \to \mathrm{Im}(A_{n-1} \cdots A_{n-i})\big).$$

Again, here we even get that the relative position stratification equals the Spaltenstein stratification.

(iii) *characteristic flag for cosimples:*
is given by the underlying Q_0-graded flag of the following flag of submodules

$$0 \subset \mathrm{rad}(M) \cap \mathrm{soc}(M) \subset \mathrm{rad}(M) \cap \mathrm{soc}^2(M) \subset \cdots \subset \mathrm{rad}(M) \cap \mathrm{soc}^{n-1}(M) \subset \mathrm{rad}(M) \subset M$$

where $\mathrm{rad}(M) \cap \mathrm{soc}^i(M)$ is given by

$$\ker(A_i \cdots A_1) \to \mathrm{Im}(A_1) \cap \ker(A_{i+1} \cdots A_2) \to \mathrm{Im}(A_2) \cap \ker(A_{i+3} \cdots A_3) \to \cdots \to \mathrm{Im}(A_{n-i-1}) \cap \ker(A_{n-1} \cdots A_{n-i}) \to \mathrm{Im}(A_{n-i}) \cap \ker(A_{n-1} \cdots A_{n-i+1}) \to \cdots \to \mathrm{Im}(A_{n-2}) \cap \ker(A_{n-1}) \to \mathrm{Im}(A_{n-1})$$

(iv) *co-characteristic flag for simples:*

is given by the underlying Q_0-graded flag of the following flag of submodules

$$0 \subset \operatorname{soc}(M) \cap \operatorname{rad}^n(M) \subset \cdots \subset \operatorname{soc}(M) \cap \operatorname{rad}^2(M) \subset \operatorname{soc}(M) \cap \operatorname{rad}(M) \\ \subset \operatorname{soc}(M) \subset M$$

where as in the previous examples $\operatorname{soc}(M) \cap \operatorname{rad}^i(M)$ equals

$$\begin{aligned} &\ker(A_1) \to \ker(A_2) \to \cdots \to \ker(A_i) \\ &\to \ker(A_{i+1}) \cap \operatorname{Im}(A_i \cdots A_1) \to \ker(A_{i+2}) \cap \operatorname{Im}(A_{i+1} \cdots A_2) \to \cdots \\ &\to \ker(A_{n-1}) \cap \operatorname{Im}(A_{n-2} \cdots A_{n-i-1}) \to \operatorname{Im}(A_{n-1} \cdots A_{n-i}) \end{aligned}$$

In example (iii) and (iv) there is a recursive relation to Reineke/Spaltenstein stratification, and for (iv) for complete flags to rb-stratification. We focus on example (iv).

The special property for example (iii) and (iv) are that they allow us to pass from quiver-graded to usual Grassmannians:
Just recall the following for a module M we have

1) For a hyperplane $H \subset \underline{M}$ it holds:

$$H \text{ is } M-\text{invariant} \iff \underline{\operatorname{rad}(M)} \subset H$$

this is saying for a dimension vector $\underline{e}$ with $\underline{d}-\underline{e} = e_i$ for one $i \in Q_0$, $\underline{r} := \underline{\dim} \operatorname{rad}(M)$ we have an isomorphism $gr\colon \operatorname{Gr}_Q\binom{M}{\underline{e}} \cong \operatorname{Gr}\binom{d_i-r_i}{d_i-r_i-1}$.

2) For a line $L \subset \underline{M}$ it holds:

$$L \text{ is } M-\text{invariant} \iff L \subset \underline{\operatorname{soc}(M)}$$

this is saying for a dimension vector $\underline{e}$ with $\underline{e} = e_i$ for one $i \in Q_0$, $\underline{s} := \operatorname{soc}(M)$ we have an isomorphism $gr\colon \operatorname{Gr}_Q\binom{M}{\underline{e}} \cong \operatorname{Gr}\binom{s_i}{1} = \mathbb{P}^{s_i-1}$.

Then, we can prove the following analogue of a result of Spaltenstein [Spa76], Lemma on page 453.

Proposition 9. *(Spaltenstein's fibration) For $Q = A_n$-equioriented, and $\underline{\mathbf{d}} = (\underline{d}^0, \underline{d}^1, \ldots, \underline{d}^\nu = \underline{d})$ a complete dimension filtration (i.e. for every $k \in \{1, \ldots, \nu\}$ there is $i \in Q_0$ such that $\underline{d}^k = \underline{d}^{k-1} + e_i$). Let $M \in \mathrm{R_Q}(\underline{\mathrm{d}})$ and F_M be a complete co-characteristic flag for simples refining the flag*

$$0 \subset soc(M) \cap rad^n(M) \subset \cdots \subset soc(M) \cap rad^2(M) \subset soc(M) \cap rad(M) \\ \subset soc(M) \subset M.$$

Then, for every $w \in \prod_{i \in Q_0} \operatorname{Mat}\,(1 \times (\nu+1))$ there is an isomorphism of algebraic varieties

$$f\colon p^{-1}(\operatorname{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w) \to \operatorname{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w \times \operatorname{Fl_Q}\binom{N}{\underline{\mathbf{e}}}$$

with $\underline{\mathbf{e}} := (\underline{d}^1 - \underline{d}^1, \underline{d}^2 - \underline{d}^1, \dots, \underline{d}^\nu - \underline{d}^1)$*,* $p\colon \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}} \to \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}$ *the* forgetting all other than the 1-st subspace *map,* $N = M/U_0$ *with* $(U_0 \subset M) \in \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w$ *arbitrary, such that the following diagram is commutative*

$$\begin{array}{ccccc} & & \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w & & \\ & \overset{p}{\nearrow} & & \overset{pr_1}{\nwarrow} & \\ p^{-1}(\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w) & & \xrightarrow{\quad f \quad} & & \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w \times \mathrm{Fl_Q}\binom{N}{\underline{\mathbf{e}}} \\ & \underset{sp}{\searrow} & & \underset{([N],sp)\circ pr_2}{\swarrow} & \\ & & \mathcal{S}. & & \end{array}$$

Proof: Let $L \in \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w = \{[0 : \dots : 0 : 1 : x_1 : \dots : x_r] \in \mathbb{P}^{s_j - 1} \mid x_i \in K, 1 \leq i \leq r\}$. Then we can define an automorphism $\phi_L \in \mathrm{Aut}(M)$ such that $\phi_L(L) = U_0$. We define

$$\begin{aligned} f\colon p^{-1}(\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1})_w &\to \mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w \times \mathrm{Fl_Q}\binom{N}{\underline{\mathbf{e}}} \\ U = (U^1 \subset \dots \subset U^\nu = M) &\mapsto (U^1, \phi_{U^1}(U)/U_0). \end{aligned}$$

This is a morphism of algebraic maps. To find the inverse, we consider $\pi\colon M \to M/U_0 = N$ the canonical projection and define

$$\begin{aligned} &\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w \times \mathrm{Fl_Q}\binom{N}{\underline{\mathbf{e}}} \to p^{-1}(\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}_w) \\ &\big(L, V = (V^1 \subset \dots \subset V^{\nu-1} = N)\big) \\ &\quad \mapsto (L \subset \phi_L^{-1}\pi^{-1}(V^1) \subset \phi_L^{-1}\pi^{-1}V^2 \subset \dots \subset \phi_L^{-1}\pi^{-1}V^{\nu-1} = M). \end{aligned}$$

□

Corollary 7.3.0.3. *Every* $\mathcal{S}_{\underline{a},\underline{\alpha}} \subset \mathrm{Fl_Q}\binom{M}{\underline{\mathbf{d}}}$ *is a locally closed subsets and is an affine space.*

Proof: Using the Spaltenstein fibration iteratively one can write $\mathcal{S}_{\underline{\alpha},\underline{a}}$ as a pullback of locally closed subset, so they are locally closed themselves. Also the Spaltenstein map shows that they are affine spaces. Let us shortly recall the induction step. We consider $\mathrm{Gr_Q}\binom{M}{\underline{\mathbf{d}}^1}) = \mathbb{P}^{s_j-1} = \{$ lines in $M^1_{j,j} \oplus \dots \oplus M^{m_{1,j}}_{1,j}\}$ and the restriction of the map p from the previous lemma

$$\begin{aligned} \mathcal{S}_{\underline{\alpha},\underline{a}} &\to \mathbb{P}^{s_j-1} \\ U &\mapsto U^1. \end{aligned}$$

We consider the Schubert cell $C_{\alpha_1,a_1} := \{[0 : \dots : 0 : 1 : x_1 : \dots : x_r] \in \mathbb{P}^{s_j-1} \mid x_i \in K, 1 \leq i \leq r\}$ where the first nonzero entry is in position a_1 in the root block corresponding to α_1. Let $\underline{a}' = (a_2, \dots, a_\nu), \underline{\alpha}' = (\alpha_2, \dots, \alpha_r)$. Then, the map p restricts as follows

$$\begin{array}{ccc} \mathcal{S}_{\underline{\alpha},\underline{a}} & \xrightarrow{\quad p \quad} & \mathbb{P}^{s_j-1} \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle \subset} \\ \mathcal{S}_{\underline{\alpha}',\underline{a}'} \times C_{\alpha_1,a_1} & \xrightarrow[pr_2]{} & C_{\alpha_1,a_1} \end{array}$$

where the map f' is an isomorphism, it is the restriction of the map f^{-1} from the previous lemma. □

Now, let us come to the split modules. Let $M = \bigoplus_\alpha \bigoplus_k M_\alpha^k$ be a root blocked decomposition,

$$T := \{\bigoplus_\alpha \bigoplus_k \lambda_\alpha^k \operatorname{id}_{M_\alpha^k} \mid \lambda_\alpha^k \in K^*\} = \bigoplus_\alpha \bigoplus_k \operatorname{Aut}(M_\alpha^k) \subset \operatorname{Aut}(M)$$

Lemma 75. *Let* $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ *a complete* A_n*-equioriented quiver flag variety. The split modules in* $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ *are precisely the* T*-fixed points.*

Proof: By definition split modules are fixed under the torus operation. On the other hand, fix a vector space basis adapted to the root blocked decomposition. A flag of submodules fixed by the torus action is a split module because each subspace of the flag has to have up to scalar multiples a subset of the base vectors of the bigger space. □

7.3.1 Betti numbers for complete A_n-equioriented quiver flag varieties

We explain the notion of an affine cell decomposition and why this is a desireable property by giving applications on calculating *(co)homology*. More precisely, for schemes with an affine cell decomposition we state

Definition 30. Let X be a scheme. An affine cell decomposition is a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes, with each $X_i \setminus X_{i-1}$ is a disloint union of finitely many schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. We call V_{ij} the closure of U_{ij} in X.

Properties of Chow groups

This is a citation of results of [Ful98].
Let $A_*(X)$ the (graded) Chow group.

1) [Ful98], Example 19.1.11, p.378
   Let X be a complex algebraic variety. Let $H_*(X)$ be Borel-Moore homology.
   It is stated that for a scheme with an affine cell decomposition, the cycle class map

$$cl_X \colon A_*(X) \to H_*(X)$$

   is an isomorphism. All odd homology groups vanish.

2) [Ful98], Example 1.9.1 on page 23
   If X has an affine cell decomposition as in the definition. Then the $[V_{ij}]$ form a basis for $A_*(X)$.

Betti numbers

This is a citation of a citation taken from [Fre09], section 4.
Let X be an algebraic variety. Let $H^*(X,\mathbb{Q})$ denote the sheaf cohomology for the constant sheaf $\mathbb{Q}_X$, let $H_c^*(X,\mathbb{Q})$ denote the sheaf cohomology with compact support for the constant sheaf $\mathbb{Q}_X$.
Let X be a complex algebraic variety with an affine cell decomposition as in the definition, i.e. $X = \bigcup_{i,j} U_{i,j}$ with $U_{ij} \subset X$ locally closed and isomorphic to an affine space. For $m \in \mathbb{N}_0$ let r_m be the number of m-dimensional cells. Then it holds

a) $H_c^l(X,\mathbb{Q}) = 0$ for l odd and $H_c^{2m}(X,\mathbb{Q}) = r_m$ for any $m \in \mathbb{N}_0$.

b) If X is projective, then $H^l(X,\mathbb{Q}) = 0$ for l odd and $H^{2m}(X,\mathbb{Q}) = r_m$ for any $m \in \mathbb{N}_0$.

Recall that we have found an affine cell decomposition for any complete $\mathbb{A}_n$-quiver flag variety $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ which is parametrized by the set of row root tableau of the shape Y_M, where Y_M is the root diagramm associated to the representation M.

Let $K = \mathbb{C}$.

Corollary 7.3.0.4. *Let Q be the quiver $1 \to \cdots \to n$, let M be any finite dimensional $\mathbb{C}Q$-module and $\underline{\mathbf{d}}$ be a dimension filtration of $\underline{\dim}\, M$. Let $\mathcal{T} := \{\tau \mid \tau$ rr-tableau of shape $Y_M, \underline{\mathbf{d}}_{T(\tau)} = \underline{\mathbf{d}}\}$ and let $c_i := \#\{\tau \in \mathcal{T} \mid \dim \tau = i\}, i \in \mathbb{N}_0$. Then, the following holds true for $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$*

(1) There is a $\mathbb{C}$-vector space basis of $A_*(\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}})$ parametrized by $\mathcal{T}$.

(2) There is a $\mathbb{C}$-vector space basis of $H_*^{BM}(\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}})$ parametrized by $\mathcal{T}$.

(3) The Betti numbers are given by

$$h^m_{Betti}(\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}) = \begin{cases} c_{\frac{m}{2}}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

7.4 Conjectural part

From here on, proofs are incomplete. It was planned as part of the chapter but had been forgotten.

7.4.1 Canonical decomposition for A_n-equioriented quiver flag varieties

Any rb-stratum is contained in a Reineke stratum and a Spaltenstein stratum. We want to show the following.

Lemma 76. *Let $\mathcal{S}_\tau \subset \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ be the rb-stratum corresponding to a row root tableau τ and let V_τ be the unique split module in $\mathcal{S}_\tau$. It holds*

$$[\mathcal{S}_\tau, \mathcal{S}_\tau] = [V_\tau, V_\tau]$$
$$[\mathcal{S}_\tau, M/\mathcal{S}_\tau] = [V_\tau, M/V_\tau]$$

It has the following corollaries.

Corollary 7.4.0.5. *Let $\mathcal{S}_\tau \subset \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ be the rb-stratum corresponding to a row root tableau τ and let V_τ be the unique split module in $\mathcal{S}_\tau$. Assume that $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ is generically reduced. Then*

(1) It holds that $\overline{\mathcal{S}_\tau}$ is an irreducible component of $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ if and only if

$$[V_\tau, M/V_\tau]_\Lambda = \sum_{k=0}^{r} [V_\tau^{k-1}, V_\tau^k] - [V_\tau^{k-1}, V_\tau^{k-1}].$$

Recall, that the right hand side is the formular for $\dim T(\tau)$.

(2) We consider for a moment arbitrary A_n-equioriented quiver flag varieties. Let $C_1 = \overline{\mathcal{S}_{\tau_1}} \subset \mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$, $C_2 = \overline{\mathcal{S}_{\tau_2}} \subset \mathrm{Fl}_Q\binom{N}{\underline{\mathbf{e}}}$, then the following are equivalent

(i) $\overline{C_1 \oplus C_2}$ is an irreducible component of $\mathrm{Fl}_Q\binom{M\oplus N}{\underline{\mathbf{e}}+\underline{\mathbf{d}}}$.

(ii) For $i \neq j$ it holds

$$[V_{\tau_i}, M_j/V_{\tau_j}]_\Lambda = \sum_{k=0}^{r} [V_{\tau_i}^{k-1}, V_{\tau_j}^k] - [V_{\tau_i}^{k-1}, V_{\tau_j}^{k-1}].$$

(iii) We call an irreducible component $C = \overline{C_1 \oplus \cdots \oplus C_t}$ with C_i is an irreducible component such that

$$[C_i, C_i] - [M_i, M_i] + [C_i, M_i/C_i] = 0, \quad 1 \leq i \leq t$$

preprojective. *It holds*

(1) Every preprojective component is the orbit closure of a split Λ-module V_C.

(2) The canonical decompostion for preprojective irreducible components in the sense of [Hub13] is determined by the composition into indecomposables in $S\mathbb{X}$. This means if $V = V_1 \oplus \cdots \oplus V_t$ is the decomposition into indecomposable in $S\mathbb{X}$, then the orbit closure of V is an irreducible component if and only if

$$[V_i, V_j] - [M_i, M_j] + [V_i, M_j/V_j] = 0$$

for all $i \neq j$.

In particular, the indecomposable preprojective irreducible components are orbit closures of indecomposables $E_{ij}(\lambda)$ (see previous paragraph).

