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SEMIFREE ACTIONS OF FINITE GROUPS ON HOMOTOPY SPHERES

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ABSTRACT. We show that for any finite group the group of semifree actions on homotopy spheres of some fixed even dimension is finite, provided that the dimension of the fixed point set is greater than 2. The argument shows that for such an action the normal bundle to the fixed point set is equivariantly, stably trivial.

0. Introduction. A group G is said to act semifreely on a space X if every point is either fixed by every element of G or fixed only by the identity. The classification of smooth semifree actions of finite groups on homotopy spheres has been discussed by Browder and Petrie [3] and Rothenberg [6]. We briefly summarize the basic scheme.

Given a finite group G we fix a representation $\rho\colon G \to O(d)$ such that ρ restricted to the unit sphere S^{d-1} is fixed point free. A (G, ρ)-manifold M is a smooth manifold together with a smooth, semifree action of G on M such that the fixed point set F is nonempty and locally the representation of G on the normal bundle of F in M is equivalent to ρ. In a natural way this defines a reduction of the structure group of the normal bundle to $Z(\rho)$, the centralizer of $\rho(G)$ in $O(d)$. A (G, ρ)-orientation is a specific reduction of the structure group of the normal bundle to $Z(\rho)$. We then define $C^N(\rho)$ to be the set of (G, ρ)-oriented h-cobordism classes of (G, ρ)-oriented manifolds which are homotopy N-spheres with fixed point set a homotopy $(N-d)$-sphere. The set $C^N(\rho)$ has the structure of an abelian group under the connected sum operation.

The object of the present paper is to prove the following qualitative result about the groups $C^N(\rho)$.

THEOREM A. *Let $\rho\colon G \to O(2d)$ be a fixed point free representation of a finite group G and suppose that $2N - 2d > 2$. Then $C^{2N}(\rho)$ is a finite group.*

We point out that the condition that ρ has even dimension is a restriction only when $|G| = 2$. This case is already well understood [3], [6].

The essential ingredient for proving this theorem is well known. According

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to [7] there is a long exact sequence

$$\to HS(D^{2n+1} \times L, S^{2n} \times L) \to C^{2N}(\rho) \xrightarrow{\varphi} \Gamma_{2n} \oplus \pi_{2n}(A(\rho), Z(\rho)) \to$$

where $2n = 2N - 2d$, HS denotes the group of homotopy smoothings, L is the orbit space of S^{2d-1} under ρ, Γ_{2n} is the group of homotopy spheres and $A(\rho)$ is the monoid of G-equivariant self-maps of S^{2d-1}. From the surgery exact sequence and the finiteness of the odd dimensional Wall groups it follows that $HS(D^{2n+1} \times L, S^{2n} \times L)$ is a finite group. It is well known that Γ_{2n} and $\pi_{2n}(A(\rho))$ are finite groups also. Therefore we must consider the composite:

$$\begin{array}{ccc} C^{2N}(\rho) \xrightarrow{\varphi} \Gamma_{2n} \oplus \pi_{2n}(A(\rho), Z(\rho)) & & \\ \downarrow \text{proj} & & \\ \pi_{2n}(A(\rho), Z(\rho)) & \xrightarrow{\partial} & \pi_{2n-1}(Z(\rho)) \end{array}$$

which assigns to an element of $C^{2N}(\rho)$ the element of $\pi_{2n-1}(Z(\rho))$ which classifies the equivariant normal bundle to the fixed point set. In order to prove Theorem A we must prove the following.

THEOREM B. *Let* $\rho: G \to O(2d)$ *be a fixed point free representation of a finite group* G *and suppose* $2N - 2d > 2$. *Then the image of* $\partial \circ \text{proj} \circ \varphi: C^{2N}(\rho) \to \pi_{2N-2d-1}(Z(\rho))$ *contains no elements of infinite order.*

A number of special cases of this result have previously been shown. In [**8**] Schultz obtained the result for G cyclic of prime order p and either p small (3, 5 or 7) or $N - d$ large ($N - d > 2 \log_2 p - 1$). (There is a sign error in [**8**] which does *not* affect the conclusions except for $p = 7$; see [**9**].) Schultz' work easily extends to cyclic groups of nonprime power order, but again for $N - d$ sufficiently large. In [**12**] Wang has extended the calculations using a different approach; but again there is a sign error which invalidates some of the results, especially Corollary 3.8 and Theorem 4.7 for even order groups. The work of [**4**] is essentially a proof of Theorem B in the case where G is cyclic of prime order. Our proof here is similar, but both the algebraic and topological arguments require more careful analysis.

