Knot Invariants and Chern-Simons Theory

José M. F. Labastida

Abstract. A brief review of the development of Chern-Simons gauge theory since its relation to knot theory was discovered in 1988 is presented. The presentation is done guided by a dictionary which relates knot theory concepts to quantum field theory ones. From the basic objects in both contexts the quantities leading to knot and link invariants are introduced and analyzed. The quantum field theory approaches that have been developed to compute these quantities are reviewed. Perturbative approaches lead to Vassiliev or finite type invariants. Non-perturbative ones lead to polynomial or quantum group invariants. In addition, a brief discussion on open problems and future developments is included.

In 1988, Edward Witten established the connection between Chern-Simons gauge theory and the theory of knot and link invariants [34]. Since then the theory has been intensively studied, making important progress as a result of the application of standard field theory methods. The development of the theory of knot and link invariants has been also very impressive in the last fifteen years and at some stages has occurred parallel to Chern-Simons gauge theory. There is a natural correspondence between both developments. This has led to the construction of the dictionary introduced in [22], and reproduced in table I, which will be used as a guide in this presentation.

Chern-Simons gauge theory was first analyzed from a non-perturbative point of view. The original paper by Witten presented a series of non-perturbative methods which led him to establish the equivalence between vacuum expectation values (vevs) of Wilson loops and polynomial invariants like the Jones polynomial [16] and its generalizations. Perturbative studies started one year later and soon their connection to Vassiliev invariants [31, 8] was pointed out. It turned out that the coefficients of the perturbative series correspond to these invariants [6, 9, 24].

The perturbative series expansion has been studied for different gauge-fixings. The first analysis in the covariant Landau gauge [14, 7] was later extended in a general framework [2, 1], reobtaining the formulation by Bott and Taubes of their configuration space integral [10]. Their integral corresponds precisely to the perturbative series expansion of the vev of a Wilson loop in Chern-Simons gauge theory in the Landau gauge. Before the work by Bott and Taubes, Kontsevich presented a different integral [21] for Vassiliev invariants which turned out to correspond to the perturbative series expansion of the vev of a Wilson loop in the light-cone gauge [11, 23, 19].

Additional studies of the perturbative series expansion have been performed in the temporal gauge [11, 25]. This gauge has the important feature that the integrals which are present in the expressions for the coefficients of the perturbative series expansion can be carried out, leading to combinatorial expressions [25]. This has been shown to be the case up to order four and it seems likely that the approach can be generalized. In this analysis a crucial role is played by the factorization theorem for Chern-Simons gauge theory proved in [4]. The resulting expressions are better presented when written in terms of Gauss diagrams for knots [26]. Recent results demonstrate the existence of a combinatorial formula of this type [13]. Chern-Simons gauge theory has provided combinatorial expressions for all the Vassiliev invariants up to order four [25]. Further work is needed to obtain a general combinatorial expression.

The invariants obtained in the perturbative framework for each gauge-fixing are the same. This is guaranteed by the fact that the theory is gauge invariant and Wilson loops are gauge invariant operators. In fact, this property is the responsible, from a field theory point of view, of the connection between Vassiliev invariants and polynomial invariants, as they appear in the non-perturbative approach, and of the existence of different representations of Vassiliev invariants. The correspondence between the Chern-Simons gauge theory description and the knot theory one is listed in the following table.

Table I

Knot Theory	**Chern-Simons Gauge Theory**
Knots and links	Wilson loops
Knot and link polynomial invariants	Vevs of products of Wilson loops
Singular knots	Operators for singular knots
Invariants for singular knots	Vevs of the new operators
Finite type or Vassiliev invariants	Coeffs. of the perturbative series
Chord diagram	First coeff. of the perturbative series
{1T,4T} and {1T,AS,IHX,STU}	Lie-algebra structure of group factors
Configuration space integral	Landau gauge
Kontsevich integral	Light-cone gauge
??	Temporal gauge

Before entering into the review of the developments of the last years guided by this table it is worth to remark two facts. First, the entry in the knot-theory column corresponding to the temporal gauge has not been filled in yet. Further work is needed to complete it. Second, recent developments based on the application of Maldacena's conjecture has led to introduce a new context in which knot invariants are organized differently [29]. It is possible that some important new boxes are still missing in the table.

Chern-Simons gauge theory on a smooth three-manifold M is defined by the action,

$$S_{\rm CS}(A) = \frac{k}{4\pi}\int_M {\rm Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\,, \tag{1}$$

where k is an integer constant. The exponential of this action is gauge invariant. Wilson loops are some of the relevant operators of the theory. They are defined as:

$$W_\gamma^R(A) = {\rm Tr}_R\left({\rm Hol}_\gamma(A)\right) = {\rm Tr}_R\,{\rm P}\exp\int_\gamma A\,, \tag{2}$$

where R denotes a representation of the gauge group G and γ a 1-cycle. Products of these operators are the natural candidates to obtain topological invariants after computing their vev respect to the action (1).