7.4.2 Submodules in terms of matrix normal forms

Let $N \subset M$ be a submodule, we call $M_{(j)}$ = sum of all direct summands with socle j, $1 \leq j \leq n$. Then we have $M = M_{(1)} \oplus \cdots \oplus M_{(n)}$, $N = N_{(1)} \oplus \cdots \oplus N_{(n)}$ and $(N_{(i)}, M_{(j)}) = 0$

for $i < j$. Let $\mathrm{Inj}(N, M) \subset (N, M)$ be denote the subset of monomorphisms. Then, we have

$$\mathrm{Inj}(N, M) = \begin{bmatrix} \mathrm{Inj}(N_{(1)}, M_{(1)}) & (N_{(2)}, M_{(1)}) & \cdots & (N_{(n)}, M_{(1)}) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & (N_{(n)}, M_{(n-1)}) \\ 0 & \cdots & 0 & \mathrm{Inj}(N_{(n)}, M_{(n)}) \end{bmatrix}$$

We fix direct sum decompositions for

$$N = \bigoplus_{j=1}^{n} \underbrace{\bigoplus_{t=0}^{j-1} \underbrace{[N_{j-t,j}^{1} \oplus \cdots \oplus N_{j-t,j}^{n_{j-t,j}}]}_{N_{(j-t,j)}}}_{:=N_{(j)}}$$

with $N_{kj}^{t} \cong E_{kj}$ and respectively for M and we fix a graded vector space basis for $\underline{N}, \underline{M}$ adapted to the direct sum decompositions. For a monomorphism $\phi \colon N \to M$ we denote by $A_{(ij)}[k]$ the matrix (wrt to the fixed basis) of the induced morphism $N_{(j)} \to M_{(i)}$ at the vertex k and by $A_{(i,s),(j,t)}[k]$ the matrix of $N_{(j,t)} \to M_{(i,s)}$ at the vertex k. Observe, $A_{(ij)}[k] = 0$ for $k > i$

$$A_{(ij)}[k] = \left[\begin{array}{c|ccc} A_{(k,i),(k,j)}[k] & 0 & \cdots & 0 \\ \hline A_{(k-1,i),(k,j)}[k] & & & \\ \vdots & & & \\ A_{(1,i),(k,j)}[k] & & & A_{(ij)}[k-1] \end{array}\right], \text{ for } k \leq i$$

so, $A_{(ij)}[k-1]$ is just a minor of $A_{(ij)}[k]$ for $k \leq i$, all are lower triangular block matrices. The map ϕ is given by n matrices $A[1], \ldots, A[n]$ for the vertices $1, \ldots, n$ respectively with

$$A[i] = \begin{bmatrix} A_{(i,i)}[i] & \cdots & A_{(i,n)}[i] \\ & \ddots & \vdots \\ 0 & & A_{(n,n)}[i] \end{bmatrix}, 1 \leq i \leq n$$

A monomorphism $\phi \colon N_{(j)} \to M_{(j)}$ is determined by its induced morphism $\mathrm{soc}(\phi) \colon \operatorname{soc} N_{(j)} \to \operatorname{soc} M_{(j)}$, i.e. $A[j]$. More precisely, ϕ is given by j matrices $A[1], \ldots, A[j]$ with

$$A[k] = A_{(jj)}[k] = \left[\begin{array}{c|c} & 0 \\ \cline{2-2} A^{k} & \\ & A[k-1] \end{array}\right], A[1] =: A^{1}$$

so $A[j] = \mathrm{soc}(\phi)$ determines ϕ. Now, we can use Gauss-elimination on $A[j]$, but we only allow elemtary column operation which are socles of automorphisms of $N_{(j)}$, that is arbitrary elementary column operation within the columns of $A[j]$ where A^t lies plus addition of a multiple of a column where A^t lies to any column where A^s lies, $s > t$. Then we find an automorphism $\kappa\colon N_{(j)} \to N_{(j)}$ such that $\mathrm{soc}(\phi \circ \kappa)$ is given (in the previous basis) by a matrix of the form $B[j]$ with

$$B[t] = \left[\begin{array}{c|c} B^t & \begin{array}{c} 0 \\ \hline \\ B[t-1] \end{array} \end{array}\right]$$

with each B^t is a column echolon form same number of columns and rows as A^t

$$B^t = \begin{bmatrix} 0 & & \cdots & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & & & 0 \\ 1 & 0 & & & & \\ * & 0 & & & & \\ \vdots & \vdots & & & & \\ * & 0 & & & & \\ 0 & 1 & 0 & & & \\ * & * & 0 & & & \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 & 0 \\ * & \cdots & * & * & * & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ * & \cdots & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ * & \cdots & & & & * \\ \vdots & & & & & \vdots \\ * & \cdots & & & & * \end{bmatrix}$$

with (row-)pivot positions $a_{tj}^1 < a_{tj}^2 < \cdots < a_{tj}^{n_{tj}}$ where $\sum_{s=t+1}^{j} m_{sj} < a_{tj}^1, a_{tj}^{n_{tj}} < \sum_{s=1}^{j} m_{sj}$ and $e_r^T B^t = 0$ for any $r \in \{a_{sj}^x \mid s < t, 1 \leq x \leq n_{sj}\}$. We say, the matrices $B[1] = B^1, B[2], \ldots, B[j]$ are in normal form.

Now, back to the general case. We find an automorphism κ of N such that all diagonal block matrices $A_{(jj)}[k]$ in any $A[k]$ are in normal form.
Again with Gauss-elimination corresponding to composing with automorphism of N we can assume all entries in each pivot-row (i.e. a row containing a pivot position) are zero

except in the pivot position in all $A[1], \dots, A[n]$, that means for $1 \le k \le i < j$

$$A_{(ij)}[k] = \left[\begin{array}{c|c} A^k_{(ij)} & \begin{array}{c} 0 \\ \hline \\ A_{(ij)}[k-1] \end{array} \end{array} \right]$$

with $e_r^T A^k_{(ij)} = 0$ for $r \in \{a^x_{si} \mid s \ge k, 1 \le x \le n_{si}\}$. Under this assumptions we say that $A[1], \dots, A[n]$ are in normal form. Once, we have fixed the basis (for n and M) every monomorphism has a unique associated normal form and any submodule $X \subset M, X \cong N$ is image of a monomorphism in normal form.

7.4.3 Remarks on partial $\mathbb{A}_n$-quiver flags

We look at quiver Grassmannians and use the *normal forms* for matrices describing submodules.

Definition 31. Let $N \subset M$ be a submodule and fix a basis for $\underline{N}$ and $\underline{M}$ adapted to direct sum decompositions as before, $\underline{d} := \underline{\dim}\, M, \underline{e} := \underline{\dim}\, N$. Then, for $\underline{a} := (a^x_{sj})_{s,j,x}$ we consider the subset of the Reineke stratum $\mathcal{F}_{[N_*]} \subset \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{e}}}$ defined by

$$\mathcal{S}_{\underline{a}} := \{(U \subset M) \in \mathcal{F}_{[N_*]} \mid U = \mathrm{Im}\,\phi, \phi \in \mathrm{Inj}(N, M) \text{ in normal form, with pivot positions } \underline{a}\}$$

This is the natural generalization of rb–stratification in this situation, so we continue to call it rb–stratification.

We do not work with this but if one is interested in this stratification, one can work on the following.

(i) Each $\mathcal{S}_{\underline{a}}$ is a locally closed subset of $\mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{e}}}$.

(ii) $\mathcal{S}_{\underline{a}}$ is isomorphic to an affine space of dimension ??

(iii) The closure of $\mathcal{S}_{\underline{a}}$ is a union of other rb–strata.

(iv) The tangent dimension $\dim T_x \mathrm{Gr}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{e}}}$ is constant for any $x \in \mathcal{S}_{\underline{a}}$.

(v) Find explicit formulas for Hall numbers.

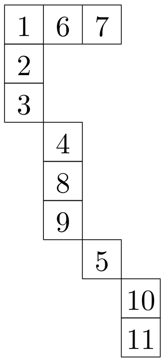
7.5 Root tableau of hook type

We translate all of Fung's result on classical Springer fibres of hook type to the A_n-equioriented quiver flags of hook type. The literature we use is in most part [Fun03]. Also a reference is [Var79].

Definition 32. M is called of **hook type** if it is isomorphic to $E_{1i} \oplus S$ with S semisimple for some $i \in \{1, \dots, n\}$. We call a root diagram of **of hook type** if it is associated to a module of hook type.

We remark that for a root diagram of hook type all sub-root diagrams are of hook type. This a complete flag example.

Example.



Remark. Let M be a representation, we fix $b \in \mathbb{N}_0$ such that $\mathrm{rad}^b(M) = 0$, $\mathrm{rad}^{b-1}(M) \neq 0$. Then, it holds $\mathrm{rad}(\mathrm{soc}^{i+1}(M)) \subset \mathrm{rad}(M) \cap \mathrm{soc}^i(M)$. If M is of hook type, we have $\mathrm{rad}(M) \cap \mathrm{soc}^i(M) = \mathrm{rad}^{b-i}(M)$, which implies

$$\mathrm{rad}(\mathrm{soc}^{i+1}(M)) \subset \mathrm{rad}^{b-i}(M).$$

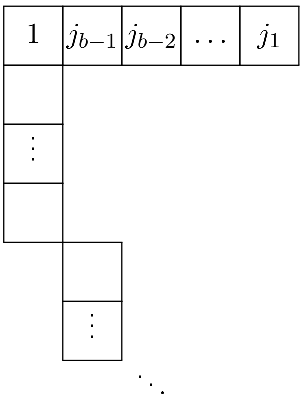
This property we use usually in the following way: Given a flag of Q_0-graded vector spaces $0 = W_0 \subset W_1 \subset \cdots \subset W_s = \underline{M}$ such that

$$\begin{array}{ccccc}
\underline{\mathrm{rad}^{b-1}(M)} & \subset & W_{i_1} & \subset & \underline{\mathrm{soc}^1(M)} \\
\underline{\mathrm{rad}^{b-2}(M)} & \subset & W_{i_2} & \subset & \underline{\mathrm{soc}^2(M)} \\
 & & \vdots & & \\
\underline{\mathrm{rad}(M)} & \subset & W_{i_{b-1}} & \subset & \underline{\mathrm{soc}^{b-1}(M)}.
\end{array}$$

for certain $1 \leq i_1 < i_2 < \cdots < i_{b-1}$, then each W_k is M-invariant, $0 \leq k \leq s$.

Let us assume M is not semi-simple of hook type, then a root tableau for with underlying root diagram Y_M is uniquely determined by the entries in the long root. Let us associate for a sequence $1 \leq i_1 < i_2 < \cdots < i_{b-1}$, set $j_k := n + 1 - i_k$, a root tabeaux

$T_{\underline{i}} :=$



The assumption for having the 1 in the left corner is just for convenience, else there would be an entry x and the numbers $1, \ldots, x-1$ would be fixed points corresponding to a semi-simple direct summand, when we pass to the quotient we get a root tableau in the form above with 1 in the upper left corner box. Let us recall Fung's definitions.

Definition 33. A space X is an **iterated fibre bundle** of base type $(B_1, \dots, B_n)$ if there exists spaces $X = X_1, B_1, X_2, B_2, \dots, X_n, B_n, X_{n+1} = pt$ and maps $p_1, p_2, \dots, p_n$ such that $p_j \colon X_j \to B_j$ is a fibre bundle with typical fibre X_{j+1}.

For simplifications of the computations of the intersection homology polynomials it is convenient to define for $d, k \in \mathbb{N}_0, \underline{d} := (d_1, \dots, d_n) \in \mathbb{N}_0^n$

$$[d] := T^{-(d-1)}(1 + T^2 + T^4 + \cdots + T^{2(d-1)}), [0] := 1, [\underline{d}] := [d_1][d_2]\cdots[d_n],$$

$$[d]! := [1]\,[2]\cdots[d], \quad [\underline{d}]! := [d_1]![d_2]!\cdots[d_n]!,$$
$$\binom{[d]}{[k]} := \frac{[d]!}{[k]![d-k]!}, \quad \binom{[\underline{d}]}{[\underline{k}]} := \binom{[d_1]}{[k_1]}\binom{[d_2]}{[k_2]}\cdots\binom{[d_n]}{[k_n]}$$

For a scheme of finite type over the $\mathbb{C}$, we write $\mathrm{IP}(X)$ to denote the intersection homology Poincare polynomial of X.

This is Fung's main example, compare [Fun03], corollary 3.1.

Example. Let V be an $\underline{d}$-dimensional Q_0-graded vector space over $\mathbb{C}$ and Fl(V) be the variety of complete flags in V (i.e. the product of complete flag varieties in each V_i, $i \in Q_0$). Fix a choice of a complete dimension filtration $\underline{\mathbf{d}}$ of $\underline{d}$, this gives an embedding of Fl(V) into the variety of (not graded) complete flags in the vector space $\oplus_{i \in Q_0} V_i$, we will always use such an embedding to write down an element of Fl(V) as a flag of Q_0-graded subvectorspaces $0 \subset W_1 \subset \cdots \subset W_r = V$ with $\dim W_i = i$. It holds

$$\mathrm{IP}(\mathrm{Fl(V)}) = [\underline{\mathrm{d}}]!.$$

Let $\mathrm{Gr}_{\mathrm{Q}\underline{k}}(V)$ be the Grassmannian variety of $\underline{k}$-dimensional Q_0-graded subvector spaces in V, then

$$\mathrm{IP}(\mathrm{Gr}_{\mathrm{Q}\underline{k}}(V)) = \binom{[\underline{d}]}{[\underline{k}]}.$$

Let I be an $\underline{a}$-dimensional Q_0-graded subvector space of V. We fix as before a complete dimension filtration $\underline{\mathbf{d}} = (\underline{d}^0, \dots, \underline{d}^r = \underline{d})$. Let $X_i(I) \subset \mathrm{Fl(V)}$ be the subvariety given by flags $0 \subset W_1 \subset \cdots \subset W_r = V$ with $I \subset W_i$. Then

$$\mathrm{IP}(X_i(I)) = \binom{[\underline{d} - \underline{a}]}{[\underline{d}^i - \underline{a}]}[\underline{d}^i]![\underline{d} - \underline{d}^i]!.$$

We first state all our main results and prove some corollaries, then give the proofs at the end of this subsection.

Theorem 7.5.1. *Let M be a module of hook type, let $\underline{\mathbf{d}}$ be a complete dimension filtration of $\underline{\dim}\, M$ and $T_{\underline{i}}$ be a root tableux of shape Y_M with $\underline{\mathbf{d}}_{T_{\underline{i}}} = \underline{\mathbf{d}}$.*
The closure of the Spaltenstein stratum $\overline{\mathcal{S}_{T_{\underline{i}}}} \subset \mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$ is given by Q_0-graded flags $(0 =$

$W_0 \subset W_1 \subset \cdots \subset W_s) \in \mathrm{F}(\underline{\mathbf{d}})$ *such that*

$$\begin{array}{ccccc} \underline{\mathrm{rad}^{b-1}(M)} & \subset & W_{i_1} & \subset & \underline{\mathrm{soc}^1(M)} \\ \underline{\mathrm{rad}^{b-2}(M)} & \subset & W_{i_2} & \subset & \underline{\mathrm{soc}^2(M)} \\ & & \vdots & & \\ \underline{\mathrm{rad}(M)} & \subset & W_{i_{b-1}} & \subset & \underline{\mathrm{soc}^{b-1}(M)}. \end{array}$$

Corollary 7.5.1.1. *In the situation of the previous theorem, set $i_b := \dim M$, $i_0 := 0$. Then, $\overline{\mathcal{S}_{T_{\underline{i}}}}$ is an iterated fibre bundle with*

$$\begin{aligned} B_{2j+1} &= \mathrm{Gr}_{\mathrm{Q}_{\underline{d}^{i_{j+1}}-\underline{d}^{i_j}-e_{b-j}}}\left(\underline{{}^{\mathrm{soc}(M/U_{i_j})}\!/\!{}_{\mathrm{rad}^{b-1-j}(M/U_{i_j})}}\right) \\ B_{2j+2} &= \mathrm{Fl}(\mathrm{W_{i_{j+1}}}/\mathrm{W_{i_j}}) \end{aligned}$$

with $j = 0, \ldots, b-1$ and . In particular, it is smooth. Its intersection homology Poincare polynomial is

$$\begin{aligned} \mathrm{IP}(\overline{\mathcal{S}_{T_{\underline{i}}}}) =& [\underline{d}^{i_1}]! \binom{[\underline{d} - (\sum_{t=1}^{b} e_t)]}{[\underline{d}^{i_1} - e_b]} [\underline{d}^{i_2} - \underline{d}^{i_1}]! \binom{[\underline{d} - \underline{d}^{i_1} - (\sum_{t=1}^{b-1} e_t)]}{[\underline{d}^{i_2} - \underline{d}^{i_1} - e_{b-1}]} \\ & [\underline{d}^{i_3} - \underline{d}^{i_2}]! \binom{[\underline{d} - \underline{d}^{i_2} - (\sum_{t=1}^{b-2} e_t)]}{[\underline{d}^{i_3} - \underline{d}^{i_2} - e_{b-2}]} \cdots [\underline{d}^{i_{b-1}} - \underline{d}^{i_{b-2}}]! \binom{[\underline{d} - \underline{d}^{i_{b-2}} - (\sum_{t=1}^{2} e_t)]}{[\underline{d}^{i_{b-1}} - \underline{d}^{i_{b-2}} - e_2]} \\ & \cdot [\underline{d} - \underline{d}^{i_{b-1}}]! \end{aligned}$$

Proof of corollary: We define

$$p_1\colon \quad X_1 := \overline{\mathcal{S}_{T_{\underline{i}}}} \to \mathrm{Gr}_{\mathrm{Q}_{\underline{d}^{i_1}-\underline{r}^{(b-1)}}}\left(\underline{{}^{\mathrm{soc}(M)}\!/\!{}_{\mathrm{rad}^{b-1}(M)}}\right) =: B_1, \qquad W_\bullet \mapsto {}^{W_{i_1}}\!/\underline{{}_{\mathrm{rad}^{b-1}(M)}}$$

$$p_2\colon \quad X_2 := p_1^{-1}(W_{i_1}/\mathrm{rad}^{b-1}(M)) \to \mathrm{Fl}(\mathrm{W_{i_1}}) =: \mathrm{B}_2, \qquad W_\bullet \mapsto W_0 \subset \cdots \subset W_{i_1}$$

where p_2 is welldefined because two Q_0-graded subvector spaces W_{i_1}, W'_{i_1} containing $\underline{\mathrm{rad}^{b-1}(M)}$ having the same quotient are equal. It is easy to check that both are fibre bundles. Now, it holds

$$\begin{aligned} \mathrm{rad}^d(M/U_{i_1}) &= (\mathrm{rad}^d(M) + U_{i_1})/U_{i_1} \\ \mathrm{soc}^d(M/U_{i_1}) &= \mathrm{soc}^{d+1}(M)/U_{i_1} \end{aligned}$$

for any d and $U_{i_1} \subset M$ with $\underline{U_{i_1}} = W_{i_1}$. Then, $X_3 := p_2^{-1}(W_0 \subset W_1 \subset \cdots \subset W_{i_1}) \cong$

$$\left\{ 0 \subset {}^{W_{i_1+1}}\!/{}_{W_{i_1}} \subset \cdots \subset {}^{\underline{M}}\!/{}_{W_{i_1}} \;\middle|\; \begin{array}{ccccc} \underline{\mathrm{rad}^{b-2}(M/U_{i_1})} & \subset & W_{i_2}/W_{i_1} & \subset & \underline{\mathrm{soc}^1(M/U_{i_1})} \\ \underline{\mathrm{rad}^{b-3}(M/U_{i_1})} & \subset & W_{i_3}/W_{i_1} & \subset & \underline{\mathrm{soc}^2(M/U_{i_1})} \\ & & \vdots & & \\ \underline{\mathrm{rad}(M/U_{i_1})} & \subset & W_{i_{b-1}}/W_{i_1} & \subset & \underline{\mathrm{soc}^{b-2}(M/U_{i_1})} \end{array} \right\}$$

by the previous theorem this is the $\overline{\mathcal{S}_{T_{\mathbf{i}-i_1}}} \subset \mathrm{Fl_Q}\binom{M/U_{i_1}}{\underline{\mathbf{d}}-\underline{d}^{i_1}}$ where $T_{\mathbf{i}-i_1}$ is root tableau of shape $Y_{M/U_{i_1}}$ associated to the sequence $i_2 - i_1 < i_3 - i_1 < \cdots < i_{b-1} - i_1$. Using *multiplicativity* of IP in locally trivial fibre bundles we get

$$\begin{aligned}\mathrm{IP}(\overline{\mathcal{S}_{T_{\mathbf{i}}}}) &= \mathrm{IP}(\mathrm{Fl}(\mathrm{W_{i_1}}))\mathrm{IP}(\mathrm{Gr_{Q_{\underline{d}^{i_1}-\underline{r}^{(b-1)}}}}(\underline{\frac{\mathrm{soc(M)}}{\mathrm{rad^{b-1}(M)}}}))\mathrm{IP}(\overline{\mathcal{S}_{\mathrm{T_{\mathbf{i}-i_1}}}})\\ &= [\underline{d}^{i_1}]!\binom{[\underline{d}-(\sum_{t=1}^{b} e_t)]}{[\underline{d}^{i_1}-e_b]}\mathrm{IP}(\overline{\mathcal{S}_{T_{\mathbf{i}-i_1}}})\end{aligned}$$