The proof of Theorem B will proceed in several stages. In §1 we make some easy observations about the centralizers of representations which show that it is sufficient to prove the theorem for G cyclic. In §2 we prove the result for cyclic G using the Atiyah-Singer G-signature Theorem together with an algebraic lemma. The proof of the algebraic lemma is then given in §3.

It should be remarked that the techniques of the present paper yield considerable information about the groups $C^N(\rho)$ when N is *odd*. In particular, these groups, even when they are finite, tend to be of rather large order, and the orders are related to both the homotopy groups of spheres (which is expected) and to the order of certain ideal class groups (which is unexpected).

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1. Reduction to the cyclic case. In this section we will show that it is sufficient to prove Theorem B in the case where G is cyclic. From our previous remarks it is evident we must consider the commutative square

$$\begin{array}{ccc} C^{2N}(\rho) & \longrightarrow & \pi_{2n-1}(Z(\rho)) \\ \big\downarrow \text{res} & & \big\downarrow \text{res} \\ \bigoplus\limits_{\substack{H\subseteq G \\ H \text{ cyclic}}} C^{2N}(\rho|_H) & \longrightarrow & \bigoplus\limits_{\substack{H\subseteq G \\ H \text{ cyclic}}} \pi_{2n-1}(Z(\rho|_H)) \end{array}$$

PROPOSITION 1.1. *Let ρ: $G \to O(2d)$ be a fixed point free representation of a finite group G. For any integer $n \geq 1$ the restriction map*

$$\text{res: } \pi_{2n-1}(Z(\rho)) \otimes \mathbf{Q} \to \bigoplus_{\substack{H\subseteq G \\ H \text{ cyclic}}} \pi_{2n-1}(Z(\rho|_H)) \oplus \mathbf{Q}$$

is a monomorphism.

We prepare for the proof of the proposition by making some easy observations about the centralizers of representations.

Given an irreducible representation ρ: $G \to O(d)$ let V denote the representation space. By Schur's Lemma we know that $\text{Hom}_G(V, V) = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. We call such a representation type I, II, or III accordingly. Now it is easy to see that the transpose operation defines an involution on $\text{Hom}_G(V, V)$ which is either trivial, complex conjugation or quaternionic conjugation in each case. It follows that $Z(\rho) \subset O(d)$ is isomorphic to $O(1)$, $U(1)$ or $\text{Sp}(1)$ respectively.

Similarly we may consider any real representation ρ: $G \to O(d)$ and write

$$\rho = \sum_i a_i\rho_i + \sum_j b_j\sigma_j + \sum_k c_k\tau_k,$$

where the ρ_i, σ_j and τ_k are irreducible representations of type I, II or III respectively. Then the centralizer $Z(\rho) \subset O(d)$ is isomorphic to the product

$$\prod_i O(a_i) \times \prod_j U(b_j) \times \prod_k \text{Sp}(c_k).$$

The situation for complex representations is, as usual, easier. If ρ: $G \to U(d)$ is a complex representation we can write $\rho = \sum_i a_i\rho_i$ where the ρ_i are irreducible. Exactly as before we see that $Z(\rho) \subseteq U(d)$ is isomorphic to $\prod_i U(a_i)$. There is then an obvious homomorphism

$$\eta: \pi_{2n-1}(Z(\rho)) \to R[G]$$

given by $\eta([f]) = \Sigma_i f^*(\sigma^* c_n^{(i)})\rho_i$, where $\sigma^* c_n^{(i)}$ is the cohomology suspension of the nth Chern class of $BU(a_i)$. It is clear that $\eta \otimes \mathbf{Q}$ is a monomorphism.