The Wilson loop (2) and the vevs of its products are the first two items of the right column of table I. The first two in the left column are the basic ingredients of knot theory. Knot theory studies inequivalent embeddings $\gamma\colon S^1 \to M$. Each of these is a knot. Knots are classified constructing knot invariants or quantities which can be computed taking a representative of a class and are invariant within the class. The problem of the classification of knots in $M = S^3$ can be reformulated in a two-dimensional framework using regular knot projections. The previous equivalence problem then translates into the equivalence of regular knot projections under Reidemeister moves.

The study of knot and link invariants experimented important progress in the eighties after the discovery of the Jones polynomial [16] and its generalizations like the HOMFLY [12] and the Kauffman [20] polynomial invariants. Witten showed in 1988 that the vevs of products of Wilson loops correspond to the Jones polynomial when one considers $SU(2)$ as gauge group and all the Wilson loops entering in the vev are taken in the fundamental representation F. For example, if one considers a knot K, with Jones polynomial $V_K(t)$, he showed that, $V_K(t) = \langle W_K^F\rangle$, provided that one performs the identification $t = \exp(2\pi i/k+h)$ where $h = 2$ is the dual Coxeter number of the gauge group $SU(2)$. He also showed that if instead of $SU(2)$ one considers $SU(N)$ and the Wilson loop carries the fundamental representation, the resulting invariant is the HOMFLY polynomial. If, instead, one considers $SO(N)$ as gauge group and Wilson loops carrying the fundamental representation one is led to the Kauffman polynomial. The case of $SU(2)$ as gauge group and Wilson loops carrying a representation of spin $j/2$ leads to the Akutsu-Wadati [5] polynomials. The framework generated by Chern-Simons gauge theory leads to an enormous variety of knot and link invariants. They can be also obtained from a quantum group approach [17], and from more general formalisms [18].

In our discussion of the next five items in table I we will deal first with the left column. In 1990, V. A. Vassiliev [31] introduced new knot invariants based on singular knots which were reformulated later by Birman and Lin [9] from an axiomatic point of view. A singular knot with j double points consists of the image of a map from S^1 into S^3 with j simple self-intersections. The key ingredient in the construction by Birman and Lin is the observation that any knot invariant extends to generic singular knots by the Vassiliev resolution:

$$\nu(K^{j+1}) = \nu(K^j_+) - \nu(K^j_-)\,, \tag{3}$$

where K^{j+1} is a singular knot with $j+1$ double points which differs from the knots K^j_+ and K^j_- only in the region where the double point is resolved by an overcrossing $(+)$ and an undercrossing $(-)$. Using this extension Birman and Lin [9] characterized the invariants of finite type or Vassiliev invariants introducing the following definition: a Vassiliev or finite type invariant of order m is a knot invariant which is zero on the unknot and that, after extending it to singular knots, it is zero on singular knots K^j with $j > m$ double points.

Besides introducing an axiomatic approach to Vassiliev invariants, Birman and Lin proved an important theorem in 1993 [9]. Any polynomial invariant $P_K(t)$ for a knot K can be expanded as:

$$Q_K(x) = P_K(\mathrm{e}^x) = \sum_{m=0}^{\infty} \nu_m(K) x^m\,. \tag{4}$$

Birman and Lin proved that if one extends the quantities $\nu_m(K)$ to Vassiliev invariants for singular knots using Vassiliev resolution (3), then $\nu_m(K)$ are Vassiliev invariants of order m. An immediate consequence of this theorem is that the coefficients of the perturbative expansion associated to the vev of a Wilson loop in Chern-Simons gauge theory are Vassiliev invariants. This property of the coefficients of the perturbative series expansion has been proved using standard quantum field theory methods [24].

From a singular knot with m double points one can construct a particular object which determines Vassiliev invariants of order m: its chord diagram [6]. Given a singular knot K^m, its chord diagram, $CD(K^m)$, is built in the following way. Take a base point and draw the preimages of the map associated to a given representative of K^m on a circle. Then join by straight lines the pairs of preimages which correspond to each singular point. If $\nu(K^m)$ is a Vassiliev invariant of order m then it is completely determined by $CD(K^m)$.

Chord diagrams play an important role in the theory of Vassiliev invariants [6]. Since Vassiliev invariants of order m for singular knots with m double points are codified by chord diagrams one could ask if there are as many independent invariants of this kind as chord diagrams. The answer to this question is no. Chord diagrams are associated to knot diagrams and these diagrams must be considered modulo the equivalence relation dictated by the Reidemeister moves. These relations indeed impose some relations among chord diagrams, the so-called 1T and 4T relations [6]. The general expression for the dimensions of the spaces of chord diagrams is an open problem which has challenged many people. These dimensions correspond in fact to the dimensions of the spaces of primitive Vassiliev invariants.