The rest follows by induction. □

Theorem 7.5.2. *Let M be a module of hook type, let $\underline{\mathbf{d}}$ be a complete dimension filtration of $\underline{\dim} M$ and $T_{\underline{i}}, T_{\underline{i}'}$ be root tableu of shape Y_M with $\underline{\mathbf{d}}_{T_{\underline{i}}} = \underline{\mathbf{d}}_{T_{\underline{i}'}} = \underline{\mathbf{d}}$.*
Then, the intersection $\overline{\mathcal{S}_{T_{\underline{i}}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'}}}$ is nonempty if and only if
$\beta_j := \max(i_j, i'_j) < \min(i_{j+1}, i'_{j+1}) =: \alpha_{j+1}$ ($\beta_0 := 0$), in which case it is given by Q_0-graded flags $(0 = W_0 \subset W_1 \subset \cdots \subset W_s) \in \mathrm{F}(\underline{\mathbf{d}})$ such that

$$\begin{array}{ccccccc}\underline{\mathrm{rad}^{b-1}(M)} & \subset & W_{\alpha_1} & \subset & W_{\beta_1} & \subset & \underline{\mathrm{soc}^1(M)}\\ \underline{\mathrm{rad}^{b-2}(M)} & \subset & W_{\alpha_2} & \subset & W_{\beta_2} & \subset & \underline{\mathrm{soc}^2(M)}\\ & & & \vdots & & & \\ \underline{\mathrm{rad}(M)} & \subset & W_{\alpha_{b-1}} & \subset & W_{\beta_{b-1}} & \subset & \underline{\mathrm{soc}^{b-1}(M)}.\end{array}$$

Corollary 7.5.2.1. *In the situation of the previous theorem. If $\overline{\mathcal{S}_{T_{\underline{i}}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'}}}$ is nonempty it is an iterated fibre bundle with*

$$\begin{aligned}B_{2j+1} &= \mathrm{Gr_{Q_{\underline{d}^{\beta_{j+1}}-\underline{d}^{\beta_j}-e_{b-j}}}}(\underline{\mathrm{soc}(M/U_{\beta_j})/\mathrm{rad}^{b-1-j}(M/U_{\beta_j})})\\ B_{2j+2} &= X_{\alpha_{j+1}-\beta_j}(\mathrm{rad}^{b-1-j}(M/U_{\beta_j})) \quad (\subset \mathrm{Fl}(W_{\beta_{j+1}}/\mathrm{W}_{\beta_j})),\end{aligned}$$

where $\alpha_b = \beta_b = \dim M, \alpha_0 = \beta_0 = 0$, $j = 0, \ldots, b-1$. In particular, it is smooth. Its intersection homology Poincare polynomial is

$$\begin{aligned}\mathrm{IP}(\overline{\mathcal{S}_{T_{\underline{i}}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'}}}) = &\binom{[\underline{d}^{\beta_1}-e_b]}{[\underline{d}^{\alpha_1}-e_b]}[\underline{d}^{\alpha_1}]![\underline{d}^{\beta_1}-\underline{d}^{\alpha_1}]!\binom{[\underline{d}-(\sum_{t=1}^{b} e_t)]}{[\underline{d}^{\beta_1}-e_b]}\\ &\binom{[\underline{d}^{\beta_2}-\underline{d}^{\beta_1}-e_{b-1}]}{[\underline{d}^{\alpha_2}-\underline{d}^{\beta_1}-e_{b-1}]}[\underline{d}^{\alpha_2}-\underline{d}^{\beta_1}]![\underline{d}^{\beta_2}-\underline{d}^{\alpha_2}]!\binom{[\underline{d}-\underline{d}^{\beta_1}-(\sum_{t=1}^{b-1} e_t)]}{[\underline{d}^{\beta_2}-\underline{d}^{\beta_1}-e_{b-1}]}\\ \cdots &\binom{[\underline{d}^{\beta_{j+1}}-\underline{d}^{\beta_j}-e_{b-j}]}{[\underline{d}^{\alpha_{j+1}}-\underline{d}^{\beta_j}-e_{b-j}]}[\underline{d}^{\alpha_{j+1}}-\underline{d}^{\beta_j}]![\underline{d}^{\beta_{j+1}}-\underline{d}^{\alpha_{j+1}}]!\binom{[\underline{d}-\underline{d}^{\beta_j}-(\sum_{t=1}^{b-j} e_t)]}{[\underline{d}^{\beta_{j+1}}-\underline{d}^{\beta_j}-e_{b-j}]}\\ \cdots &\binom{[\underline{d}^{\beta_{b-1}}-\underline{d}^{\beta_{b-2}}-e_2]}{[\underline{d}^{\alpha_{b-1}}-\underline{d}^{\beta_{b-2}}-e_2]}[\underline{d}^{\alpha_{b-1}}-\underline{d}^{\beta_{b-2}}]![\underline{d}^{\beta_{b-1}}-\underline{d}^{\alpha_{b-1}}]!\\ \cdot &\binom{[\underline{d}-\underline{d}^{\beta_{b-2}}-(\sum_{t=1}^{2} e_t)]}{[\underline{d}^{\beta_{b-1}}-\underline{d}^{\beta_{b-2}}-e_2]}[\underline{d}-\underline{d}^{\beta_{b-1}}]!\end{aligned}$$

Proof of corollary: We define

$$\begin{aligned}p_1\colon X_1 = \overline{\mathcal{S}_{T_{\underline{i}}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'}}} &\to \mathrm{Gr_{Q_{\underline{d}^{\beta_1}-e_b}}}(\mathrm{soc}(M)/\underline{\mathrm{rad}^{b-1}(M)}) =: B_1, \qquad W_\bullet \mapsto W_{\beta_1}/\underline{\mathrm{rad}^{b-1}(M)}\\ p_2\colon X_2 = p_1^{-1}(W_{\beta_1}/\underline{\mathrm{rad}^{b-1}(M)}) &\to X_{\alpha_1}(\mathrm{rad}^{b-1}(M)) =: B_2, \qquad W_\bullet \mapsto W_0 \subset \cdots \subset W_{\beta_1}\end{aligned}$$

where $X_{\alpha_1}(\mathrm{rad}^{b-1}(M)) \subset \mathrm{Fl}(\mathrm{W}_{\beta_1})$ is defined as in the example. Both maps define fibre bundles. Then, $X_3 := p_2^{-1}(W_0 \subset W_1 \subset \cdots \subset W_{i_1}) \cong$

$$\begin{array}{lccccccc} \{ & 0 \subset {}^{W_{\beta_1+1}}/_{W_{\beta_1}} \subset \cdots \subset {}^{\underline{M}}/_{W_{\beta_1}} \mid & & & & & & \\ & \underline{\mathrm{rad}^{b-2}({}^{M}/_{U_{\beta_1}})} & \subset & {}^{W_{\alpha_2}}/_{W_{\beta_1}} & \subset & {}^{W_{\beta_2}}/_{W_{\beta_1}} & \subset & \underline{\mathrm{soc}^1({}^{M}/_{U_{\beta_1}})} \\ & \underline{\mathrm{rad}^{b-3}({}^{M}/_{U_{\beta_1}})} & \subset & {}^{W_{\alpha_3}}/_{W_{\beta_1}} & \subset & {}^{W_{\beta_3}}/_{W_{\beta_1}} & \subset & \underline{\mathrm{soc}^2({}^{M}/_{U_{\beta_1}})} \quad \}, \\ & & & & \vdots & & & \\ & \underline{\mathrm{rad}({}^{M}/_{U_{\beta_1}})} & \subset & {}^{W_{\alpha_{b-1}}}/_{W_{\beta_1}} & \subset & {}^{W_{\beta_{b-1}}}/_{W_{\beta_1}} & \subset & \underline{\mathrm{soc}^{b-2}({}^{M}/_{U_{\beta_1}})} \end{array}$$

this is isomorphic to $\overline{\mathcal{S}_{T_{\underline{i}-\beta_1}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'-\beta_1}}}$, where $T_{\underline{i}-\beta_1}$, $T_{\underline{i}'-\beta_1}$ are the root tableau of shape $Y_{M/U_{\beta_1}}$ associated to the sequences $i_2 - \beta_1 < i_3 - \beta_1 < \ldots < i_{b-1} - \beta_1$ and $i_2' - \beta_1 < i_3' - \beta_1 < \ldots < i_{b-1}' - \beta_1$ respectively. By multiplicativity of the intersection homology polynomial we get

$$\begin{aligned} \mathrm{IP}(\overline{\mathcal{S}_{T_{\underline{i}}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'}}}) &= \mathrm{IP}(X_{\alpha_1}(\mathrm{rad}^{b-1}(M)))\mathrm{IP}(\mathrm{Gr}_{\mathrm{Q}_{\underline{d}^{\beta_1}-e_b}}({}^{\mathrm{soc}(M)}/_{\mathrm{rad}^{b-1}(M)}))\mathrm{IP}(\overline{\mathcal{S}_{T_{\underline{i}-\beta_1}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'-\beta_1}}}) \\ &= \binom{[\underline{d}^{\beta_1} - e_b]}{[\underline{d}^{\alpha_1} - e_b]} [\underline{d}^{\alpha_1}]![\underline{d}^{\beta_1} - \underline{d}^{\alpha_1}]! \binom{[\underline{d} - (\sum_{t=1}^{b} e_t)]}{[\underline{d}^{\beta_1} - e_b]} \mathrm{IP}(\overline{\mathcal{S}_{T_{\underline{i}-\beta_1}}} \cap \overline{\mathcal{S}_{T_{\underline{i}'-\beta_1}}}). \end{aligned}$$

The rest follows by induction. □

Theorem 7.5.3. *Let M be a module of hook type, let $\underline{\mathbf{d}}$ be a complete dimension filtration of $\underline{\dim} M$. Then, there are finitely many $\mathrm{Aut}(M)$-orbits in $\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}}$. The orbits in $\overline{\mathcal{S}_{T_{\underline{i}}}}$ for $1 \le i_1 < \cdots < i_{b-1}$ are in bijection with sequences*

$$\underline{\alpha\beta} : 0 \le \alpha_1 \le i_1 \le \beta_1 < \alpha_2 \le i_2 \le \beta_2 < \cdots < \alpha_{b-1} \le i_{b-1} \le \beta_{b-1}.$$

The corresponding $\mathrm{Aut}(M)$-orbit $\mathcal{O}_{\underline{\alpha\beta}}$ is given by $(0 = W_0 \subset W_1 \subset \cdots \subset W_s) \in \mathrm{F}(\underline{\mathbf{d}})$ such that

$$\underline{\mathrm{rad}^{b-j} M} \subset W_{\alpha_j} \subset W_{\beta_j} \subset \underline{\mathrm{soc}^j M}$$
$$\underline{\mathrm{rad}^{b-j} M} \not\subseteq W_{\alpha_j - 1}, \quad W_{\beta_j + 1} \not\subseteq \underline{\mathrm{soc}^j M}.$$

The closure of the orbit $\overline{\mathcal{O}_{\underline{\alpha\beta}}}$ is given by $(0 = W_0 \subset W_1 \subset \cdots \subset W_s) \in \mathrm{F}(\underline{\mathbf{d}})$ such that

$$\underline{\mathrm{rad}^{b-j} M} \subset W_{\alpha_j} \subset W_{\beta_j} \subset \underline{\mathrm{soc}^j M}.$$

The orbit which is dense in $\overline{\mathcal{S}_{T_{\underline{i}}}}$ is given by

$$0 \le \alpha_1 = i_1 = \beta_1 < \alpha_2 = i_2 = \beta_2 < \cdots < \alpha_{b-1} = i_{b-1} = \beta_{b-1}.$$

Open problems:

- We would like to see the structure as modules for the KLR-algebra $R_{\underline{d}}$ for $Q = \mathbb{A}_n$ on the Springer fibre modules $\bigoplus_{\underline{\mathbf{d}}} H_*^{BM}(\mathrm{Fl}_{\mathrm{Q}}\binom{M}{\underline{\mathbf{d}}})$ where $\underline{\mathbf{d}}$ runs through all dimension filtration of a given dimension vector $\underline{d}$. It should be possible to write this down with the vector space basis which we have found for them. We conjecture/ would like to prove:

Conjecture. The KLR-algebras for linear oriented A_n-quiver are cellular algebras, the cell modules are given by the Springer fibre modules.

- In Fung's work [Fun03], he investigates a connection to Kazhdan-Lusztig theory. What is the analogue here?
- Does the cell decomposition of $\mathrm{Fl}_Q\binom{M}{\underline{\mathbf{d}}}$ have a corresponding decomposition for $\mathcal{O}_M \cap F_{\underline{\mathbf{d}}}$?

Chapter 8

Appendix on equivariant (co)homology

Summary. These are the topics treated shortly in this chapter:
Slodowy's Lemma. Equivariant cohomology and splitting principle, Localization Theorem. Equivariant Borel-Moore homology and convolution product. Equivariant derived categories and duality. The Serre cohomology spectral sequence applied to prove some lemmata from the survey on Springer theory and to study the equivariant cohomology of flag varieties. Equivariant perverse sheaves.

8.0.1 A Lemma from Slodowy's book

Let us recall the slightly more general version of a lemma from Slodowy's book, which I sometimes refer to as Slodowy's lemma (because I do not know its origin).

Lemma 77. *(Slodowy's lemma, [Slo80b], p.26, lemma 4) Let X be a G-scheme of finite type over K and $\phi\colon X \to G/P$ be a G-equivariant morphism, we denote by $F := \phi^{-1}(eP)$ the scheme-theoretic fibre. Then, F is a P-scheme and if $G \times^P F$ exists we have a commutative diagram*

$$\begin{array}{ccccc} G\times^P F & \xrightarrow{\psi} & X & \qquad (g,f) & \longmapsto & gf \\ & \searrow & \swarrow_{\phi} & & \searrow \quad \swarrow & \\ & G/P & & & gP & \end{array}$$

with ψ is a G-equivariant isomorphism.

Proof: There is a G-equivariant isomorphism $G \times^P X \to G/P \times X, (g,x) \mapsto gx$ because X is already a G-scheme. Now, we compose the closed immersion $G \times^P F \to G \times^P X$ with this isomorphism and get the map ψ as the composition

$$\begin{aligned} G\times^P F &\xrightarrow{\tau} G/P \times X &&\xrightarrow{p_2} X \\ (g,f) &\mapsto (gP, gf) &&\mapsto gf, \end{aligned}$$

where τ is a closed immersion. We identify $G \times^P F$ with $\mathrm{Im}(\tau)$. Therefore, it suffices to find a G-equivariant section $\gamma\colon X \to G/P \times X$ of p_2 onto the image of τ which is the *identity* restricted on F.

We consider $\gamma(x) := (\phi(x), x)$, which is obviously a G-equivariant section of p_2, therefore it is a closed immersion. Using [DG70], chapter III, remember that G/P is the sheafification of the functor on K-algebras $X\colon R \mapsto T(R) := G(R)/P(R)$ and $G\times^P F$ the sheafification of the functor $S\colon R \mapsto S(R) := (G(R) \times F(R))/P(R)$. Denote by $s\colon T \to G/P, s\colon S \to G\times^P F$ the canonical natural transformations into the sheafification, we factorize $X \to G/P \times X$ into $X \xrightarrow{r} T \times X \xrightarrow{s} G/P \times X$ as indicated above. We call $\tilde{\tau}\colon S \to T \times X$ the functor which sheafifies to τ. Then it is obviously $im(r) \subset im(\tilde{\tau})$. Let R be a K-algebra; the commutative diagram

$$\begin{array}{ccccc} X(R) & \xrightarrow{\;r\;} & T(R) \times X(R) & \xleftarrow{\;\tilde{\tau}\;} & S(R) \\ & \searrow^{(\phi,\mathrm{id})} & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ & & G/P(R) \times X(R) & \xleftarrow{\;\tau\;} & G \times^P F(R) \end{array}$$

shows that $\mathrm{Im}(\phi, \mathrm{id}) = s(\mathrm{Im}(r)) \subset s(\mathrm{Im}(\tilde{\tau})) = \mathrm{Im}(\tau \circ s) \subset \mathrm{Im}(\tau)$.

□

8.1 Equivariant cohomology

We follow the lecture notes of Fulton on this topic, notes taken by Anderson and can be found for example in [And11]. Let G be a topological group. A G-space X is a topological space endowed with a continous G-action. Let EG be a contractible G-space with a topologically free G-action, i.e.

(1) the stabilizers of all points are trivial.

(2) the quotient map is a locally trivial fibration.

Such a space always exists and is uniquely determined up to homotopy (see for example [Die91]). For any G-space X we define

$$\begin{aligned} p\colon X_G := EG \times^G X &\to BG := EG/G \\ \overline{(e,x)} &\mapsto eG \end{aligned}$$

and call X_G the *homotopy quotient*.

Definition 34. The **equivariant cohomology** of X with respect to G is the singular (ordinary) cohomology of X_G

$$H^i_G(X) := H^i(X_G).$$

It carries the structure of a ring via the cap product on singular cohomology. In particular,

$$H^*_G(pt) = H^*(BG)$$

and $H_G^*(X)$ is a $H_G^*(pt)$-module via pullback along p.