Finally we consider a real representation $\rho: G \to O(d)$ without any irreducible constituents of type I; that is, using our previous notation,

$$\rho = \sum_j b_j \sigma_j + \sum_k c_k \tau_k.$$

Let $\rho_{\mathbf{C}}$ denote the complexification of ρ. By the usual arguments each $(\sigma_j)_{\mathbf{C}}$ is the sum of an irreducible and its (distinct) conjugate while each $(\tau_k)_{\mathbf{C}}$ is twice a self conjugate irreducible. It follows that

$$Z(\rho) \approx \prod_j U(b_j) \times \prod_k \mathrm{Sp}(c_k) \quad \text{and}$$

$$Z(\rho_{\mathbf{C}}) \approx \prod_j \left[U(b_j) \times U(b_j) \right] \times \prod_k U(2c_k).$$

Letting $i: Z(\rho) \to Z(\rho_{\mathbf{C}})$ denote the inclusion, it is now easy to see that $i_\# \otimes \mathbf{Q}$ is a monomorphism on homotopy groups.

Proof of Proposition 1.1. Clearly the proposition is true for G cyclic and hence we may assume that $|G| > 2$. From the classification of fixed point free representations of finite groups given by Wolf [13], we see that for $|G| > 2$ there are only irreducible fixed point free representations of type II or III. Consider the commutative diagram:

$$\begin{array}{ccccc}
\pi_{2n-1}(Z(\rho)) & \xrightarrow{\ i_\#\ } & \pi_{2n-1}(Z(\rho_{\mathbf{C}})) & \xrightarrow{\ \eta\ } & R[G] \\
\Big\downarrow \text{res} & & \Big\downarrow \text{res} & & \Big\downarrow \text{res} \\
\bigoplus\limits_{\substack{H \subseteq G \\ H \text{ cyclic}}} \pi_{2n-1}(Z(\rho|_H)) & \longrightarrow & \bigoplus\limits_{\substack{H \subseteq G \\ H \text{ cyclic}}} \pi_{2n-1}(Z(\rho_{\mathbf{C}}|_H)) & \longrightarrow & \bigoplus\limits_{\substack{H \subseteq G \\ H \text{ cyclic}}} R[H]
\end{array}$$

We have observed that $i_\# \otimes \mathbf{Q}$ and $\eta \otimes \mathbf{Q}$ are monomorphisms. It is well known that the restriction map for the representation rings is a monomorphism. The proposition is now immediate. □

2. The proof when G is cyclic. We now give the proof of Theorem B for G a cyclic group. Since the result is well known for $G = \mathbf{Z}_2$, we can assume that G is cyclic of order $q > 2$. First, we establish some notation and recall some facts.

Let T denote a generator of G and let $\lambda = e^{2\pi i/q}$. The irreducible unitary representations of G are given by ρ_k, $0 \leqslant k < q$, where $\rho_k(T) = \lambda^k$. Clearly

ρ_k is fixed point free if and only if $(k, q) = 1$. We note that ρ_k is *real* equivalent to $\bar{\rho}_k = \rho_{q-k}$.

Given a fixed point free representation $\rho\colon G \to O(2d)$, it is a standard fact that ρ is the realization of a unitary representation. Hence ρ is a real equivalent to $\Sigma_{k \in A} d_k \rho_k$, where the d_k are nonnegative integers whose sum is d and $A = \{k \in \mathbf{Z} |\ 1 \leqslant k < q/2, (k, q) = 1\}$. As in the previous section it follows that $Z(\rho) \subset O(2d)$ is isomorphic to $\Pi_{k \in A} U(d_k)$.

Now given an element of $C^{2N}(\rho)$ we choose a representative homotopy sphere Σ^{2N} with (G, ρ)-action, and let Σ^{2n} denote the fixed point set, where $2n - 2N - 2d$. Let ν denote the normal bundle of the fixed point set. The reduction of the structure group of ν to $Z(\rho)$ gives ν the structure of a *complex* G-vector bundle, and as in **[11]** we may write

$$\nu = \sum_{k \in A} \xi_k \otimes \rho_k,$$

where ξ_k is a complex vector bundle of dimension d_k. In order to prove Theorem B we need only show that the rational Chern classes $c_n(\xi_k) \in H^{2n}(\Sigma^{2n}, \mathbf{Q})$ are zero for $k \in A$.

