Global existence and uniqueness of solutions for the equations of third grade fluids

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ABSTRACT

The motion of third grade fluids is modeled by the following equation:

$$\partial_t(u - \alpha_1 \triangle u) - \nu \triangle u + u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A + L^t A + AL) - \alpha_2 \operatorname{div} A^2 - \beta \operatorname{div}(|A|^2 A) = f - \nabla p, \quad \operatorname{div} u = 0, \tag{1}$$

where $L = (\partial_j u_i)_{i,j}$, $A = (\partial_i u_j + \partial_j u_i)_{i,j}$ and the material coefficients ν, α_1, α_2 and β must satisfy the conditions

$$\nu \geq 0, \quad \alpha_1 > 0, \quad \beta \geq 0 \quad \text{and} \quad |\alpha_1 + \alpha_2| \leq (24\nu\beta)^{1/2}.$$

If $\beta = 0$, we obtain the equation of second grade fluids which is studied by many authors. We therefore assume that $\beta \neq 0$. The mathematical results available in the literature consider these equations in a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ and show local existence and uniqueness of solutions for arbitrary size of initial data, or global existence and uniqueness if $\nu > 0$ and if the initial data is small compared with the viscosity ν. The regularity of the initial data needed in order to obtain these results is at least H^3 or $W^{2,r}$, $r > 3$.

Here, we prove that global solutions exist without any smallness assumption and with less regularity requirements as before. We also prove the uniqueness of such bidimensional solutions. More precisely,

Theorem. *Consider equation (1) in $\mathbb{R}^n$, $n = 2, 3$, with $f \in L^\infty_{loc}\big([0,\infty); L^2\big)$ and $u_0 \in H^2$, $\operatorname{div} u_0 = 0$. There exists a global solution $u \in C_w\big([0,\infty); H^2\big) \cap C\big([0,\infty); H^s\big)$ for all $s < 2$. Moreover, if $n = 2$ then this solution is unique.*

The starting point of the proof is the remark that the term $-\beta \operatorname{div}(|A|^2 A)$ is more "regularizing" then the viscosity one. Making H^2 energy estimates, we prove that this term has the good sign and can be used to cancel all other "bad terms". We obtain in this way global H^2 solutions. As for the uniqueness of the bidimensional H^2 solutions, we first cancel some of the "bad terms" with the same idea as in the H^2 energy estimates. Next, we remark that the remaining terms may be "controlled" by using H^s norms of the solutions with $s > 2$ instead of $s = 2$. We conclude by using a kind of limiting process.

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