Remark. This definition is independent of the choice of EG. Usually the spaces EG are infinite dimensional manifolds, to remain in the categories of algebraic varieties one writes them as limits of finite dimensional manifolds (or varieties). For more details and examples look at [And11].
In the older literature it is often assumed that the group is compact. The standard way to apply these results more generally is: By Homotopy invariance of singular cohomology one can substitute a reductive group G by a maximal compact subgroup to which it is homotopy equivalent because of the Iwasawa decomposition.

We assume G to be a reductive group over $\mathbb{C}$, $T \subset G$ is a maximal torus and W is the Weyl group of (G, T).

Theorem 8.1.1. *(Splitting principle) Let X be a quasi-projective G-variety, then*

$$H_G^*(X) \cong (H_T^*(X))^W.$$

For a proof see [Bri98].
We denote by K the quotient field of $H_T^*(pt)$ and for any T-variety X by

$$H_T^*(X) \to \mathcal{H}^*(X) := H_T^*(X) \otimes_{H_T^*(pt)} K, \alpha \mapsto \alpha \otimes 1$$

the equivariant homology tensored over K. According to [Bri00], Lemma 2, a complex variety X with finitely many T-fixed points is *T-equivariantly formal for cohomology* if and only of $H^{odd}(X) = 0$.

Theorem 8.1.2. *(Localization Theorem - weaker version) Let X be a equivariantly formal T-variety for cohomology (see later remark 8.4.3 for the definition) with finitely many T-fixed points. Then the pullback along $i\colon X^T \to X$ induces an isomorphism*

$$i^* \otimes \mathrm{id}_K \colon \mathcal{H}^*(X) \to \mathcal{H}^*(X^T) = \bigoplus_{x \in X^T} K \cdot x$$

of K-algebras.

In fact, the stronger statement would be that the pullback is injective and becomes an isomorphism after inverting the Euler classes of the T-fixed points (see below) which occur as factors in formulas for push-forwards to T-fixed points.
With stronger assumption on the varieties (also assuming finitely many 1-dimensional T-orbits) there is an explicit description of the image of the injective map i^*, this is the main theorem of Goretzky, Kottwitz, MacPherson see [GKM98]. The application of this theorem is called GKM-theory.

Definition 35. (see [And11]) If F is a G-equivariant complex vector bundle on X, then it has **equivariant Chern classes**

$$c_i^G(F) \in H_G^{2i}(X),$$

defined as follows. Since F is equivariant, $F_G = EG \times^G F \to X_G$ is a vector bundle on X_G and one defines

$$c_i^G(F) := c_i(F_G).$$

The **Euler class** at an isolated G-fixed point $x \in X^G$ is by definition

$$e_G(x, X) := c_{top}^G(T_x X) \quad \in H_G^*(\{x\}) = H_G(pt).$$

Following [CG97], 2.6.42, p.107, we also call the top Chern class $c_{top}^G(F)$ the *equivariant Euler class* of F.

The inverses of Euler classes of T-fixed points occur in formulas for pushforwards to T-fixed points and in multiplicity formulas for cycles after tensoring $- \otimes K$ and using the isomorphism from theorem 8.1.2.
The Euler class for a vector bundle occurs in the Thom isomorphism later.

8.2 Equivariant Borel-Moore homology

Definition 36. Let X be a complex algebraic G-variety embedded into a complex manifold M, which is equi-dimensional $\dim_{\mathbb{C}} M = m$. Pick a G-representation Y and an G-equivariant open subset $U \subset Y$ with G operates freely on U and the (complex) codimension of $Y \setminus U$ in Y is greater or equal $\dim X - \frac{i}{2}$. In particular $U \times^G X$ exists as an algebraic variety and can be embedded into the (equi-dimensional) manifold $U \times^G M$. Then, we define the i-th G-**equivariant Borel-Moore homology group** of X with coefficients in $\mathbb{C}$ via

$$H_i^G(X) := H^{BM}_{i+2\dim U - 2\dim G}(U \times^G X) := H_{ord}^{2m-i}(U \times^G M, (U \times^G M) \setminus (U \times^G X))$$

We shorten $X_G := U \times^G X$ always assuming that U is chosen appropriately and call it *an approximation to the homotopy quotient of* X. For G-equivariant maps $f\colon X \to Y$, we write $f_G := U \times^G f$ where U is chosen appropriately for X and Y.

In particular, we have that $H_i^G(pt)$ is zero for all $i > 0$. Furthermore, $H_*^G(X) = \bigoplus_{i \in \mathbb{Z}} H_i^G(X)$ is a graded $H_G^*(X)$-module via the cap product

$$H_G^j(X) \times H_i^G(X) \to H_{i-j}^G(X), \quad (c, d) \mapsto c \cap d$$

In particular, as $H_G^*(X)$ is a $H_G^*(pt)$-module, $H_*^G(X)$ is a $H_G^*(pt)$-module with $H_G^j(X) \cdot H_i^G(X) \subset H_{i-j}^G(X)$.

8.2.1 Basic properties

Basic properties which we get from the properties of Borel-Moore homology (cp. [CG97])

(1) (proper pushforward) Let $f\colon X \to Y$ be a G-equivariant proper morphism of complex varieties. Then, the induced map $f_G := U \times^G f$ is proper and we define $f_*\colon H_i^G(X) \to$

$H_i^G(Y)$ to be the map

$$(f_G)_*\colon H^{BM}_{i+2\dim U-2\dim G}(X_G)\to H^{BM}_{i+2\dim U-2\dim G}(Y_G)$$

(2) (localization property) For any open G-equivariant subset $j\colon A\subset X$ we have $j_G\colon A_G\subset X_G$ is open. There exists a natural restriction morphism $j^*\colon H_*^G(X)\to H_*^G(A)$ defined as the natural restriction morphism

$$(j_G)^*\colon H^{BM}_{i+2\dim U-2\dim G}(X_G)\to H^{BM}_{i+2\dim U-2\dim G}(A_G).$$

Set $i\colon F:=X\setminus A\subset X$, it is proper, therefore i_* is welldefined. Then, there is a natural long exact sequence in equivariant Borel-Moore homology

$$\cdots\to H_p^G(F)\xrightarrow{i_*}H_p^G(X)\xrightarrow{j^*}H_p^G(A)\to H_{p-1}^G(F)\to\cdots$$

(3) (smooth pullback) Let $p\colon \tilde X\to X$ be a G-equivariant morphism of complex algebraic varieties, where X has a G-equivariant covering such that the restriction of p is trivial over every open with smooth fibre F of complex dimension d. Then the induced map $\tilde X_G\xrightarrow{p_G}X_G$ is a locally trivial fibre bundle with typical fibre F as well and we can define the pullback $p^*\colon H_i^G(X)\to H_{i+2d}^G(\tilde X)$ as the map

$$p_G^*\colon H^{BM}_{i+2\dim U-2\dim G}(X_G)\to H^{BM}_{i+2\dim U-2\dim G+2d}(\tilde X_G)$$

(4) (intersection pairing) Let M be a G-equivariant complex manifold, equidimensional with $\dim_{\mathbb C}M=m$. Let $X,Y\subset M$ be two G-equivariant closed subsets, then $X_G,Y_G\subset M_G$ are closed subsets. There is the map

$$\cap\colon H_i^G(X)\times H_j^G(Y)\to H_{i+j-2m}^G(X\cap Y)$$

defined via the $\cup$-product in relative singular cohomology

$$\begin{aligned}\cup\colon H^{2m-i}_{ord}(M_G,M_G\setminus X_G)\times H^{2m-j}_{ord}(M_G,M_G\setminus Y_G)&\\ \to H^{4m-j-i}_{ord}(M_G,(M_G\setminus X_G)\cup(M_G\setminus Y_G)).&\end{aligned}$$

(5) (Thom isomorphism) Given a G-equivariant vector bundle $\mu\colon E\to X$ of rank r (i.e. the complex dimension of the fibre is r) with a zero section $i\colon X\to E$. Then, the Gysin pullback morphisms i^*,μ^* give mutually inverse isomorphims in equivariant Borel-Moore homology

$$\mu^*\colon H_*^G(X)\to H_{*+2r}^G(E),\quad i^*\colon H_*^G(E)\to H_{*-2r}^G(X).$$

(For any real vector bundle $\pi\colon E\to X$ of rank r, $A\subset X$ any subset, we get an isomorphism $H^i(X,X\setminus A)\cong H^{i+r}(E,E\setminus\pi^{-1}(A))$. Now given a G-equivariant vector

bundle $E \to X$ of real rank r, assume that there exists a smooth $E \subset \mathbb{E}, X \subset \mathbb{X}$, $e = \dim_{\mathbb{R}} \mathbb{E}$, $\dim_{\mathbb{R}} \mathbb{X} = x = e - r$ and a vector bundle $\pi \colon \mathbb{E} \to \mathbb{X}$ with $\pi^{-1}(X) = E$. Then, $\mathbb{E}_G \to \mathbb{X}_G$ is still a real rank r vector bundle, the preimage of X_G is E_G and $H_i^G(E) = H^{2e-i}(\mathbb{E}_G, \mathbb{E}_G \backslash E_G) \cong H^{2e-i-r}(\mathbb{X}_G, \mathbb{X}_G \backslash X_G) = H^{2x-(i-r)}(\mathbb{X}_G, \mathbb{X}_G \backslash X_G) = H_{i-r}^G(X)$.)
Furthermore for any $c \in H_*^G(X)$ one has $i^* i_*(c) = e(E) \cup c$ where $e(E) \in H_G^r(X)$ is the equivariant Euler class of the vector bundle.

(6) (equivariant cycle) Any G-stable closed subvariety $Y \subset X$ has a fundamental class $[Y]_G \in H_{2\dim Y}^G(X)$. This yields an equivariant Poincare duality map

$$H_G^i(X) \to H_{2\dim X - i}^G(X), \quad c \mapsto c \cap [X]_G,$$

which is an isomorphism if X is smooth (even if it is rationally smooth). In particular $H_i^G(pt) = H_G^{-i}(pt)$, this makes $H_*^G(pt) = H_G^{-*}(pt)$ a ring and the equivariant homology $H_*^G(X)$ for any G-variety X is a module over $H_*^G(pt)$ where $H_i^G(pt) \cdot H_j^G(X) \subset H_{i+j}^G(X)$.

8.2.2 Set theoretic convolution

Let M_1, M_2, M_3 be connected G-equivariant complex manifolds. Let

$$Z_{12} \subset M_1 \times M_2, \quad Z_{23} \subset M_2 \times M_3, \quad p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$$

be two G-equivariant locally closed subsets and p_{ij} be the projection on the i-th and j-th factor. We define the **set theoretic convolution** of Z_{12} and Z_{23} via

$$\begin{aligned} Z_{12} \circ Z_{23} &:= p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})) \\ &= \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \colon (m_1, m_2) \in Z_{12},\ (m_2, m_3) \in Z_{23}\} \\ &= Z_{12} \times_{M_2} Z_{23} \end{aligned}$$

8.2.3 Convolution in equivariant Borel-Moore homology

Let $M_1, M_2, M_3, Z_{12} \subset M_1 \times M_2, Z_{23} \subset M_2 \times M_3, p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ defined as before. Additionally assume $Z_{12} \subset M_1 \times M_2, Z_{23} \subset M_2 \times M_3$ closed and the restriction of p_{13} denoted also by p_{13}

$$p_{13} \colon p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is proper. Then, we define the **convolution product** $*$ as follows

$$\begin{aligned} H_i^G(Z_{12}) \times H_j^G(Z_{23}) &\to H_{i+j-2m_2}^G(Z_{12} \circ Z_{23}) \\ (c_{12}, c_{23}) &\mapsto c_{12} * c_{23} := (p_{13})_*(p_{12}^* c_{12} \cap p_{23}^* c_{23}), \end{aligned}$$

8.3 Duality between equivariant cohomology and equivariant Borel-Moore homology

8.3.1 Equivariant derived category of sheaves after Bernstein and Lunts

This is based on Bernstein-Lunts [BL94]. I also used Fiebig's and Williamson's article [Fie11].

As in the subsection on equivariant cohomology, let G be a topological group and EG be a contractible G-space with a topologically free G-action (i.e. trivial stabilizers and the quotient map is a locally trivial fibration). Consider

$$\begin{array}{ccc} & EG \times X & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & X_G := EG \times^G X \end{array}$$

where q is the quotient map and p is the projection on the second factor.
Let k be any commutative ring with unit. For any topological space Y we denote by $D(Y,k)$ the derived category of sheaves of k-modules on Y and by $D^b(Y,k)$ the full subcategory of objects with bounded cohomology. For any continous map $f\colon Y \to Y'$ we have a pushforward and a pullback functor

$$f_*\colon D(Y,k) \to D(Y,k), \quad f^*\colon D(Y',k) \to D(Y,k)$$

Definition 37. The *equivariant derived category* of sheaves of X with coefficients in k is the full subcategory $D_G(X,k)$ of $D(X_G,k)$ consisting of all objects $\mathcal{F}$ such that there exists $\mathcal{F}_X \in D(X,k)$ such that $q^*\mathcal{F} \cong p^*\mathcal{F}_X$.
We denote by $D^b_G(X,k)$ the full subcategory of $D_G(X,k)$ consisting of objects with bounded cohomology.

Remark. The categories $D_G(X,k)$ and $D^b_G(X,k)$ are independent of the choice of a contractible space EG.
Since $p\colon EG \times X \to X$ is a trivial fibration with contractible fibre EG, the functor $p^*\colon D(X,k) \to D(EG \times X,k)$ is a full embedding, in particular the sheaf $\mathcal{F}_X$ appearing in the definition is unique up to unique isomorphism. We get a forgetful functor

$$For\colon D_G(X,k) \to D(X,k), \quad \mathcal{F} \mapsto \mathcal{F}_X$$

Remark. $D_G(X,k)$ is not the derived category of equivariant sheaves, which often is not sensible to consider. $D_G(X,k)$ is constructed in such a way that its properties are analogous to the derived category of equivariant sheaves, i.e. has a six-functor formalism which commutes with the forgetful functor and it is a triangulated category with its heart isomorphic to the category of equivariant sheaves.
For any complex G-variety X, the constant sheaf C_X and the dualizing object $D_X := p^!C_{pt}$ for the morphism $p\colon X \to pt$ are always objects in $D^b_G(X,k)$, see [BL94], example 3.4.2

(1).

8.3.2 The functor formalism

In order to ensure existence of all the functors we restrict to the following situation. Let G be a complex Lie group and let $f\colon X \to Y$ be a G-equivariant morphism of (algebraic) G-varieties. There exist functors

$$\begin{aligned} f_*, f_!\colon D^b_G(X,k) &\to D^b_G(Y,k) \\ f^*, f^!\colon D^b_G(Y,k) &\to D^b_G(X,k) \\ \mathcal{H}om, \otimes\colon D^b_G(X,k) \times D^b_G(X,k) &\to D^b_G(X,k) \\ \mathbb{D} := \mathcal{H}om(-, D_X)\colon D^b_G(X,k) &\to D^b_G(X,k) \end{aligned}$$

they are basically defined as the functors associated to $f_G = EG \times^G f\colon X_G \to Y_G$, but these spaces are not locally compact in general, so the problem is overcome by considering them as direct limits of locally compact spaces, see [BL94], chapter 3. For $G = \{e\}$, they coincide with the non-equivariant functors given by the same symbol.

(1) (see [BL94], Thm 3.4.1, Thm 3.5.2) The functors $f_*, f_!, f^*, f^!, \mathcal{H}om, \otimes$ and $\mathbb{D}$ commute with the forgetful functor.

(2) (see [BL94], 1.4.1-1.4.3, thm 3.4.3)

   (i) f^* is naturally left adjoint to f_*, in particular, there is a natural transformation $1 \to f_* f^*$,

   (ii) $f_!$ is naturally left adjoint to $f^!$, in particular, there is a natural transformation $f_! f^! \to 1$,

   (iii) There are natural isomorphisms of functors $(fg)^* = g^* f^*, (fg)^! = g^! f^!, (fg)_* = f_* g_*, (fg)_! = f_! g_!$,

   (iv) There are natural functorial isomorphisms $\mathcal{H}om(A \otimes B, C) = \mathcal{H}om(A, \mathcal{H}om(B,C)), \quad f^*(A \otimes B) = f^*A \otimes f^*B.$

   (v) For $f\colon X \to pt$ we define $\Gamma := f_*$ and $\Gamma_c := f_!$ to be the global section functor and the global section functor with support, respectively.

(3) (see [BL94], Thm 1.6.2, thm 3.5.2) For any morphism of complex algebraic varieties there exist canonical functorial isomorphisms

$$\mathbb{D} f_! = f_* \mathbb{D}, \quad f^! \mathbb{D} = \mathbb{D} f^*, \mathbb{D}\mathbb{D} = \mathrm{id}\,.$$

Remark. Warning, we denote the derived functors with the same symbol as the functors $f_*, f^*, f_!$ themselves. The functor $f^!$ is constructed to be the right adjoint of $f_!$, it is not the derived functor of a functor of the same name.

The following properties are commonly used, see [CG97], (8.3.13)-(8.3.16) on p.428. The properties transfer to the equivariant functors.

(1) If f is G-equivariant proper, then $f_* = f_!$. In particular, this holds for any closed embedding.

(2) If f is G-equivariant, flat with smooth fibers of complex dimension d, then $f^! = f^*[2d]$. In particular, for an open G-equivariant embedding $f^* = f^!$. Also, for $f\colon X \to pt$ smooth, G-equivariant, we get $D_X = C_X[2d]$.

(3) For any cartesian square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

of G-equivariant varieties it holds $g^! f_* = \tilde{f}_* \tilde{g}^! \colon D^b_G(X) \to D^b_G(Y)$

(4) Let $i_\Delta \colon X \to X \times X$ be the diagonal embedding of a G-variety. Recall that we can define the tensor product to be $A \otimes B := i_\Delta^*(A \times B)$ for $A, B \in D^b_G(X)$. We define a second derived tensor product via $A \otimes^! B := i_\Delta^!(A \times B)$. It holds

$$A \otimes C_X = A, \quad A \otimes^! D_X = A, \quad \mathcal{H}om(A, B) = \mathbb{D}(A) \otimes^! B$$

8.3.3 Duality

Let us first recall the appearence of equivariant cohomology in the context of the equivariant derived category.