To accomplish this we compute the G-signature (see **[1**, p. 578]) in two ways. First, since the middle dimensional cohomology of Σ^{2N} is trivial, it is clear that $\operatorname{sign}(T, \Sigma^{2N}) = 0$.

On the other hand we can compute $\operatorname{sign}(T, \Sigma^{2N})$ from the G-signature Theorem **[1**, p. 582]. We see that

$$\operatorname{sign}(T, \Sigma^{2N}) = \pm C \left\langle \prod_{k \in A} \mathfrak{M}(\lambda^k, \xi_k), [\Sigma^{2n}] \right\rangle,$$

where

$$C = 2^n \prod_{k \in A} \left(\frac{\lambda^k + 1}{\lambda^k - 1} \right)$$

and where $\mathfrak{M}(\lambda^k, \xi_k)$ is the characteristic class defined by the power series associated to

$$\left(\frac{\lambda^k - 1}{\lambda^k + 1} \right) \left(\frac{\lambda^k e^z + 1}{\lambda^k e^z - 1} \right).$$

Since the only possible nonzero, rational Chern classes are $c_0(\xi_k) = 1$ and $c_n(\xi_k)$, we can write:

$$\mathfrak{M}(\lambda^k, \xi_k) = 1 + \Phi_n(\lambda^k) c_n(\xi_k),$$

where $\Phi_n(\lambda^k)$ is some number in $\mathbf{Q}(\lambda)$.

We can "determine" the numbers $\Phi_n(\lambda^k)$ by using the defining power series for $\mathfrak{M}(\lambda^k, \ )$ and a standard trick. This gives a generating function for $\Phi_n(\lambda^k)$ as follows.

$$\sum_{n=0}^{\infty}(-1)^n\Phi_n(\lambda^k)z^n = 1 - z\,\frac{d}{dz}\log\left[\left(\frac{\lambda^k-1}{\lambda^k+1}\right)\left(\frac{\lambda^k e^z+1}{\lambda^k e^z-1}\right)\right]$$
$$= 1 + \frac{z}{2}\left[\frac{\lambda^k e^z+1}{\lambda^k e^z-1} - \frac{\lambda^k e^z-1}{\lambda^k e^z+1}\right].$$

Elementary manipulation of the generating function now yields the following two lemmas.

LEMMA 2.1. $\Phi_n(\lambda^{-k}) = (-1)^n\Phi_n(\lambda^k)$. □

LEMMA 2.2. *If q is even and $n > 0$ then* $\Phi_n(\lambda^{q/2-k}) = (-1)^{n+1}\Phi_n(\lambda^k)$.

To summarize, we have shown that

$$0 = \operatorname{sign}(T, \Sigma^{2N}) = \pm C\left\{\sum_{k\in A}\Phi_n(\lambda^k)\langle c_n(\xi_k), [\Sigma^{2n}]\rangle\right\}, \tag{2.3}$$

where, of course, $\langle c_n(\xi_k), [\Sigma^{2n}]\rangle \in \mathbf{Q}$ and $C \neq 0$. It is now clear that the following is the crucial result concerning the $\Phi_n(\lambda^k)$.

LEMMA 2.4. (i) *If $q \neq 0 \bmod 4$ then the numbers $\{\Phi_n(\lambda^k) | k \in A\}$ are linearly independent over $\mathbf{Q}$ for $n > 1$.*

(ii) *If $q \equiv 0 \bmod 4$ then the numbers $\{\Phi_n(\lambda^k | k \in \tilde{A})\}$ are linearly independent over $\mathbf{Q}$ for $n > 1$, where $\tilde{A} = \{k \in \mathbf{Z} | 1 \leqslant k \leqslant q/4 \text{ and } (k, q) = 1\}$.*

We shall defer the proof of this purely algebraic lemma until the next section.

The proof of Theorem B is now almost immediate. If $q \not\equiv 0 \bmod 4$ then from (2.3) and Lemma 2.4 we conclude that $c_n(\xi_k) = 0$ for $k \in A$.