The vector space of chord diagrams can be characterized in an equivalent way using trivalent diagrams an introducing a series of new relations. This characterization is very important because it corresponds to the one that naturally arises from the point of view of Chern-Simons gauge theory. It consists of the expansion of the set of chord diagrams to a new set in which trivalent vertices are allowed. This means that now the lines in the interior of the circle can join a point on the circle to a point on one of the internal lines. Bar-Natan showed [6] that the space of chord diagrams modulo 1T and 4T relations is equivalent to the new one after modding out by the so-called 1T, AS, IHX and STU relations.

The relations AS, STU and IHX are reminiscent of a Lie-algebra structure. If one assigns totally antisymmetric structure constants f_{abc} to the internal trivalent vertices, and group generators T_a to the vertices on the circle, the STU relation is just the defining Lie-algebra relation, while the IHX relation corresponds to the Jacobi identity. The group factors associated to the perturbative series expansion of the vev of a Wilson loop in Chern-Simons gauge theory correspond precisely to these spaces.

We will now turn our attention to the column on the right column in table I. Singular knots play a central role in the theory of Vassiliev invariants. As shown in [24], they have an operator counterpart in Chern-Simons gauge theory. It has a rather simple form. Let us consider a singular knot K^n with n double points, and let us assign to each double point i a triple $\tau_i = \{s_i, t_i, T^{a_i}\}$ where s_i and t_i, $s_i < t_i$, are the values of the K^n-parameter at the double point, and T^{a_i} is a group generator. The gauge-invariant operator associated to the singular knot K^n is:

$$\left(\frac{4\pi i}{k}\right)^n \mathrm{Tr}\Big[T^{\phi(w_1)}U(w_1,w_2)T^{\phi(w_2)}U(w_2,w_3)T^{\phi(w_3)}\cdots \\ \cdots U(w_{2n-1},w_{2n})T^{\phi(w_{2n})}U(w_{2n},w_1)\Big], \quad (5)$$

where $\{w_i; i=1,\ldots,2n\}$, $w_i < w_{i+1}$, is the set that results from ordering the values s_i and t_i, for $i=1,\ldots,n$, and ϕ is a map that assigns to each w_i the group generator in the triple to which it belongs.

Some immediate implications of the singular operators (5) are the following. First, they lead to a proof of the theorem by Birman and Lin discussed above; second, they allow to make direct contact with chord diagrams since these diagrams correspond to these operators at lowest order. This has been shown in [24]. The quantities which result after the assignment of Lie-algebra data to chord diagrams are called weight systems [6]. For each system one chooses a group and a representation. They correspond to the group theory factors in the context of Chern-Simons perturbation theory.

We will now describe the last three items of table I starting with the right column. Perturbative studies of Chern-Simons gauge theory started with the works by Guadagnini, Martellini and Mintchev [14] and by Bar-Natan [7]. These studies were made in the covariant Landau gauge. Subsequent works [2, 1] in this gauge led to a framework linked to the theory of Vassiliev invariants, which constituted the configuration space integral approach [10]. The elements of the resulting Feynman rules in this gauge are:

$$\frac{i}{4\pi}\delta_{ab}\epsilon^{\mu\nu\rho}\frac{(x-y)_\rho}{|x-y|^3}, \quad -igf_{abc}\epsilon_{\mu\nu\rho}\int d^3x, \quad g(T^a)_i^j\,, \tag{6}$$

which correspond to internal line (gauge propagator), internal vertex, and vertex on the Wilson loop, respectively. In these equations $g^2 = 4\pi/k+h$. To these rules one must include the ones related to the ghost fields present in the Landau gauge. One of the consequences of their presence is that higher-loop corrections to two- and three-point functions can be ignored, at least in some of the standard regularization schemes.

In analyzing the perturbative series one must deal with an important subtlety. If one computes the first order contribution to the perturbative series expansion of the vev of a Wilson loop one finds that the resulting quantity is not a topological invariant. In the gauge fixing of the theory we have introduced a metric dependence that could lead to quantities which are not topological. This first order contribution is just a manifestation of it. Fortunately, only in this term, and in its propagation in higher order contributions, topological invariance is lost. The rest of the perturbative series expansion is truly topological. Thus, although vevs are not topological invariant quantities, they fail to be so in a controllable way. The non-topological terms factorize and multiply a term which is topological. The factor turns out to have a framing dependence equivalent to the one obtained in non-perturbative approaches.