Definition 38. Let $\mathcal{F} \in D^b_G(X, k)$. The *equivariant cohomology* $\mathbb{H}^*_G(X, \mathcal{F})$ *of* X *with coefficients in* $\mathcal{F}$ is defined as

$$\mathbb{H}^*_G(X, \mathcal{F}) := H^*_{ord}(BG, \pi_* \mathcal{F})$$

where $\pi \colon X \to pt$, $\pi_* \mathcal{F} \in D^b_G(pt, k) \subset D^b(BG, k)$ with $BG = EG/G$ and the right hand side is ordinary sheaf cohomology, i.e. the i-th cohomology group is the i-th hypercohomology of the complex $\pi_* \mathcal{F}$ of sheaves on BG. This is naturally a graded module over $H^*_G(pt, k)$, so we have a functor

$$\mathbb{H}^*_G \colon D^b_G(X, k) \to H^*_G(pt, k) - mod^{\mathrm{gr}}$$

The following is the link to the equivariant cohomology and equivariant Borel-Moore homology from the previous two sections.

Lemma 78. *([CG97], (8.3.6) on page 426 for not equivariant) Let X be a complex G-variety. Then:*

$$H^i_G(X) = \mathbb{H}^i_G(X, C_X), \quad H^G_i(X) = \mathbb{H}^{-i}_G(X, D_X)$$

Remark. There is a sheaf theoretic convolution product defined on the equivariant cohomology using the functor formalism. This is just a rewrite of [CG97], chapter 8 where the non-equivariant case has been treated.

8.3.4 Localization for equivariant Borel-Moore homology

Let T be an algebraic torus over the complex numbers. We use the notion T-equivariantly formal for Borel-Moore homology which we define later. We only need the following remark: If the odd Borel-Moore homology vanishes for a variety, then it is T-equivariantly formal for Borel-Moore homology.

Theorem 8.3.1. *(Localization - weak version, [Bri00], Lemma 1) Let X be a complex T-variety which is T-equivariantly formal for the Borel-Moore homology (see definition 40 later). Assume X^T is finite and let $i\colon X^T \to X$ be the inclusion of T-fixed points. Then the pushforward induces an isomorphism*

$$i_* \otimes K\colon H_*^T(X^T) \otimes_{\mathbb{C}} K \to H_*^T(X) \otimes_{\mathbb{C}} K$$

of $K = H_*^T(pt)$-vector spaces.

Theorem 8.3.2. *(multiplicity formula, [Bri00], section 3) In the situation as above and assume that X is compact. Then, it holds*

$$[X]_T = \sum_{x \in X^T} e_T(x, X)[x]_T \in \quad H_*^T(X^T) \otimes_{\mathbb{C}} K.$$

8.3.5 Cellular fibration for equivariant Borel-Moore homology

We rewrite [CG97], section 5.5 for equivariant Borel-Moore homology. That is straightforward since all properties which are needed to get the analogue of [CG97], lemma 5.5.1 are fullfilled (i.e. Thom isomorphism and localization property).

Definition 39. Let G be a complex algebraic group. Let $\pi\colon Z \to X$ be a dominant morphism of complex G-varieties. We call Z a *cellular fibration* over X if Z is equipped with a finite desreasing filtration

$$Z = Z^n \supset Z^{n-1} \supset \cdots \supset Z^1 \supset Z^0 = \emptyset$$

such that for any $i = 1, \ldots, n$ the following hold

(a) $Z^{i-1} \subset Z^i$ is G-equivariant closed immersion. The restriction

$$\pi|_{Z^i}\colon Z^i \to X$$

is G-equivariant and locally trivial.

(b) Set $E^i := Z^i \setminus Z^{i-1}$. The restriction $\pi_i = \pi|_{E^i}\colon E^i \to X$ is a vector bundle (in particular also dominant).

Lemma 79. *([CG97], Lemma 5.5.1, p.270) In the setup of the previous definition.*

(a) Assume $H^G_{odd}(X) = 0$. For each $i = 1, \dots, n$ it holds $H^G_{odd}(Z^i) = 0$ and there is a canonical exact sequence

$$0 \to H^G_*(Z^{i-1}) \to H^G_*(Z^i) \to H^G_*(E^i) \to 0$$

of $H^*_G(pt)$-modules.

(b) If $H^G_{odd}(X) = 0$ and $H^G_*(X)$ is a free $H^G_*(pt)$-module with basis $c_1, \dots, c_m$, then all the short exact sequences in (a) are (non-canonically) split and $H^G_*(Z)$ is a free $H^G_*(pt)$-module of rank $n \cdot m$ with basis $(\varepsilon_i)_* \overline{\pi_i}^* c_j$, $1 \le i \le n$, $1 \le j \le m$ where

$$X \xleftarrow{\overline{\pi_i}} \overline{E^i} \xrightarrow{\varepsilon_i} Z$$

with $\overline{E^i}$ the closure of E^i, $\overline{\pi_i}$ the restriction of π and ε_i the closed embedding.

Sketch of proof:

1) Use the localization property and the Thom isomorphism to prove first the odd vanishing inductively and obtain this way the short exact sequences.

2) Again by induction on n. For $n = 1$ it holds $Z = E^1 \to X$ is a vector bundle and the claim follows from the Thom isomorphism. For the induction step use that the freeness of $H^G_*(E^i)$ and $H^G_*(Z^{i-1})$ implies the freeness of $H^G_*(Z^i)$ and the statement on the rank. The explicit basis statement comes from the splittings of the sequences.

□

8.4 The Serre cohomology spectral sequence with arbitrary coefficients

This has been explained to me by Greg Stevenson.
Let $X \to B$ a fibration of complex algebraic varieties over $\mathbb{C}$ with typical fibre F. We look at the diagram

$$\begin{array}{ccccc} F & \xrightarrow{f} & X & \xrightarrow{g} & B \\ & \searrow & \downarrow q & \swarrow p & \\ & & pt & & \end{array} .$$

The Grothendieck spectral sequence gives for any $\mathcal{F} \in D(X)$ a spectral sequence

$$E_2^{ij} = R^i p_*(R^j g_* \mathcal{F}) \Longrightarrow R^{i+1} q_* \mathcal{F}$$

Now, $R^j g_* \mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto \mathbb{H}^j(g^{-1}(U), \mathcal{F}|_{g^{-1}(U)})$$

We are interested in the two special cases

(1) $\mathcal{F} = \underline{A}_X$ is the complex concentrated in degree zero, where it is the trivial local system for a commutative ring A. In this case, for an open, contractable subset $U \subset B$ such that $g^{-1}(U) \cong F \times U$, we have

$$\mathbb{H}^j(g^{-1}(U), \underline{A}_X) = H^j_{ord}(F \times U, A) \cong H^j_{ord}(F, A)$$

where H^j_{ord} denotes ordinary (= singular) cohomology and the second isomorphism uses the Künneth formula. The spectral sequence is

$$E_2^{ij} = H^i(B, \mathcal{H}^j(F, A)) \Longrightarrow H^{i+j}_{ord}(X, A)$$

where $\mathcal{H}^j(F, A)$ is a local system with stalks $H^j_{ord}(F, A)$ and $H^i(B, -)$ is the sheaf cohomology. If the local system $\mathcal{H}^j(F, A)$ is trival and $H^j_{ord}(F, A)$ is a free A-module, then the spectral sequence simplifies to

$$E_2^{ij} = H^i_{ord}(B, A) \otimes_A H^j_{ord}(F, A)) \Longrightarrow H^{i+j}_{ord}(X, A)$$

Furthermore, if $H^{odd}_{ord}(F, A) = 0$, the spectral sequence degenerates at E_2 and we get an isomorphism of $H^*_{ord}(B, A)$-modules

$$H^*_{ord}(X, A) \cong H^*_{ord}(B, A) \otimes_A H^*_{ord}(F, A)$$

(more precisely, there exists a filtration of the graded ring $R := H^*_{ord}(X, A)$

$$R^u = F^0R^u \supset F^1R^u \supset \cdots \supset F^uR^u \supset F^{u+1}R^u = 0$$

by $H^*_{ord}(B, A)$-submodules such that the subquotients are free and $F^pR^u \cdot F^qR^v \subset F^{p+q}R^{u+v}$ and the associated graded ring $gr_F(H^*_{ord}(X, A))$ is isomorphic as bigraded ring to $H^*_{ord}(B, A) \otimes_A H^*_{ord}(F, A)$.)
Also, the two edge maps of the spectral sequence are morphisms of A-algebras

$$H^*_{ord}(X, A) \twoheadrightarrow H^*_{ord}(F, A) = E_2^{0,*}$$
$$E_2^{*,0} = H^*_{ord}(B, A) \hookrightarrow H^*_{ord}(X, A)$$

(2) If $\mathcal{F} = D_X$ is the dualizing sheaf and we assume B to be a manifold of (real) dimension $2n$, then for $\mathbb{C}^n \cong U \subset B$ open such that $g^{-1}(U) \cong F \times U$ it holds that

$$\mathbb{H}^j(g^{-1}(U), D_X|_{g^{-1}(U)}) = H^{BM}_{-j}(F \times U, \mathbb{C}) \cong H^{BM}_{-j-2n}(F, \mathbb{C})$$

where H^{BM}_j is the j-th Borel-Moore homology and the last isomorphism is given by the Künneth formula for Borel-Moore homology (cp. [CG97], p.99). Therefore $R^jg_*D_X$ is a local system with stalks isomorphic to $H^{BM}_{-j-2n}(F, \mathbb{C})$. If we asume that it is the trivial local system and use the fact that $H^{BM}_{-j-2n}(F, \mathbb{C})$ is automatically a free $\mathbb{C}$-vector space, we get a spectral sequence

$$E_2^{ij} = H^i_{ord}(B, \mathbb{C}) \otimes_{\mathbb{C}} H^{BM}_{-j-2n}(F, \mathbb{C}) \Longrightarrow H^{BM}_{-i-j}(X, \mathbb{C})$$

Furthermore, if $H^{BM}_{odd}(F, \mathbb{C}) = 0$, the spectral sequence degenerates at E_2 and we get an isomorphism of $\mathbb{C}$-vector spaces

$$H^{BM}_m(X, \mathbb{C}) \cong \bigoplus_{-i-j=m} H^i_{ord}(B, \mathbb{C}) \otimes_{\mathbb{C}} H^{BM}_{-j-2n}(F, \mathbb{C})$$

Now, let X be a complex G-variety where G is a connected algebraic group over $\mathbb{C}$. Then, the previous spectral sequence applied to the fibration $X_G \to BG$ can be written as

$$E_2^{pq} = H^p_G(BG) \otimes_{\mathbb{C}} H_{-q-2n}(X) \Longrightarrow H_{-p-q}(X_G) = H^G_{-p+(-q-2n)}(X)$$

because BG is simply connected (whenever G is connected, you get as part of the long exact sequence for homotopy groups $0 = \pi_1(EG, x) \to \pi_1(BG, y) \to \pi_0(G) = pt$). Remembering that $H^p_G(BG) = H^p_G(pt) = H^G_{-p}(pt)$, then after reindexing $s = -p, t = -q-2n$ the spectral sequence can be written as

$$E_2^{s,t} = H^G_s(pt) \otimes_{\mathbb{C}} H_t(X) \Longrightarrow H^G_{s+t}(X)$$

Definition 40. If the above spectral sequence degenerates, we say X is **G-equivariantly formal for Borel-Moore homology**.

Now, let us come to the Steinberg variety, recall that $Z = \bigsqcup_{i,j \in I} Z_{i,j}$, $Z_{i,j} = E_i \times_V E_j$ with $E_i = G \times^{P_i} F_i$ and G is a connected reductive group. We know that $H_{odd}(Z) = 0$ and that $H_*(Z)$ has a basis as $\mathbb{C}$-vector space given by algebraic cycles. We set $e_i := \dim E_i = \dim G + \dim F_i - \dim P_i$. Then, $H_*(Z)$ and $H^G_*(Z)$ become graded rings by the following definition (see [CG97], p.481)

$$H_{[p]}(Z) := \bigoplus_{i,j \in I} H_{e_i+e_j-p}(Z), \quad H^G_{[p]}(Z) := \bigoplus_{i,j \in I} H^G_{e_i+e_j-p}(Z).$$

It holds that $H_{[p]}(Z) * H_{[q]}(Z) \subset H_{[p+q]}(Z)$. For any $i, j \in I$, we have a degenerate (at E_2) spectral sequence

$$E_2^{-s, e_i+e_j-t} = H^{-s}_G(pt) \otimes_{\mathbb{C}} H_{e_i+e_j-t}(Z_{i,j}) \Longrightarrow H^G_{e_i+e_j-(s+t)}(Z_{i,j})$$

Taking the direct sum, we get a (degenerate at E_2) spectral sequence

$$E_2^{-s,t} = H^{-s}_G(pt) \otimes_{\mathbb{C}} H_{[t]}(Z) \Longrightarrow H^G_{[s+t]}(Z_{i,j})$$

Now, $E_2 = \bigoplus_{s,t} H^G_s(pt) \otimes_{\mathbb{C}} H_{[t]}(Z)$ is a bigraded ring. This is a spectral sequence which is compatible with the graded ring structure (i.e. $d_r(ab) = d_r(a)b + (-1)^{s+t} a d_r(b)$ because

$d_r = 0$), cp. for example Stricklands notes [Str08]. It implies

$$H_*^G(Z) = H_*^G(pt) \otimes H_*(Z)$$

as $H_*^G(pt)$-modules (more precisely: There exists a a filtration of the graded ring $R := H_*^G(Z)$

$$R^u = F^0R^u \supset F^1R^u \supset \cdots \supset F^uR^u \supset F^{u+1}R^u = 0$$

by $H_*^G(pt)$-submodules such that the subquotients are free and $F^pR^u \cdot F^qR^v \subset F^{p+q}R^{u+v}$ and the associated graded ring $gr_F(H_*^G(Z))$ is isomorphic as bigraded ring to $H_*^G(pt) \otimes H_*(Z)$.)

Also the two corner maps of the spectral sequence are morphisms of graded $\mathbb{C}$-algebras

$$H_*^G(Z) \twoheadrightarrow H_*(Z) = E_2^{0,*}$$
$$E_2^{*,0} = H_*^G(pt) \otimes H_{[0]}(Z) \hookrightarrow H_*^G(Z)$$

8.4.1 A lemma from the survey on Springer theory

As an application of the previously discussed Leray spectral sequences, we include the two lemmata and their proofs from the first chapter.

We set $\widetilde{W} := \bigsqcup_{i,j \in I} W_{i,j}$ with $W_{i,j} := W_i \; W/W_j$ where W is the Weyl group for (G,T) and $W_i \subset W$ is the Weyl group for (L_i, T) with $L_i \subset P_i$ is the Levi subgoup. We will fix representatives $w \in G$ for all elements $w \in \widetilde{W}$.

Let $C_w = G \cdot (eP_i, wP_j)$ be the G-orbit in $G/P_i \times G/P_i$ corresponding to $w \in W_{i,j}$.

Lemma 80. *(1) $p\colon C_w \subset G/P_i \times G/P_j \xrightarrow{pr_1} G/P_i$ is G-equivariant, locally trivial with fibre $p^{-1}(eP_i) = P_iwP_j/P_j$.*

(2) P_iwP_j/P_j admits a cell decomposition into affine spaces via Schubert cells

$$xB_jx^{-1}vwP_j/P_j, \quad v \in W_i$$

where $B_j \subset P_j$ is a Borel subgroup and $x \in W$ such that ${}^xB_j \subset P_i$. In particular, $H_{odd}(P_iwP_j/P_j) = 0$ and

$$H_*(P_iwP_j/P_j) = \bigoplus_{v \in W_i} \mathbb{C}b_{i,j}(v), \quad b_{i,j}(v) := [\overline{xB_jx^{-1}vwP_j/P_j}].$$

It holds that $\deg b_{i,j}(v) = 2\ell_{i,j}(v)$ where $\ell_{i,j}(v)$ is the length of a minimal coset representative in W for $x^{-1}vwW_j \in W/W_j$.

(3) For $A \in \{pt, T, G\}$ it holds $H^A_{odd}(C_w) = 0$ and since G/P_i is simply connected

$$H_n^A(C_w) = \bigoplus_{p+q=n} H_A^p(G/P_i) \otimes H_q(P_iwP_j/P_i),$$
$$H_*^A(C_w) = \bigoplus_{u \in W/W_i, v \in W_i} \mathbb{C}b_i(u) \otimes b_{i,j}(v),$$

where $b_i(u) = [\overline{B_iuP_i/P_i}]^*$ is of degree $2\dim_{\mathbb{C}} G/P_i - 2\ell_i(u)$ with $\ell_i(u)$ is the length of a minimal coset representative for $u \in W/W_i$ and $b_{i,j}(v)$ as in (2).

Proof:

(1) Since $p\colon C_w \subset G/P_i \times G/P_j \to G/P_i$ is G-equivariant and G/P_i a homogeneous space, we can apply Slodowy's lemma (see lemma 77) which implies there is an isomorphism of G-varieties $C_w \cong G \times^{P_i} p^{-1}(eP_i)$ over G/P_i. If the Levi group for P_i is a product of special algebraic groups, then P_i is special as well, this implies that any principal P_i-bundle is locally trivial in the Zariski topology. For the analytic topology it is locally trivial because it is a smooth map. It is easy to see $p^{-1}(eP_i) = P_iwP_j/P_j$.

(2) Now, let $B_j \subset P_j$ be a Borel subgroup of G, $x \in W$ such that ${}^xB_j \subset P_i$. It holds $P_iwP_j/P_j = \bigcup_{v\in W_i} xB_jx^{-1}vwP_j/P_j$, $xB_jx^{-1}vwP_j/P_j$ is isomorphic to an affine space. The dimension of a Schubert cell $xB_jx^{-1}yP_j/P_j$ can be found as follows: Let $(x^{-1}y)^j$ be the minimal coset representative of $x^{-1}yW_j \in W/W_j$. Then it holds $\dim xB_jx^{-1}yP_j/P_j = \ell((x^{-1}y)^j)$. It is well-known that the Schubert cells give a cellular decomposition of P_iwP_j/P_j. It follows that $H_{odd}(P_iwP_j/P_j) = 0$ and $H_*(P_iwP_j/P_j)$ has a $\mathbb{C}$-vector space basis given by the cycles $[\overline{xB_jx^{-1}vwP_j/P_j}]$ where $\deg[\overline{xB_jx^{-1}vwP_j/P_j}] = 2\ell((x^{-1}vw)^j)$.