If $q \equiv 0 \bmod 4$ we must work a little harder. We prove the result by induction on q. If $q = 4$ then A contains only one element and a rational pontrjagin class argument shows that $c_n(\xi_1) = 0$. Suppose that $q \equiv 0 \bmod 4$ and $q > 4$. Consider the subgroup H of G generated by T^2 of order $q/2$. Clearly Σ^{2N} is an $(H, \rho|_H)$-manifold and exactly as before the normal bundle of the fixed point set decomposes into: $\bigoplus_{k\in\tilde{A}}\eta_k$. By considering the restriction of the representation ρ_k to H, it is evident that $\eta_k = \xi_k \oplus \bar{\xi}_{q/2-k}$. From the induction hypothesis we conclude that

$$c_n(\eta_k) = c_n(\xi_k) + (-1)^n c_n(\xi_{q/2-k}) = 0.$$

Therefore, using Lemma 2.2, we see that

$$0 = \operatorname{sign}(T, \Sigma^{2N}) = \pm C\sum_{k\in\tilde{A}} 2\Phi_n(\lambda^k)\langle c_n(\xi_k), [\Sigma^{2n}]\rangle.$$

Finally, from Lemma 2.4 we conclude that $c_n(\xi_k) = 0$ for all $k \in A$.

The proof of Theorem B is now complete provided we demonstrate the algebraic Lemma 2.4.

3. The algebraic lemma. In order to complete the proof of the main theorem we must now prove Lemma 2.4. The proof will require some preliminary notions and lemmas. Throughout this section q will denote an integer greater than 2, $\lambda = e^{2\pi i/q}$, $\mathbf{Q}(\lambda)$ is the cyclotomic field and $\mathfrak{g}$ denotes the Galois group of $\mathbf{Q}(\lambda)/\mathbf{Q}$. The degree of $\mathbf{Q}(\lambda)$ over $\mathbf{Q}$ is $\phi(q)$, where ϕ is the Euler function. For convenience we let $m = \phi(q)/2$ and as before let $A = \{k \in \mathbf{Z} | 1 \leqslant k \leqslant q/2 \text{ and } (k, q) = 1\}$.

It is not hard to see that $\mathbf{Q}(\lambda)$ decomposes as a vector space over $\mathbf{Q}$ as $Q(\lambda) = V_0 \oplus V_1$ where V_0 is the subspace of real elements and V_1 is the subspace of purely imaginary elements. Of course, $\dim V_0 = \dim V_1 = m$ and V_0 and V_1 are each invariant under $\mathfrak{g}$. From Lemma 2.1 we see that $\Phi_n(\lambda^k) \in V_\varepsilon$, where $\varepsilon = 0$ or 1 as n is even or odd. To prove Lemma 2.4 we must investigate when a given set of elements of V_ε are linearly independent over $\mathbf{Q}$.

Let $\sigma_i \in \mathfrak{g}$ be the automorphism defined by $\sigma_i(\lambda) = \lambda^i$.

LEMMA 3.1. *Let $\{\omega_1, \ldots, \omega_m\}$ be elements of V_ε and consider the $m \times m$ matrix $B = [\sigma_i^{-1}\omega_j]$; $i \in A$, $1 \leqslant j \leqslant m$. The rank of B is equal to the dimension of the span of $\{\omega_1, \ldots, \omega_m\}$ in V_ε.*

PROOF. Let $\{\varphi_1, \ldots, \varphi_m\}$ be a basis for V_ε over $\mathbf{Q}$. It is well known [**2**, p. 405] that $\tilde{B} = [\sigma_i^{-1}\varphi_j]$; $i \in A$, $1 \leqslant j \leqslant m$, is nonsingular. Let C be the $m \times m$ matrix with entries in $\mathbf{Q}$ defined by

$$\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} = C \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix}.$$

It follows from elementary linear algebra that $\operatorname{rank} C = \dim \operatorname{span}\{\omega_1, \ldots, \omega_m\}$. Moreover, it is easy to verify that $B = \tilde{B}C$ and hence $\operatorname{rank} B = \operatorname{rank} C$. □

From the definition it is clear that $\Phi_n(\lambda^k) = \sigma_k\Phi_n(\lambda)$ for $k \in A$. We are therefore really concerned with a special case of the preceding lemma, and in this case we can compute the rank of B using characters mod q. (See [**2**, p. 415 ff.] for definitions and elementary properties of characters.)

