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## On the CA Closure of an Archimedean Field

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## ABSTRACT

Cellular automata appear to be a relevant model for massively parallel computation. This paper deals with the notion of CA closed field. The partially ordered set of CA closed fields is proved isomorphic to the ideal completion of unsolvability degrees.

Any standard cellular acceptor $\mathcal{A} = (S^m, \Sigma, N, \#, \delta, F)$ is said to be a $k-$*cellular acceptor* $(k < m)$ if the first $k$ components of each state are invariant under $\delta$. Given $F \supseteq A$ ($A$ being an archimedean field) a field extension, an element $f \in F$ is said to be *CA over* $A$ if there are $m \in \mathbb{N}$, $w \in A^m$ and a $m-$cellular acceptor $\mathcal{A}$ such that $\mathcal{A}$ accepts the input $\ldots[\#\ldots\#][w_1^1 \ldots w_m^1\#\ldots\#]\ldots[w_1^k \ldots w_m^k\# \ldots\#]\ldots$ and if $\ldots[\#\ldots\#][w_1^1 \ldots w_m^1 e_1^1 \ldots e_r^1]\ldots[w_1^k \ldots w_m^k e_1^k \ldots e_r^k]\ldots$ is the configuration at the moment of acceptance corresponding to the input considered above then $e_r^1 \ldots e_r^k \ldots$ is the binary representation of $f$. If all elements of $F$ are CA over $A$ then $F$ is a *CA extension* of $A$. A field $A$ is said to be *CA closed* iff it does not have any proper CA extension. The intersection of all CA closed fields containing the field $A$ is the *CA closure* of $A$.

The CA closed fields and degrees of unsolvability are related from classical recursion theory. We show that the ideal completion of the partially ordered set of degrees is isomorphic to the set of CA closed subfields of $\mathbb{R}$. For any subfield $A$ of $\mathbb{R}$, a real number $f$ is CA over $A$ iff $\mathrm{dg}\, f \in [\mathrm{dg}\, A]$ (the ideal generated by $\mathrm{dg}\, A = \{\mathrm{dg}\, h \mid h \in A\}$). We mention the relation between ideals (the "ideal" means here "ideal over the poset of degrees") and CA closed fields. Thus for each CA closed field $A$, $\mathrm{dg}\, A$ is an ideal of degrees, and for each ideal $I$, the set $\mathrm{dg}^{-1} I = \{f \in \mathbb{R} \mid \mathrm{dg}\, f \in I\}$ is a CA closed field. Given $A$ an archimedean field and $C$ its CA closure we obtain that $[\mathrm{dg}\, A] = \mathrm{dg}\, C$. Observe that the map dg establishes an isomorphism between the poset of ideals and the poset of CA closed fields.

It is known that every countable sup-semilattice with a least element is isomorphic to a countable ideal of degrees. This implies that the ideal completion of every countable sup-semilattice with a least element is isomorphic to the poset of CA closed subfields of some CA closed field. Conclude the paper with a characterization of a countable CA closed field. More exactly, for a countable CA closed field $A$, one of the following holds: (a) there are two principal CA closed fields $A_1, A_2$ such that $A = A_1 \cap A_2$; (b) there is some $g \in \mathbb{R}$ such that $A = \{f \mid \mathrm{dg}\, f < \mathrm{dg}\, g\}$.

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