(3) Consider the Serre cohomology spectral sequence for the fibration in (1)

$$E_2^{p,q} = H^p(G/P_i, \mathcal{H}_{-q-2f_i}(P_iwP_j/P_j)) \Rightarrow H_{-p-q}(C_w)$$

where $\mathcal{H}_{-q-2f_i}(P_iwP_j/P_j)$ is a local system with stalk $H_{-q-2f_i}(P_iwP_j/P_j)$ and $f_i = \dim(G/P_i)$. Since $H_{odd}(P_iwP_j/P_j) = 0$ we obtain that the spectral sequence degenerates, which implies that $H_{odd}(C_w) = 0$. Since G/P_i is simply connected the local system $\mathcal{H}_{-q-2f_i}(P_iwP_j/P_j)$ is trivial and $H^p(G/P_i, \mathcal{H}_{-q-2f_i}(P_iwP_j/P_j)) = H^p(G/P_i) \otimes_{\mathbb{C}} H_{-q-2f_i}(P_iwP_j/P_j)$. Then the degeneration gives

$$H_m(C_w) = \bigoplus_{q-p=m-2f_i} H^p(G/P_i) \otimes_{\mathbb{C}} H_q(P_iwP_j/P_j)$$

Now, for $A \in \{T, G\}$ we have the Serre spectral sequence associated to the fibration $(C_w)_A \to BA$

$$E_2^{p,q} := H_p^A(pt) \times_{\mathbb{C}} H_q(C_w) \Rightarrow H_{p+q}^A(C_w)$$

which degenerates because $H_{odd}(C_w) = 0$, therefore we have

$$\begin{aligned} H_m^A(C_w) &= \bigoplus_{p+q=m} H_p^A(pt) \otimes_{\mathbb{C}} H_q(C_w) \\ &= \bigoplus_{p+s-t=m-2f_i} H_p^A(pt) \otimes_{\mathbb{C}} H^s(G/P_i) \otimes_{\mathbb{C}} H_t(P_iwP_j/P_j) \end{aligned}$$

and $H_{odd}^A(C_w) = 0$.

□

8.4.2 The cohomology rings of flag varieties.

We review some known results.
Let G be a connected reductive group over $\mathbb{C}$ and $P \subset G$ a parabolic subgroup. We write $H^*(-, \mathbb{C})$ for the ordinary cohomology ring (wrt the cap product) and for any algebraic group H we write $H^*_H(-, \mathbb{C})$ for the H-equivariant cohomology ring (wrt. cap product). As usual, if we leave out the coefficients in (co)homology groups, they are assumed to be the complex numbers.
Furthermore, fix $T \subset P$ a maximal torus and let W be the Weyl group associated to (G, T). Let $W_P \subset W$ be the subgroup stabilizing P. Let us first remark the following

Remark. Let $I_W \subset \mathbb{C}[\mathfrak{t}]$ and $I_W^{(P)} \subset \mathbb{C}[\mathfrak{t}]^{W_P}$ be the ideals generated by the kernel of the algebra homomorphism $\mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}, f \mapsto f(0)$. Then it holds

$$I_W^{(P)} = I_W \cap \mathbb{C}[\mathfrak{t}]^{W_P} = I_W^{W_P} \quad \subset \mathbb{C}[\mathfrak{t}]^{W_P}.$$

Furthermore, it holds that

$$(\mathbb{C}[\mathfrak{t}]/I_W)^{W_P} = \mathbb{C}[\mathfrak{t}]^{W_P}/I_W^{(P)}$$

This follows because $()^{W_P}$ is an exact functor on finite dimensional complex W_P-modules and since W_P operates by graded vector space maps on $\mathbb{C}[\mathfrak{t}]$ and I_W, we can restrict on the graded parts to use the exactness.

Theorem 8.4.1. *There are isomorphisms*

$$\begin{aligned} H^*(G/P) &= (\mathbb{C}[\mathfrak{t}]/I_W)^{W_P} = \mathbb{C}[\mathfrak{t}]^{W_P}/I_W^{(P)} && \textit{as } \mathbb{C}-\textit{algebras,} \\ H^*_T(G/P) &= \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}]^{W_P} && \textit{as } H^*_T(pt) \cong \mathbb{C}[\mathfrak{t}]-\textit{algebras,} \\ H^*_G(G/P) &= \mathbb{C}[\mathfrak{t}]^{W_P} && \textit{as } H^*_G(pt) \cong \mathbb{C}[\mathfrak{t}]^W-\textit{algebras,} \end{aligned}$$

where the degree of the elements in $\mathrm{Hom}_{\mathbb{C}-vs}(\mathfrak{t}, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}]$ *is* 2.

Proof: Let $B \subset P$ be a Borel subgroup. Recall that the Borel homomorphism $c\colon \mathbb{C}[t] \to H^*(G/B)$ is a surjective algebra W-linear algebra homomophism with kernel I_W. Let $\alpha\colon G/B \to G/P$ be the natural map. We write α^* for the pullback map in the different cohomology groups. Recall from [BGG73b] that $\alpha^*\colon H^*(G/P) \to H^*(G/B) = \mathbb{C}[\mathfrak{t}]/I_W$ is a monomorphism of finite dimensional $\mathbb{C}$-algebras which identifies $H^*(G/P)$ with the subalgebra of W_P-invariants in $H^*(G/B)$.
The third identity $H^*_G(G/P) = \mathbb{C}[\mathfrak{t}]^{W_P}$ follows from [Bri00], rem. 3), page 4, which is originally due to Arabia (cp. [Ara85]). In our situation it states as

$$H^*_G(G/P) \cong H^*_P(pt).$$

Now let L be the Levi group in P, by definition W_P is the Weyl group of (L,T). The Levi-decomposition shows that L is homotopy equivalent to P. This implies $H_P^*(pt) = H_L^*(pt)$, then the splitting principle gives $H_L^*(pt) = (H_T^*(pt))^{W_P}$.
Finally $H_T^*(G/P)$ is calculated with the knowledge of $H_G^*(G/P)$ also in [Bri00], Prop 1 (iii), page 6 for the isomorphism as $H_G(pt)$-modules. To see that the ring structure is also the same one uses a localization to the T-fixed points, for the case $P=B$ see the example on p.14 in [Bri00]. □.

8.4.3 Forgetful maps

Recall that by the forgetful maps the commutative diagrams of $\mathbb{C}$-algebras

$$\begin{array}{ccc} H_G^*(G/P) & \longrightarrow & H_T^*(G/P) \\ & \searrow \quad \swarrow & \\ & H^*(G/P) & \end{array}$$

is given by the ring homomorphisms

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{t}]^{W_P} = \mathbb{C}[\mathfrak{t}]^W \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}]^{W_P} & \xrightarrow{\ incl\ } & \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}]^{W_P} \\ & \searrow^{\pi} \qquad \swarrow^{??} & \\ & \mathbb{C}[\mathfrak{t}]^{W_P}/I_W^{(P)}. & \end{array}$$

where $\pi\colon \mathbb{C}[\mathfrak{t}]^{W_P} \to \mathbb{C}[\mathfrak{t}]^{W_P}/I_W^{(P)}$ is the canonical surjection. The map ?? is a surjective ring homomorphism which makes the diagram commutative but I do not know it explicitly. The claim that the forgetful maps to usual cohomology are surjective is equivalent to the degeneration of the Serre spectral sequence which we discuss in the following.

Remark. (The Serre cohomology spectral sequence from ordinary to equivariant cohomology)
This is another special case of Serre cohomology spectral sequence, this time for singular cohomology.
Let X be a complex algebraic variety with an action of an algebraic group G and let $c \in H^*(X)$. We say that $\tilde{c} \in H_G^*(X)$ **lifts** c **to** $H_G^*(X)$ if it maps under the forgetful map $H_G^*(X) \to H^*(X)$ to c.

The Serre cohomology spectral sequence for the fibration $X_G := X \times^G EG \to BG$ with fibre X is of the form

$$E_2^{pq} = H^p(BG, \mathcal{H}^q(X)) \Rightarrow H^{p+q}(X_G)$$

The second sheet of the spectral sequence is the tensor product is $E_2^{pq} = H^p(BG) \otimes_{\mathbb{C}} H^q(X)$. because $\pi_1(BG)$ is trivial.The following statements are equivalent

(1) The Serre cohomology spectral sequence degenerates at E_2.

(2) The forgetful ring homomorphism $H_G^*(X) \to H^*(X)$ is surjective.

(3) $H_G^*(X) = H_G^*(pt) \otimes_{\mathbb{C}} H^*(X)$ as $H_G(pt)$-modules.

(4) $H_G^*(X)$ is a free $H_G^*(pt)$-module and every lift of every $\mathbb{C}$-vector space basis of $H^*(X)$ under the forgetful map is a basis of $H_G^*(X)$ as $H_G^*(pt)$-module.

If one of the equivalent condition is fulfilled we say that X is **G-equivariantly formal for cohomology**.

Now, for G/P seen as T-variety the previous remark applies, in particular $H_T^*(G/P) \cong H_T^*(pt) \otimes H^*(G/P) = \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} (\mathbb{C}[\mathfrak{t}]/I_W)$ as $\mathbb{C}[\mathfrak{t}]$-module. The basis of Schubert cycles in $H^*(G/P)$ can be lifted to the basis of (T-equivariant) Schubert cycles in $H_T^*(G/P)$ (recall that all these cycles are invariant under T, the cellular fibration method can be applied to see this).

For G/P as G-variety the previous remark applies, the Serre spectral sequence degenerates at E_2 because the odd cohomology of G/P vanishes (which implies $\mathcal{H}^q(G/P) = 0$ for q odd). In other words we know

$$H_G^n(G/P) = \bigoplus_{p+q=n} H^p(BG, \mathcal{H}^q(G/P))$$

and since $\pi_1(BG) = \{pt\}$ because G is connected we get $\mathcal{H}^q(G/P)$ is the trivial local system. Hence we see

$$H_G^*(G/P) \cong H^*(G/P) \otimes_{\mathbb{C}} H_G^*(pt) \text{ as } H_G^*(pt)\text{-module.}$$

Using the statement (4) in the remark we obtain the following corollary.

Corollary 8.4.1.1. *(1) $H_G^*(G/B) = \mathbb{C}[\mathfrak{t}]$ is a free module over $H_G^*(G/P) = \mathbb{C}[\mathfrak{t}]^{W_P}$ of rank $\#W_P$.*
A basis is given by a set b_w, $w \in W_P$ where b_w is a lift of $[\overline{B_L w B_L/B_L}] \in H^*(L/B_L)$ to $H_L^*(L/B_L) = \mathbb{C}[\mathfrak{t}]$ where $L \subset P$ is the Levi subgroup and $B_L = B \cap L$.

(2) $H_G^*(G/P) = \mathbb{C}[\mathfrak{t}]^{W_P}$ is a free module over $H_G^*(pt) = \mathbb{C}[\mathfrak{t}]^W$ of rank $\#W^P$.
A basis is given by a set c_w, $w \in W^P$ where c_w is a lift of $[\overline{BwP/P}] \in H^*(G/P)$ to $H_G^*(G/P) = \mathbb{C}[\mathfrak{t}]^{W_P}$.

Sketch of proof:

(1) Consider the locally trivial fibre bundle $G/B \to G/P$ with typical fibre P/B. Use the degeneration of a Serre cohomology spectral sequence to get

$$H_G^n(G/B) = \bigoplus_{p+q=n} H^p(P/B) \otimes_{\mathbb{C}} H_G^q(G/P).$$

Identify this with the degeneration of the Serre cohomology spectral sequence for the map $(L/B_L)_L \to BL$ with typical fibre L/B_L, i.e. with

$$H^n_L(L/B_L) = \bigoplus_{p+q=n} H^p(L/B_L) \otimes_{\mathbb{C}} H^q_L(pt).$$

to get the statement of the corollary.

(2) Use the degeneration of the Serre cohomology spectral sequence for the map $(G/P)_G \to BG$ with typical fibre G/P.

□

8.5 Equivariant perverse sheaves

We will only consider the middle perversity function. We use the convention of Arabia for the perverse t-structure (compare [Ara01]).

8.5.1 Perverse sheaves

Let X be an algebraic variety over $\mathbb{C}$ of complex dimension d_X. Recall that we have Verdier duality $\mathbb{D}$ on the category of derived category of constructible sheaves $D^b(X)$. Also we set the support of a sheaf to be the set of points where the stalks are nonzero. We denote the shift functor by $F^\bullet \mapsto F^\bullet[d]$ defined via $(F^\bullet[d])^n := F^{n+d}$ (and on cohomology by $\mathcal{H}^n(F^\bullet[d]) = \mathcal{H}^{n+d}(F^\bullet))$.

Definition 41. We define the perverse t-structure on $D^b(X)$ via

$$F^\bullet \in {}^pD_{\geq 0} \quad \iff \quad \forall S \subset X \colon \mathcal{H}^n(i_S^! F^\bullet) = 0, \quad n > -d_S$$

where $S \subset X$ runs through all locally closed analytic subsets, d_S is its complex dimension and $i_S \colon S \to X$ the inclusion.

We say a $F^\bullet \in D^b(X)$ is a **perverse sheaf** if it is in the heart of the perverse t-structure, i.e. if the supports of $\mathcal{H}^n(F^\bullet)$ and $\mathcal{H}^n(\mathbb{D}_X(F^\bullet))$ have dimension $\leq -n$. In particular, for a perverse sheaf $F^\bullet$ it holds that $\mathcal{H}^n(F^\bullet) = 0, \mathcal{H}^n(\mathbb{D}_X F^\bullet) = 0$ for $n > 0$.

We denote by $\mathcal{P}(X) \subset D^b(X)$ the category of perverse sheaves on X.

For example, if X is smooth, the constant sheaf $\underline{\mathbb{C}}_X[d_X]$ (i.e. the complex concentrated in degree $-d_X$) is a perverse sheaf, where we use $\mathbb{D}_X(\underline{\mathbb{C}}_X[d_X]) = \underline{\mathbb{C}}_X[d_X]$. More generally for any local systems $\mathcal{L}$ on a smooth variety X, the shifted complex $\mathcal{L}[d_X]$ is perverse sheaf, it holds that $\mathbb{D}_X(\mathcal{L}[d_X]) = \mathcal{L}^*[d_X]$ where $\mathcal{L}^* := \mathcal{H}om(\mathcal{L}, \underline{C}_X)$ is the dual local system. We call a local system $\mathcal{L}$ simple if the via monodromy associated representation $\pi_1(X,x) \to \mathbf{Gl}(\mathcal{L}_x)$ of the fundamental group is simple.

Let X again be an arbitrary complex variety. Let $U \subset X$ be a dense smooth subset and $\mathcal{L}$ a local system on U, then there exists a complex $\mathcal{IC}(X, \mathcal{L}) \in D^b(X)$ called the minimal (or intermediate) extension[1] uniquely determined by the properties

[1] it is defined by applying Deligne's minimal extension functor

1) $\mathcal{IC}(X,\mathcal{L})[d_X] \in \mathcal{P}(X)$

2) $i_U^*\mathcal{IC}(X,\mathcal{L}) = \mathcal{L}$

It holds that $\mathcal{IC}(X,\mathcal{L}) = \mathcal{IC}(X,\mathcal{L}')$ for a local system $\mathcal{L}$ on $U \subset X, \mathcal{L}'$ on $U' \subset X$ if and only if there exists an open smooth subset $V \subset U \cap U'$ with $\mathcal{L}|_V = \mathcal{L}'|_V$.

For every closed irreducible subset $Z \xrightarrow{i_Z} X$ of dimension d_Z (as complex variety) and a local system $\mathcal{L}$ on an open smooth subset $U \subset Z$ we get a perverse sheaf

$$IC_{(Z,\mathcal{L})} := (i_Z)_* \left(\mathcal{IC}(Z,\mathcal{L})[d_Z]\right) \quad \in \mathcal{P}(X).$$

Theorem 8.5.1. *(for this version see [Ara01], but it is due to [BBD82].)*

(a) The category $\mathcal{P}(X)$ is an abelian category.

(b) It is stable under $\mathbb{D}_X$,

(c) It is admissible (i.e. for every $0 \to E^\bullet \xrightarrow{a} F^\bullet \xrightarrow{b} G^\bullet \to 0$ with $ba = 0$ it holds: (a,b) short exact sequence in $\mathcal{P}(X)$ if and only if $E^\bullet \xrightarrow{a} F^\bullet \xrightarrow{b} G^\bullet \xrightarrow{[+1]}$ is a distinguished triangle.),

(d) it is extension closed (i.e. for any distinguished triangle $E^\bullet \xrightarrow{a} F^\bullet \xrightarrow{b} G^\bullet \xrightarrow{[+1]}$ with $E^\bullet, G^\bullet \in \mathcal{P}(X)$ it holds $F^\bullet \in \mathcal{P}(X)$.).

(e) The simple objects of $\mathcal{P}(X)$ are precisely $IC_{(Z,\mathcal{L})}$ for $Z \subset X$ closed and $\mathcal{L}$ a simple local system on an open in Z.

(f) Every perverse sheaf has a composition series (i.e. filtration with simple subquotients) of finite lengths. We say that $\mathcal{P}(X)$ is a finite length category. (In the literature this is referred to saying that: $\mathcal{P}(X)$ is artinian and noetherian.)

Theorem 8.5.2. *(BBD-Decomposition theorem, [BBD82]) Let $\pi\colon X \to Y$ be a proper map between complex algebraic varieties. Then, $\pi_* IC^G_{(X,\underline{\mathbb{C}})} \in D^b_G(Y)$ is a direct sum of shifts of simple objects.*

8.5.2 Equivariant perverse sheaves

Here we follow the definition of [BL94], p.41.
Let G be an algebraic group operating on a complex algebraic variety X. Recall that we have a forgetful functor

$$For\colon D^b_G(X) \to D^b(X), \quad F \mapsto F_X.$$

Definition 42. We define the category of G-equivariant perverse sheaves $\mathcal{P}_G(X)$ to be the following full subcategory of $D^b_G(X)$

$$\mathcal{P}_G(X) := \{F \in D^b_G(X) \mid F_X \in \mathcal{P}(X)\}.$$

By definition we get through restriction a forgetful functor

$$For\colon \mathcal{P}_G(X) \to \mathcal{P}(X), \quad F \mapsto F_X.$$

Remark. ([BL94], loc cit) It follows from the properties of $\mathcal{P}(X)$ that the category $\mathcal{P}_G(X)$ has the following properties:
It is the *heart of the perverse t-structure* on $D^b_G(X)$ and therefore abelian. Every object in $\mathcal{P}_G(X)$ has finite length, and we can describe the simple objects as IC-sheaves in the following way.