LEMMA 3.2. *Let ω be an element of V_ε. Then the dimension of the span of $\{\sigma_k(\omega) | k \in A\}$ is equal to the number of nonzero sums $\{\sum_{k \in A} \chi(k)\sigma_k(\omega)\}$, where χ runs over all characters* mod q *such that $\chi(-1) = (-1)^\varepsilon$. (Such a character is called even or odd as $\varepsilon = 0$ or 1.)*

PROOF. As in the preceding lemma let

$$B = \left[\sigma_i^{-1}\sigma_j(\omega)\right]; \qquad i \in A, j \in A.$$

We can explicitly compute the eigenvalues and eigenvectors of B as follows. Given a character χ mod q such that $\chi(-1) = (-1)^{\varepsilon}$ we see that

$$\begin{aligned}\sum_{j\in A} \sigma_i^{-1}\sigma_j(\omega)\chi(j) &= \chi(i) \sum_{j\in A} \sigma_i^{-1}\sigma_j(\omega)\bar{\chi}(i)\chi(j) \\ &= \chi(i) \sum_{k\in A} \chi(k)\sigma_k(\omega).\end{aligned}$$

It follows that for each such character mod q the vector with components $\chi(i)$, $i \in A$, is an eigenvector of B with eigenvalue $\sum_{k\in A} \chi(k)\sigma_k(\omega)$. From the Dedekind independence of characters theorem these vectors are linearly independent. The result now follows immediately from Lemma 3.1. □

We have now reduced our problem to the problem of evaluating certain character sums. The computation is reasonably complex and it is not only more convenient but also more illuminating to break it down into several stages.

First, we can describe the numbers $\Phi_n(\lambda)$ in terms of certain Dirichlet series.

DEFINITION. For any integer $n \geqslant 1$ and real number $0 < t < 1$ let

$$\varphi(n, t) = \sum_{\nu=0}^{\infty} \left[\frac{1}{(\nu + 1 - t)^n} + \frac{(-1)^n}{(\nu + t)^n} \right].$$

It is easy to see that $\varphi(n, t)$ is convergent for $n \geqslant 1$ and absolutely convergent for $n > 1$.

PROPOSITION 3.3. *For any real number* $0 < t < 1$

$$\frac{e^{2\pi i t}e^z + 1}{e^{2\pi i t}e^z - 1} = \sum_{n=0}^{\infty} \frac{2}{(2\pi i)^{n+1}} \varphi(n + 1, t)z^n.$$

PROOF. The proof follows easily from the identity

$$\frac{e^{2\pi i x} + 1}{e^{2\pi i x} - 1} = ictn(\pi x),$$

and the well-known partial fraction decomposition

$$\pi ctn(\pi x) = \frac{1}{x} + \sum_{\nu=1}^{\infty} \left(\frac{1}{x + \nu} + \frac{1}{x - \nu} \right).$$

We leave the manipulation to the reader. □

COROLLARY 3.4. *For any real number* $0 < t < \frac{1}{2}$,

$$\frac{e^{2\pi i t}e^z - 1}{e^{2\pi i t}e^z + 1} = \sum_{n=0}^{\infty} \frac{2}{(2\pi i)^{n+1}} \varphi\left(n + 1, t + \tfrac{1}{2}\right)z^n.$$

PROOF. Replace t by $t + \frac{1}{2}$ in the previous proposition. □

Using Proposition 3.3 and Corollary 3.4 together with the generating function for $\Phi_n(\lambda^k)$, we conclude:

COROLLARY 3.5. *For* $k \in A$,

$$\Phi_n(\lambda^k) = \frac{(-1)^n}{(2\pi i)^{n+1}}\left[\varphi(n, k/q) - \varphi\left(n, k/q + \tfrac{1}{2}\right)\right]. \qquad \square$$

We can now quite easily evaluate the requisite character-sums. Our answer will involve the Dirichlet L-series defined by

$$L(s, \chi) = \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^s};$$

where χ is a character mod q.

PROPOSITION 3.6. *Let* χ *be a character* mod q *such that* $\chi(-1) = (-1)^n$. *Then*

(i) $\sum_{k \in A} \chi(k)\varphi(n, k/q) = (-1)^n q^n L(n, \chi)$,
(ii) $\sum_{k \in A} \chi(k)\varphi(n, k/q + \frac{1}{2}) = (-1)^n q^n (2^n\bar{\chi}(2) - 1)L(n, \chi)$, *$q$ odd,*
(iii) $\sum_{k \in A} \chi(k)\varphi(n, k/q + \frac{1}{2}) = (-1)^n q^n \bar{\chi}(1 + q/2)L(n, \chi)$, $q \equiv 0 \bmod 4$.

PROOF. The calculations are all similar. For the first we use the fact that $\chi(\nu) = 0$ if $(\nu, q) \neq 1$. From the definition of $\varphi(n, t)$ we see that

$$\begin{aligned}
\sum_{k \in A} \chi(k)\varphi\left(n, \frac{k}{q}\right) &= \sum_{k \in A} \chi(k) \sum_{\nu=0}^{\infty}\left[\frac{1}{(\nu + 1 - k/q)^n} + \frac{(-1)^n}{(\nu + k/q)^n}\right] \\
&= q^n \sum_{k \in A}\sum_{\nu=0}^{\infty}\left[\frac{\chi(k)}{(q\nu + q - k)^n} + \frac{(-1)^n\chi(k)}{(q\nu + k)^n}\right] \\
&= q^n \sum_{k \in A}\sum_{\nu=0}^{\infty}\left[\frac{(-1)^n\chi(q - k)}{(q\nu + q - k)^n} + \frac{(-1)^n\chi(k)}{(q\nu + k)^n}\right] \\
&= (-1)^n q^n \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^n} = (-1)^n q^n L(n, \chi).
\end{aligned}$$

The second sum is similar. Using the definition of $\varphi(n, t)$ we see that

$$\sum_{k\in A} \chi(k)\varphi\left(n, k/q + \tfrac{1}{2}\right)$$

$$= \sum_{k\in A} \chi(k) \sum_{\nu=0}^{\infty} \left[\frac{1}{\left(\nu + \frac{1}{2} - k/q\right)^n} + \frac{(-1)^n}{\left(\nu + \frac{1}{2} + k/q\right)^n} \right]$$

$$= 2^n q^n \sum_{k\in A} \sum_{\nu=0}^{\infty} \left[\frac{\chi(k)}{(2\nu q + q - 2k)^n} + \frac{(-1)^n \chi(k)}{((2\nu + 1)q + 2k)^n} \right]$$

$$= (-1)^n 2^n q^n \bar{\chi}(2) \sum_{k\in A} \sum_{\nu=0}^{\infty} \left[\frac{\chi(q-2k)}{(2\nu + q - 2k)^n} + \frac{\chi(2k)}{((2\nu+1)q + 2k)^n} \right]$$

$$= (-1)^n 2^n q^n \bar{\chi}(2) \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{\infty} \frac{\chi(\nu)}{\nu^n}$$

$$= (-1)^n 2^n q^n \bar{\chi}(2) \left\{ \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^n} - \sum_{\nu=1}^{\infty} \frac{\chi(2\nu)}{(2\nu)^n} \right\}$$

$$= (-1)^n q^n (2^n \bar{\chi}(2) - 1) L(n, \chi).$$

Finally, for the third case we use the fact that for $q \equiv 0 \bmod 4$ and k odd, $k(1 + q/2) \equiv k + q/2 \equiv k - q/2 \bmod q$. Then

$$\sum_{k\in A} \chi(k)\varphi\left(n, k/q + \tfrac{1}{2}\right)$$

$$= \sum_{k\in A} \chi(k) \sum_{\nu=0}^{\infty} \left[\frac{1}{\left(\nu + \frac{1}{2} - k/q\right)^n} + \frac{(-1)^n}{\left(\nu + \frac{1}{2} + k/q\right)^n} \right]$$