Recall that we shortly mentioned the following result: By definition $D^b_G(X)$ is a full triangulated subcategory of $D^b(X_G)$. The heart of the natural t-structure (i.e. the one defined by the truncation functors) is the category of G-equivariant constructible sheaves on X, we denote this by $Sh_G(X)$, cf. [BL94], Prop: 2.5.3, p.25.
With the analogue definition using truncation functors one can define minimal (or intermediate) extension functors in the equivariant situation. It can be characterized as before:

Let $j\colon U \to X$ be the inclusion of a locally closed irreducible G-invariant smooth dense subset of complex dimension d_U. Let $\mathcal{L} \in Sh_G(U)$ be a G-equivariant local system. There exists $j_{!*}\mathcal{L} = \mathcal{IC}^G(X,\mathcal{L}) \in D^b_G(X)$, called the **intermediate extension**, uniquely determined by the following properties

1) $\mathcal{IC}^G(X,\mathcal{L})[d_V] \in \mathcal{P}_G(X)$
2) $j^*\mathcal{IC}^G(X,\mathcal{L}) = \mathcal{L}$.

Then for every closed irreducible G-equivariant subset $Z \xrightarrow{i_Z} X$ of dimension d_Z and a G-equivariant local system $\mathcal{L}$ on an open smooth subset of Z we get a perverse sheaf

$$IC^G_{(Z,\mathcal{L})} := (i_Z)_* \left(\mathcal{IC}^G(Z,\mathcal{L})[d_Z]\right) \quad \in \mathcal{P}_G(X).$$

The analogue of theorem 8.5.1 holds in the equivariant situation. Furthermore, we have the following.

Lemma 81. *(Folklore, see also [CG97], p.438, also used in [Kat13]) The forgetful functor $For\colon \mathcal{P}_G(X) \to \mathcal{P}(X)$ fulfills $IC^G_{(Z,\mathcal{L})} \mapsto IC_{(Z,\mathcal{L})}$. It induces an equivalence of categories between the semisimple category spanned by the simple objects in $\mathcal{P}_G(X)$ and full additive subcategory of $\mathcal{P}(X)$ generated by $IC_{(Z,\mathcal{L})}$ where Z is G-invariant and $\mathcal{L}$ is G-equivariant.*

In the Appendix of [Cla08], it is stated that the forgetful functor in the previous lemma is even fully faithful.

Theorem 8.5.3. *(equivariant BBD-decomposition theorem, [BL94], p.42) Let $\pi\colon X \to Y$ be a proper, G-equivariant map of complex algebraic varieties. Then, $\pi_* IC^G_{(X,\underline{\mathbb{C}})} \in D^b_G(Y)$ is a direct sum of shifts of simple objects.*

Remark. Since the forgetful functor commute with pushforward, the direct summands in the BBD-decomposition theorem are the *same* after applying the forgetful functor as in the equivariant BBD-decomposition theorem.
Also observe that if X is smooth, then $IC_{(X,\underline{\mathbb{C}})} = \underline{\mathbb{C}}_X[d_X]$. In fact, the BBD-decomposition theorem and its equivariant version hold true not just for the pushforward of $IC_{(X,\underline{\mathbb{C}})}$ but for the pushforward of $IC_{(X,\mathcal{L})}$ with $\mathcal{L}$ local system of *geometric origine*, see [BBD82] for the definition. We do not use this here.

Bibliography

[ADE08] ADE chain: *workshop.* Website, 2008. – Available online at http://www.math.uni-bielefeld.de/ sek/ADE/ visited on September 26th 2013. 182

[AH08] Achar, Pramod N. ; Henderson, Anthony: Orbit closures in the enhanced nilpotent cone. In: *arxiv[math.RT]* 07121079 (2008) 48

[And11] Anderson, Dave: *Introduction to Equivariant Cohomology in Algebraic Geometry.* Dec 2011. – arXiv:1112.1421v1 247, 248

[Ara85] Arabia, Alberto: Cycles de Schubert et cohomologie équivariante de K/T. In: *C. R. Acad. Sci. Paris Sér. I Math.* 301 (1985), Nr. 2, S. 45–48. – ISSN 0249–6291 63, 261

[Ara01] Arabia, Alberto: Faisceaux pervers sur les variétés algébriques complexes. Correspondance de Springer (d'après Borho-MacPherson). In: *preprint* (2001), 1–63. http://www.math.jussieu.fr/~arabia/ 18, 23, 28, 36, 50, 100, 264, 265

[Ass06a] Assem, Ibrahim: Simple connectedness and homology of Schurian algebras. In: *Ann. Sci. Math. Québec* 30 (2006), Nr. 2, S. 143–174 (2008). – ISSN 0707–9109 181, 182, 189

[ASS06b] Assem, Ibrahim ; Simson, Daniel ; Skowroński, Andrzej: *London Mathematical Society Student Texts.* Bd. 65: *Elements of the representation theory of associative algebras. Vol. 1.* Cambridge : Cambridge University Press, 2006. – x+458 S. http://dx.doi.org/10.1017/CBO9780511614309. http://dx.doi.org/10.1017/CBO9780511614309. – ISBN 978–0–521–58423–4; 978–0–521–58631–3; 0–521–58631–3. – Techniques of representation theory 165, 185, 187, 210

[BB81] Beĭlinson, Alexandre ; Bernstein, Joseph: Localisation de g-modules. In: *C. R. Acad. Sci. Paris Sér. I Math.* 292 (1981), Nr. 1, S. 15–18. – ISSN 0151–0509 36

[BBD82] Beĭlinson, A. A. ; Bernstein, J. ; Deligne, P.: Faisceaux pervers. In: *Analysis and topology on singular spaces, I (Luminy, 1981)* Bd. 100. Paris : Soc. Math. France, 1982, S. 5–171 18, 265, 267

[BBM89] Borho, W. ; Brylinski, J.-L. ; MacPherson, R.: *Progress in Mathematics*. Bd. 78: *Nilpotent orbits, primitive ideals, and characteristic classes*. Boston, MA : Birkhäuser Boston Inc., 1989. – vi+131 S. http://dx.doi.org/10.1007/978-1-4612-4558-2. http://dx.doi.org/10.1007/978-1-4612-4558-2. – ISBN 0–8176–3473–8. – A geometric perspective in ring theory 36, 50

[BD01] Bongartz, Klaus ; Dudek, Dirk: Decomposition classes for representations of tame quivers. In: *J. Algebra* 240 (2001), Nr. 1, 268–288. http://dx.doi.org/10.1006/jabr.2000.8705. – DOI 10.1006/jabr.2000.8705. – ISSN 0021–8693 146, 151, 169

[Bez06] Bezrukavnikov, Roman: Noncommutative counterparts of the Springer resolution. In: *International Congress of Mathematicians. Vol. II.* Eur. Math. Soc., Zürich, 2006, S. 1119–1144 33, 49, 51

[BGG73a] Bernšteĭn, I. N. ; Gel′fand, I. M. ; Gel′fand, S. I.: Schubert cells, and the cohomology of the spaces G/P. In: *Uspehi Mat. Nauk* 28 (1973), Nr. 3(171), S. 3–26. – ISSN 0042–1316 100

[BGG73b] Bernšteĭn, I. N. ; Gel′fand, I. M. ; Gel′fand, S. I.: Schubert cells, and the cohomology of the spaces G/P. In: *Uspehi Mat. Nauk* 28 (1973), Nr. 3(171), S. 3–26. – ISSN 0042–1316 261

[BH00a] Brüstle, Thomas ; Hille, Lutz: Finite, tame, and wild actions of parabolic subgroups in $\mathrm{GL}(V)$ on certain unipotent subgroups. In: *J. Algebra* 226 (2000), Nr. 1, 347–360. http://dx.doi.org/10.1006/jabr.1999.8180. – DOI 10.1006/jabr.1999.8180. – ISSN 0021–8693 215

[BH00b] Brüstle, Thomas ; Hille, Lutz: Matrices over upper triangular bimodules and Δ-filtered modules over quasi-hereditary algebras. In: *Colloq. Math.* 83 (2000), Nr. 2, S. 295–303. – ISSN 0010–1354 33, 215

[BHRR99] Brüstle, Thomas ; Hille, Lutz ; Ringel, Claus M. ; Röhrle, Gerhard: The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n\rangle$. In: *Algebr. Represent. Theory* 2 (1999), Nr. 3, 295–312. http://dx.doi.org/10.1023/A:1009999006899. – DOI 10.1023/A:1009999006899. – ISSN 1386–923X 33, 215

[BK81] Brylinski, J.-L. ; Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems. In: *Invent. Math.* 64 (1981), Nr. 3, 387–410. http://dx.doi.org/10.1007/BF01389272. – DOI 10.1007/BF01389272. – ISSN 0020–9910 36

[BK09] Brundan, Jonathan ; Kleshchev, Alexander: Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. In: *Invent. Math.* 178 (2009), Nr. 3, 451–484. http://dx.doi.org/10.1007/s00222-009-0204-8. – DOI 10.1007/s00222–009–0204–8. – ISSN 0020–9910 51

[BKM12] BRUNDAN, J. ; KLESHCHEV, A. ; MCNAMARA, P.J.: Homological Properties of finite type Khovanov-Lauda-Rouquier algebras. In: *arxiv[math.RT]* 12106900 (2012) 50

[BL94] BERNSTEIN, Joseph ; LUNTS, Valery: *Lecture Notes in Mathematics*. Bd. 1578: *Equivariant sheaves and functors*. Berlin : Springer-Verlag, 1994. – iv+139 S. – ISBN 3–540–58071–9 12, 18, 96, 97, 98, 115, 252, 253, 265, 266

[BLB10] BUCHWEITZ, Ragnar-Olaf ; LEUSCHKE, Graham J. ; BERGH, Michel Van d.: Non-commutative desingularization of determinantal varieties I. In: *Invent. Math.* 182 (2010), Nr. 1, 47–115. http://dx.doi.org/10.1007/s00222-010-0258-7. – DOI 10.1007/s00222–010–0258–7. – ISSN 0020–9910 33, 51

[BLB11] BUCHWEITZ, Ragnar-Olaf ; LEUSCHKE, Graham J. ; BERGH, Michel Van d.: Non-commutative desingularization of determinantal varieties, II. In: *arXiv [math.AC]* 11061833 (2011), S. 1–20 33, 51

[BM83] BORHO, Walter ; MACPHERSON, Robert: Partial resolutions of nilpotent varieties. In: *Analysis and topology on singular spaces, II, III (Luminy, 1981)* Bd. 101. Paris : Soc. Math. France, 1983, S. 23–74 105

[Bon83] BONGARTZ, Klaus: Algebras and quadratic forms. In: *J. London Math. Soc. (2)* 28 (1983), Nr. 3, 461–469. http://dx.doi.org/10.1112/jlms/s2-28.3.461. – DOI 10.1112/jlms/s2–28.3.461. – ISSN 0024–6107 184

[Bon84a] BONGARTZ: A criterium for finite representation type. In: *Math. Ann.* 269 (1984), S. 1–12 182

[Bon84b] BONGARTZ: Critical simply connected algebras. In: *Manuscripta Math.* 46 (1984), S. 117–136 183

[Bon98] BONGARTZ: Some geometric aspects of representation theory. In: *Canadian Mathematical Society, Conference Proceedings* 23 (1998), S. 1–27 30, 210

[Bri97] BRION, M.: Equivariant Chow groups for torus actions. In: *Transform. Groups* 2 (1997), Nr. 3, 225–267. http://dx.doi.org/10.1007/BF01234659. – DOI 10.1007/BF01234659. – ISSN 1083–4362 100

[Bri98] BRION, Michel: Equivariant cohomology and equivariant intersection theory. In: *Representation theories and algebraic geometry (Montreal, PQ, 1997)* Bd. 514. Dordrecht : Kluwer Acad. Publ., 1998, S. 1–37. – Notes by Alvaro Rittatore 248

[Bri00] BRION, Michel: Poincaré duality and equivariant (co)homology. In: *Michigan Math. J.* 48 (2000), 77–92. http://dx.doi.org/10.1307/mmj/1030132709. – DOI 10.1307/mmj/1030132709. – ISSN 0026–2285. – Dedicated to William Fulton on the occasion of his 60th birthday 16, 78, 79, 248, 255, 261, 262

[BZ01] Bobiński, Grzegorz ; Zwara, Grzegorz: Normality of orbit closures for Dynkin quivers of type $\mathbb{A}_n$. In: *Manuscripta Math.* 105 (2001), Nr. 1, 103–109. http://dx.doi.org/10.1007/PL00005871. – DOI 10.1007/PL00005871. – ISSN 0025–2611 175

[BZ02] Bobiński, Grzegorz ; Zwara, Grzegorz: Schubert varieties and representations of Dynkin quivers. In: *Colloq. Math.* 94 (2002), Nr. 2, 285–309. http://dx.doi.org/10.4064/cm94-2-10. – DOI 10.4064/cm94–2–10. – ISSN 0010–1354 175

[BZ06] Bobiński, Grzegorz ; Zwara, Grzegorz: Normality of orbit closures for directing modules over tame algebras. In: *J. Algebra* 298 (2006), Nr. 1, 120–133. http://dx.doi.org/10.1016/j.jalgebra.2005.06.023. – DOI 10.1016/j.jalgebra.2005.06.023. – ISSN 0021–8693 175

[Car85a] Carter, Roger W.: *Finite groups of Lie type*. New York : John Wiley & Sons Inc., 1985 (Pure and Applied Mathematics (New York)). – xii+544 S. – ISBN 0–471–90554–2. – Conjugacy classes and complex characters, A Wiley-Interscience Publication 31, 36, 166

[Car85b] Carter, Roger W.: *Finite groups of Lie type*. New York : John Wiley & Sons Inc., 1985 (Pure and Applied Mathematics (New York)). – xii+544 S. – ISBN 0–471–90554–2. – Conjugacy classes and complex characters, A Wiley-Interscience Publication 53

[CB92] Crawley-Boevey, William: *Lectures on Representations of Quivers*. Website, 1992. – Available online at http://www1.maths.leeds.ac.uk/~pmtwc/; visited on August 19th 2013. 163

[CBS02] Crawley-Boevey, William ; Schröer, Jan: Irreducible components of varieties of modules. In: *J. Reine Angew. Math.* 553 (2002), S. 201–220 205, 206, 207, 208, 209

[CC05] Caldero ; Chapoton: Cluster algebras as Hall algebras of quiver representations. In: *arxiv:* 0410187 (2005) 49

[CG97] Chriss ; Ginzburg: *Representation Theory and Complex Geometry*. Birkhäuser, 1997 9, 11, 12, 13, 15, 22, 23, 25, 34, 36, 37, 50, 64, 71, 80, 96, 97, 102, 109, 133, 166, 169, 249, 254, 255, 256, 257, 258, 266

[Cla08] Clausen, Dustin: *The Springer correspondence*, Harvard University, Cambidge USA, Diss., 2008 27, 28, 35, 266

[CLO92] Cox, David ; Little, John ; O'Shea, Donal: *Ideals, varieties, and algorithms*. New York : Springer-Verlag, 1992 (Undergraduate Texts in Mathematics). – xii+513 S. – ISBN 0–387–97847–X. – An introduction to computational algebraic geometry and commutative algebra 147

[CM07] Cataldo, Mark A. ; Migliorini, Luca: *The Decomposition Theorem and the topology of algebraic maps*. Dec 2007. – arXiv:0712.0349 97, 102

[DD05] Deng, Bangming ; Du, Jie: Monomial bases for quantum affine sl_n. In: *Adv. Math.* 191 (2005), Nr. 2, 276–304. http://dx.doi.org/10.1016/j.aim.2004.03.008. – DOI 10.1016/j.aim.2004.03.008. – ISSN 0001–8708 146, 152, 157

[Dem73] Demazure, Michel: Invariants symétriques entiers des groupes de Weyl et torsion. In: *Invent. Math.* 21 (1973), S. 287–301. – ISSN 0020–9910 83, 86, 91, 100

[DG70] Demazure ; Gabriel: *Groupes Algebriques*. Bd. 1. Masson and Cie, North-Holland, 1970 247

[Die91] Dieck, Tammo tom: *Topologie*. Berlin : Walter de Gruyter & Co., 1991 (de Gruyter Lehrbuch. [de Gruyter Textbook]). – x+401 S. – ISBN 3–11–013187–0; 3–11–012463–7 247

[DM11] Davydov, Alexei ; Molev, Alexander: A categorical approach to classical and quantum Schur-Weyl duality. Version: 2011. http://dx.doi.org/10.1090/conm/537/10572. In: *Groups, algebras and applications* Bd. 537. Providence, RI : Amer. Math. Soc., 2011. – DOI 10.1090/conm/537/10572, 143–171 111

[Dou98] Douglass, J. M.: Irreducible components of fixed point subvarieties of flag varieties. In: *Math. Nachr.* 189 (1998), 107–120. http://dx.doi.org/10.1002/mana.19981890107. – DOI 10.1002/mana.19981890107. – ISSN 0025–584X 181

[DR92] Dlab, Vlastimil ; Ringel, Claus M.: The module theoretical approach to quasi-hereditary algebras. In: *Representations of algebras and related topics (Kyoto, 1990)* Bd. 168. Cambridge : Cambridge Univ. Press, 1992, S. 200–224 187

[DR09] Douglass, J. M. ; Röhrle, Gerhard: The Steinberg variety and representations of reductive groups. In: *J. Algebra* 321 (2009), Nr. 11, 3158–3196. http://dx.doi.org/10.1016/j.jalgebra.2008.10.027. – DOI 10.1016/j.jalgebra.2008.10.027. – ISSN 0021–8693 105

[Drä94] Dräxler: Completely Separating Algebras. In: *Journal of Algebra* 165 (1994), S. 550–565 183, 189

[DW02] Derksen, Harm ; Weyman, Jerzy: Generalized quivers associated to reductive groups. In: *Colloq. Math.* 94 (2002), Nr. 2, 151–173. http://dx.doi.org/10.4064/cm94-2-1. – DOI 10.4064/cm94–2–1. – ISSN 0010–1354 52, 124

[EG98a] Edidin, Dan ; Graham, William: Equivariant intersection theory. In: *Invent. Math.* 131 (1998), Nr. 3, 595–634. http://dx.doi.org/10.1007/s002220050214. – DOI 10.1007/s002220050214. – ISSN 0020–9910 30, 32

[EG98b] Edidin, Dan ; Graham, William: Localization in equivariant intersection theory and the Bott residue formula. In: *Amer. J. Math.* 120 (1998), Nr. 3, 619–636. http://muse.jhu.edu/journals/american_journal_of_mathematics/v120/120.3edidin.pdf. – ISSN 0002–9327 78

[Fie11] Fiebig, Williamson P.: Parity sheaves, moment graphs and the p-smooth locus of Schubert varieties. In: *arxiv:* 10080719 (2011) 252

[Fre09] Fresse, Lucas: Betti numbers of Springer fibers in type A. In: *J. Algebra* 322 (2009), Nr. 7, 2566–2579. http://dx.doi.org/10.1016/j.jalgebra.2009.07.008. – DOI 10.1016/j.jalgebra.2009.07.008. – ISSN 0021–8693 216, 235

[Ful98] Fulton, William: *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Bd. 2: *Intersection theory*. Second. Berlin : Springer-Verlag, 1998. – xiv+470 S. – ISBN 3–540–62046–X; 0–387–98549–2 234

[Fun03] Fung, Francis Y. C.: On the topology of components of some Springer fibres and their relation to Kazhdan-Lusztig thoery. In: *Advances in Mathematics* 178 (2003), S. 244–276 216, 239, 241, 245

[Gab75] Gabriel: Finite representation type is open. In: *Representations of Algebras, Springer Lecture Notes in Mathematics* 488 (1975), S. 132–155 29

[GKM98] Goresky, Mark ; Kottwitz, Robert ; MacPherson, Robert: Equivariant cohomology, Koszul duality, and the localization theorem. In: *Invent. Math.* 131 (1998), Nr. 1, 25–83. http://dx.doi.org/10.1007/s002220050197. – DOI 10.1007/s002220050197. – ISSN 0020–9910 16, 248

[GM03] Gelfand, Sergei I. ; Manin, Yuri I.: *Methods of homological algebra*. Second. Berlin : Springer-Verlag, 2003 (Springer Monographs in Mathematics). – xx+372 S. – ISBN 3–540–43583–2 28

[Gon] Gonzales, Richard: *Research statement*. http://www.math.uwo.ca/~rgonzal3/research_statement_gonzales.pdf 16

[Hap87] Happel, Dieter: On the derived category of a finite-dimensional algebra. In: *Comment. Math. Helv.* 62 (1987), Nr. 3, 339–389. http://dx.doi.org/10.1007/BF02564452. – DOI 10.1007/BF02564452. – ISSN 0010–2571 182

[Hil82] Hiller, Howard: *Research Notes in Mathematics*. Bd. 54: *Geometry of Coxeter groups*. Boston, Mass. : Pitman (Advanced Publishing Program), 1982. – iv+213 S. – ISBN 0–273–08517–4 100

[HJ05] Hinich, Vladimir ; Joseph, Anthony: Orbital variety closures and the convolution product in Borel-Moore homology. In: *Selecta Math. (N.S.)* 11

(2005), Nr. 1, 9–36. http://dx.doi.org/10.1007/s00029-005-0002-0. – DOI 10.1007/s00029–005–0002–0. – ISSN 1022–1824 73

[HMM12] Hill, David ; Melvin, George ; Mondragon, Damien: Representations of quiver Hecke algebras via Lyndon bases. In: *J. Pure Appl. Algebra* 216 (2012), Nr. 5, 1052–1079. http://dx.doi.org/10.1016/j.jpaa.2011.12.015. – DOI 10.1016/j.jpaa.2011.12.015. – ISSN 0022–4049 42

[Hot81] Hotta, Ryoshi: On Springer's representations. In: *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 28 (1981), Nr. 3, S. 863–876 (1982). – ISSN 0040–8980 36, 50

[HTT08] Hotta, Ryoshi ; Takeuchi, Kiyoshi ; Tanisaki, Toshiyuki: *Progress in Mathematics*. Bd. 236: *D-modules, perverse sheaves, and representation theory*. Boston, MA : Birkhäuser Boston Inc., 2008. – xii+407 S. http://dx.doi.org/10.1007/978-0-8176-4523-6. http://dx.doi.org/10.1007/978-0-8176-4523-6. – ISBN 978–0–8176–4363–8. – Translated from the 1995 Japanese edition by Takeuchi 169

[Hub13] Hubery, Andrew: *Irreducible components of quiver Grassmannians*. Jun 2013. – arXiv:1303.4585v2 178, 201, 203, 205, 206, 236

[Hum75] Humphreys, James E.: *Linear algebraic groups*. New York : Springer-Verlag, 1975. – xiv+247 S. – Graduate Texts in Mathematics, No. 21 52

[HV83] Happel, Dieter ; Vossieck, Dieter: Minimal algebras of infinite representation type with preprojective component. In: *Manuscripta Math.* 42 (1983), Nr. 2-3, 221–243. http://dx.doi.org/10.1007/BF01169585. – DOI 10.1007/BF01169585. – ISSN 0025–2611 183, 189, 190

[Här99] Härterich, Martin: *Kazhdan-Lusztig-Basen, unzerlegbare Bimoduln und die Topologie der Fahnenmannigfaltigkeit einer Kac-Moody-Gruppe*, Albert-Ludwigs-Universität Freiburg im Breisgau, Diss., 1999 64

[Jos98] Joshua, Roy: Modules over convolution algebras from equivariant derived categories. I. In: *J. Algebra* 203 (1998), Nr. 2, 385–446. http://dx.doi.org/10.1006/jabr.1997.7346. – DOI 10.1006/jabr.1997.7346. – ISSN 0021–8693 13

[Kas91] Kashiwara, M.: On crystal bases of the Q-analogue of universal enveloping algebras. In: *Duke Math. J.* 63 (1991), Nr. 2, 465–516. http://dx.doi.org/10.1215/S0012-7094-91-06321-0. – DOI 10.1215/S0012–7094–91–06321–0. – ISSN 0012–7094 45

[Kat09] Kato, Syu: An exotic Deligne-Langlands correspondence for symplectic groups. In: *Duke Math. J.* 148 (2009), Nr. 2, 305–371. http://dx.doi.org/10.1215/00127094-2009-028. – DOI 10.1215/00127094–2009–028. – ISSN 0012–7094 48, 49

[Kat11] KATO, Syu: Deformations of nilpotent cones and Springer correspondences. In: *Amer. J. Math.* 133 (2011), Nr. 2, 519–553. http://dx.doi.org/10.1353/ajm.2011.0014. – DOI 10.1353/ajm.2011.0014. – ISSN 0002–9327 48

[Kat13] KATO, Syu: An algebraic study of extension algebras. In: *arxiv[math.RT]* 12074640 (2013) 22, 50, 266

[Kem76] KEMPF: On the Collapsing of Homogeneous Bundles. In: *Inventiones math.* 37 (1976), S. 229–239 50

[Kho99] KHOVANOV, Mikhail: *NilCoxeter algebras categorify the Weyl algebra.* Jun 1999. – arXiv:math/9906166v1 116

[KK86] KOSTANT, Bertram ; KUMAR, Shrawan: The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. In: *Adv. in Math.* 62 (1986), Nr. 3, 187–237. http://dx.doi.org/10.1016/0001-8708(86)90101-5. – DOI 10.1016/0001–8708(86)90101–5. – ISSN 0001–8708 100

[KL80] KAZHDAN, David ; LUSZTIG, George: A topological approach to Springer's representations. In: *Adv. in Math.* 38 (1980), Nr. 2, 222–228. http://dx.doi.org/10.1016/0001-8708(80)90005-5. – DOI 10.1016/0001–8708(80)90005–5. – ISSN 0001–8708 36, 50

[KL87] KAZHDAN, David ; LUSZTIG, George: Proof of the Deligne-Langlands conjecture for Hecke algebras. In: *Invent. Math.* 87 (1987), Nr. 1, 153–215. http://dx.doi.org/10.1007/BF01389157. – DOI 10.1007/BF01389157. – ISSN 0020–9910 37

[KL09] KHOVANOV, Mikhail ; LAUDA, Aaron D.: A diagrammatic approach to categorification of quantum groups. I. In: *Represent. Theory* 13 (2009), 309–347. http://dx.doi.org/10.1090/S1088-4165-09-00346-X. – DOI 10.1090/S1088–4165–09–00346–X. – ISSN 1088–4165 20, 39, 47, 114, 118, 138

[Kwo09] KWON, Namhee: Borel-Moore homology and K-theory on the Steinberg variety. In: *Michigan Math. J.* 58 (2009), Nr. 3, 771–781. http://dx.doi.org/10.1307/mmj/1260475700. – DOI 10.1307/mmj/1260475700. – ISSN 0026–2285 16

[LS79] LESZCZYNSKI ; SIMSON: On triangular matrix rings of finite representation type. In: *J. London Math. Soc. (2)* 20 (1979), S. 396–402 183

[Lus81] LUSZTIG, G.: Green polynomials and singularities of unipotent classes. In: *Adv. in Math.* 42 (1981), Nr. 2, 169–178. http://dx.doi.org/10.1016/0001-8708(81)90038-4. – DOI 10.1016/0001–8708(81)90038–4. – ISSN 0001–8708 36, 50

[Lus90] Lusztig, G.: Canonical bases arising from quantized enveloping algebras. II. In: *Progr. Theoret. Phys. Suppl.* (1990), Nr. 102, 175–201 (1991). http://dx.doi.org/10.1143/PTPS.102.175. – DOI 10.1143/PTPS.102.175. – ISSN 0375–9687. – Common trends in mathematics and quantum field theories (Kyoto, 1990) 45

[Lus91] Lusztig, G.: Quivers, perverse sheaves, and quantized enveloping algebras. In: *J. Amer. Math. Soc.* 4 (1991), Nr. 2, 365–421. http://dx.doi.org/10.2307/2939279. – DOI 10.2307/2939279. – ISSN 0894–0347 33, 39, 43, 44, 49, 50, 114, 117, 119, 120, 222

[LZ00] Li, Libin ; Zhang, Pu: Twisted Hopf algebras, Ringel-Hall algebras, and Green's categories. In: *J. Algebra* 231 (2000), Nr. 2, 713–743. http://dx.doi.org/10.1006/jabr.2000.8362. – DOI 10.1006/jabr.2000.8362. – ISSN 0021–8693. – With an appendix by the referee 42

[MP06] Melnikov ; Pagnon: On intersections of orbital varieties and components of Springer fiber. In: *Journal of Algebra* 298 (2006), S. 1–14 29, 31

[NO82] Năstăsescu, C. ; Oystaeyen, F. van: *North-Holland Mathematical Library*. Bd. 28: *Graded ring theory*. Amsterdam : North-Holland Publishing Co., 1982. – ix+340 S. – ISBN 0–444–86489–X 19

[NS97] Nörenberg ; Skowronski: Tame minimal non-polynomial growth simply connected algebras. In: *Colloquium Mathematicum* 73 (1997), Nr. 2, S. 301–330 183, 184

[Pop09] Popov, V. L.: Two orbits: when is one in the closure of the other? In: *Tr. Mat. Inst. Steklova* 264 (2009), Nr. Mnogomernaya Algebraicheskaya Geometriya, 152–164. http://dx.doi.org/10.1134/S0081543809010179. – DOI 10.1134/S0081543809010179. – ISSN 0371–9685 147

[PS99] Pena de l. ; Skowronski: The Tits and Euler form of a tame algebra. In: *Mathematische Annalen* 315 (1999), S. 37–59 184

[Rei02] Reineke, Markus: The monoid of families of quiver representations. In: *Proc. London Math. Soc.* 84 (2002), Nr. 3, S. 663–685 143, 150, 153

[Rei03] Reineke, Markus: Quivers, desingularizations and canonical bases. In: *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)* Bd. 210. Boston, MA : Birkhäuser Boston, 2003, S. 325–344 6, 29, 41, 49, 50, 104, 117, 146, 154, 155, 195, 216, 217, 219, 221, 222, 226

[Rin84] Ringel: *Tame algebras and integral quadratic forms*. Bd. 1099. Springer-Verlag. Lecture Notes in Mathematics., 1984 157

[Rin91] Ringel, C.M.: The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. In: *Mathematische Zeitschrift* 208 (1991), S. 209–223 185

[Rin93] Ringel, Claus M.: *The Hall algebra approach to quantum groups.* 1993. – http://www.math.uni-bielefeld.de/ ringel/opus/elam.pdf 47

[Ros91] Rossmann, W.: Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety. II. Representations of Weyl groups. In: *J. Funct. Anal.* 96 (1991), Nr. 1, 155–193. http://dx.doi.org/10.1016/0022-1236(91)90077-I. – DOI 10.1016/0022–1236(91)90077–I. – ISSN 0022–1236 36, 50

[Rou11] Rouquier, Raphael: *Quiver Hecke algebras and 2-Lie algebras.* 2011. – http://people.maths.ox.ac.uk/rouquier/papers.html 39, 44, 45, 46, 49, 50, 118, 119, 120, 121, 135

[RZ12] Ringel, Claus M. ; Zhang, Pu: *From submodule categories to preprojective algebras.* 2012. – arXiv:1211.4858 [math.RT] 33, 49

[Sau13] Sauter, Julia: *Generalized Quiver Hecke Algebras.* June 2013. – arXiv:1306.3892v2 [math.RT] 100, 104, 105, 115, 117, 118, 119

[Sko97] Skowronski: Simply connected algebras of polynomial growth. In: *Compositio Mathematica* 109 (1997), S. 99–103 183

[Slo80a] Slodowy, Peter: *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht.* Bd. 11: *Four lectures on simple groups and singularities*. Utrecht : Rijksuniversiteit Utrecht Mathematical Institute, 1980. – ii+64 S. 36, 50

[Slo80b] Slodowy, Peter: *Simple Singularities and Simple Algebraic Groups.* Bd. 815. Lecture Notes in Math. ,Springer-Verlag, Berlin Heidelberg New York, 1980 10, 35, 70, 246

[Spa76] Spaltenstein, N.: The fixed point set of a unipotent transformation on the flag manifold. In: *Indag. Math.* 38 (1976), Nr. 5, S. 542–456 50, 219, 220, 228, 232

[Spa77] Spaltenstein, N.: On the fixed point set of a unipotent element on the variety of Borel subgroups. In: *Topology* 16 (1977), S. 203–204 29, 50

[Spr76] Springer, T. A.: Trigonometric sums, Green functions of finite groups and representations of Weyl groups. In: *Invent. Math.* 36 (1976), S. 173–207. – ISSN 0020–9910 35, 36, 50

[Spr78] Springer, T. A.: A construction of representations of Weyl groups. In: *Invent. Math.* 44 (1978), Nr. 3, S. 279–293. – ISSN 0020–9910 36, 50

[Ste74] STEINBERG, Robert: *Conjugacy classes in algebraic groups*. Berlin : Springer-Verlag, 1974 (Lecture Notes in Mathematics, Vol. 366). – vi+159 S. – Notes by Vinay V. Deodhar 50

[Str08] STRICKLAND, N.: *Spectral Sequences*. Website, 2008. – Available online at http://neil-strickland.staff.shef.ac.uk/courses/bestiary/ss.pdf; visited on September 27th 2013. 259

[SW11] STROPPEL ; WEBSTER: *Quiver Schur algebras and q-Fock space*. Oct 2011. – arXiv:1110.1115v1 [math.RA] 26, 48, 49

[Tu03] TU, Loring W.: *Characteristic numbers of a Homogeneous Space*. Dec 2003. – arXiv:math/0102013v3 [math.AG] 62

[Tym09] TYMOCZKO, Julia: *Divided difference operators for partial flag varieties*. Dec 2009. – arXiv:0912.2545v1 [math.AT] 100, 101

[Ung90] UNGER, Luise: The concealed algebras of the minimal wild type, hereditary algebras. In: *Bayreuth. Math. Schr.* 31 (1990), S. 145–154 183, 184

[Var79] VARGAS, J. A.: Fixed points under the action of unipotent elements of SL_n in the flag variety. In: *Bol. Soc. Mat. Mexicana (2)* 24 (1979), Nr. 1, S. 1–14 29, 50, 239

[Var09] VARAGNOLO, Vasserot: *Canonical basis and Khovanov-Lauda algebras*. 2009. – arXiv:0901.3992v2 12, 39, 47, 49, 50, 52, 71, 118

[VV11] VARAGNOLO, M. ; VASSEROT, E.: Canonical bases and affine Hecke algebras of type B. In: *Invent. Math.* 185 (2011), Nr. 3, 593–693. http://dx.doi.org/10.1007/s00222-011-0314-y. – DOI 10.1007/s00222–011–0314–y. – ISSN 0020–9910 15, 52, 72, 83, 114, 121, 143

[Web13] WEBSTER, Ben: *A note on isomorphisms between Hecke algebras*. 2013. – arXiv:1305.0599v1 51

[Wol09] WOLF, Stefan: *The Hall algebra and the Composition Monoid*, Universität Paderborn, Diss., 2009 39, 49, 50, 116, 143, 148, 152, 153, 154, 177, 180, 182, 191, 195, 196, 197, 219

[You12] YOUNG, Matthew: *The Hall module of an exact category with duality*. Dec 2012. – arXiv:1212.0531v1 142

[Zwa02a] ZWARA, G.: Smooth morphisms of module schemes. In: *Proc. London Math. Soc. (3)* 84 (2002), Nr. 3, 539–558. http://dx.doi.org/10.1112/S0024611502013382. – DOI 10.1112/S0024611502013382. – ISSN 0024–6115 197, 198

[Zwa02b] ZWARA, Grzegorz: Unibranch orbit closures in module varieties. In: *Ann. Sci. École Norm. Sup. (4)* 35 (2002), Nr. 6, 877–895. http://dx.doi.org/10.1016/S0012-9593(02)01110-2. – DOI 10.1016/S0012–9593(02)01110–2. – ISSN 0012–9593 175

[Zwa03] ZWARA, Grzegorz: An orbit closure for a representation of the Kronecker quiver with bad singularities. In: *Colloq. Math.* 97 (2003), Nr. 1, 81–86. http://dx.doi.org/10.4064/cm97-1-8. – DOI 10.4064/cm97–1–8. – ISSN 0010–1354 175

[Zwa05a] ZWARA, Grzegorz: Orbit closures for representations of Dynkin quivers are regular in codimension two. In: *J. Math. Soc. Japan* 57 (2005), Nr. 3, 859–880. http://projecteuclid.org/getRecord?id=euclid.jmsj/1158241938. – ISSN 0025–5645 175

[Zwa05b] ZWARA, Grzegorz: Regularity in codimension one of orbit closures in module varieties. In: *J. Algebra* 283 (2005), Nr. 2, 821–848. http://dx.doi.org/10.1016/j.jalgebra.2004.09.021. – DOI 10.1016/j.jalgebra.2004.09.021. – ISSN 0021–8693 175