$$= q^n \sum_{k\in A} \sum_{\nu=0}^{\infty} \left[\frac{\chi(k)}{(q\nu + q/2 - k)^n} + \frac{(-1)^n \chi(k)}{(q\nu + q/2 + k)^n} \right]$$

$$= (-1)^n q^n \bar{\chi}(1 + q/2) \sum_{k\in A} \sum_{\nu=0}^{\infty} \left[\frac{\chi(q/2 - k)}{(q\nu + q/2 - k)^n} + \frac{\chi(q/2 + k)}{(q\nu + q/2 + k)^n} \right]$$

$$= (-1)^n q^n \bar{\chi}(1 + q/2) \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^n} = (-1)^n q^n \bar{\chi}(1 + q/2) L(n, \chi). \qquad \square$$

We are now in a position to prove Lemma 2.4. Combining Corollary 3.5 and Proposition 3.6, we see that for any character χ mod q such that $\chi(-1) = (-1)^n$,

$$\sum_{k\in A} \chi(k)\Phi_n(\lambda^k) = \begin{cases} \dfrac{2}{(2\pi i)^{n+1}} q^n\left(1 - 2^{n-1}\bar{\chi}(2)\right)L(n,\chi) & q \text{ odd}, \\ \dfrac{1}{(2\pi i)^{n+1}} q^n\left(1 - \bar{\chi}(1 + q/2)\right)L(n,\chi) & q \equiv 0 \bmod 4. \end{cases}$$

By Lemma 3.2 it is enough to determine when these sums are nonzero.

It is immediate from the Euler product formula that $L(n, \chi) \neq 0$ for $n > 1$. For the remaining part we must consider the various possibilities for q separately.

If q is odd then we simply note that $|\bar{\chi}(2)| = 1$. Hence $1 - 2^{n-1}\bar{\chi}(2) \neq 0$ for $n > 1$ and the lemma is proved.

The case when $q \equiv 2 \bmod 4$ is easily reduced to the preceding one by noting that $\Phi_n(-\lambda^k) = -\Phi_n(\lambda^k)$.

If $q \equiv 0 \bmod 4$ we note that by Lemma 2.2

$$\operatorname{span}\{\Phi_n(\lambda^k) | k \in A\} = \operatorname{span}\{\Phi_n(\lambda^k) | k \in \tilde{A}\}.$$

Now the character sum vanishes precisely for those characters χ mod q for which $\chi(1 + q/2) = 1$. It is easy to show that this is true if and only if χ is induced from a character mod $q/2$. Moreover, there are precisely $\phi(q)/4$ even characters and $\phi(q)/4$ odd characters with this property. Since the cardinality of A is $\phi(q)/2$, we conclude that the dimension of the span of $\{\Phi_n(\lambda^k) | k \in A\}$ is $\phi(q)/4$. However, the cardinality of $\tilde{A}$ is also $\phi(q)/4$. It follows that $\{\Phi_n(\lambda^k) | k \in \tilde{A}\}$ are linearly independent over $\mathbf{Q}$. □

REMARK. Lemma 2.4 is in general false for $n = 1$, and it is not hard to show that Theorems A and B are consequently false in general in case $2N - 2d = 2$. (See [**10**].)

However Lemma 2.4 *does* hold for $n = 1$ if $\chi(2) \neq 1$ for every odd character mod q when q is odd. This is true if and only if -1 is a power of 2 mod q. The following facts are all elementary except the last, which follows from a theorem of Tchebotarev [**5**, p. 169].

(1) Write $q = p_1^{t_1}, \ldots, p_k^{t_k}$, where p_i is prime, and suppose the order of 2 mod p_i is 2^{l_i}(odd). Then -1 is a power of 2 mod q if and only if $l_1 = l_2 = \cdots = l_k > 0$.

Let p be an odd prime and suppose the order of 2 mod p is 2^l (odd). Then

(2) If $p \equiv 7 \bmod 8$ then $l = 0$.

(3) If $p \equiv 5 \bmod 8$ then $l = 2$.

(4) If $p = 3 \bmod 8$ then $l = 1$.

(5) If $p \equiv 1 \bmod 8$ then l can be any nonnegative integer. Moreover, for each fixed value of l there are an infinite number of primes p for which 2 has order 2^l (odd) mod p.

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