The Feynman rules allow to split the contributions to each order in two factors: a geometrical factor which includes all the space dependence, and a group factor which includes all the group theoretical dependence. The general form is:

$$\langle W_K^R\rangle = \dim R\sum_{i=0}^{\infty}\sum_{j=1}^{d_i}\alpha_{ij}(K)r_{ij}(R)x^i, \quad x=\frac{2\pi i}{k+h}=ig^2/2\,, \tag{7}$$

where R is the dimension of the representation R, $\alpha_{0,1}=r_{0,1}=1$, and $d_0=1$. The factors $\alpha_{ij}(K)$ and $r_{ij}(R)$ appearing at each order i incorporate all the dependence dictated from the Feynman rules apart from the dependence on the coupling constant, which is contained in x. Of these two factors, in the $r_{ij}(R)$ all the group-theoretical dependence is collected. The rest is contained in the $\alpha_{ij}(K)$ or geometrical factors. They have the form of integrals over the Wilson loop corresponding to the knot K of products of propagators, as dictated by the Feynman rules. The first index in $\alpha_{ij}(K)$ denotes the order in the expansion and the second index labels the different geometrical factors which can contribute at the given order. Similarly, $r_{ij}(R)$ stands for the independent group structures which appear at order i, which are also dictated by the Feynman rules. The object d_i in (7) is the dimension of the space of invariants at a given order.

Among the basis of group factors which can be chosen there is a special class called canonical basis which turns out to be very useful. Basically, it consists of connected diagrams. If we denote by $r_{ij}^c(R)$ the group factors associated to this basis, and $\alpha_{ij}^c(K)$ the corresponding geometrical factors, the perturbative series expansion (7) can be written as [4]:

$$\langle W_K^R \rangle = \dim R \exp \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\hat{d}_i} \alpha_{ij}^c(K)\, r_{ij}^c(R)\, x^i \right\} , \tag{8}$$

where $\hat{d}_i$ stands for the number of connected elements in the canonical basis at order i. The result (8) is known as the factorization theorem, and it holds for arbitrary gauges. The geometrical factors $\alpha_{ij}^c(K)$ are a selected set of Vassiliev invariants. They are called primitive Vassiliev invariants. They have been computed for general classes of knots as torus knots [3, 33] up to order six.

The contribution at first order in (8) is precisely the framing factor. The rest of the terms in the exponent of (8) are knot invariants. The series contained in that exponent was analyzed by Bott and Taubes [10] in their work on the configuration space for Vassiliev invariants (listed on the left column in table I). They showed that the integral expression entering the geometrical factors $\alpha_{ij}^c(K)$ are convergent [10, 30]. Further work on the subject has led to a proof of their invariance [1, 35].

The explicit expression for the integrals entering in the second order contribution was first presented in [14]. It was later analyzed in detail by Bar-Natan [7]. This invariant turns out to be the total twist of the knot and coincides mod 2 with the Arf invariant. The integral expression for the order three invariant, $\alpha_{31}^c(K)$ was first presented in [2]. Properties of the primitive Vassiliev invariants $\alpha_{21}^c(K)$ and $\alpha_{31}^c(K)$ have been studied in [15]. In these works the integral expressions for $\alpha_{21}^c(K)$ and $\alpha_{31}^c(K)$ were studied in the flat-knot limit and combinatorial expressions were obtained.

The perturbative analysis of Chern-Simons gauge theory in the light-cone gauge leads to the Kontsevich integral, which constitutes a particular representation of Vassiliev invariants. Non-covariant gauges are characterized by a unit constant vector n and have the form $n^\mu A_\mu = 0$. In the case of the light-cone gauge the unit vector n satisfies the condition $n^2 = 0$. In this gauge there is only one Feynman rule to be taken into account to compute the vevs of operators: the one associated to the propagator. The group factors that remain in this case correspond just to chord diagrams. The fact that in this gauge no group factors with trivalent vertices have to be taken into account is a quantum field theory ratification of Bar-Natan theorem among the equivalence of the two representations of the space of diagrams. Non-covariant gauges share the problem of the presence of unphysical poles in their propagators [27]. Several prescriptions have been proposed to avoid these unphysical poles. Usually, a prescription is chosen so that some particular properties of the correlation functions are fulfilled. In the light-cone gauge there is a natural prescription which is motivated by the simple form that the elements of the perturbative expansion take after performing a Wick rotation. This prescription leads [23] to the Kontsevich integral.

The studies in the Landau and in the light-cone gauge provide integral expression for Vassiliev invariants. It is difficult to obtain information on these invariants from these expressions. Combinatorial formulas are much preferred. It is known that a general combinatorial formula for Vassiliev invariants exists [13, 28]. The search for an explicit construction of the combinatorial formula has led to the study of Chern-Simons gauge theory in the temporal gauge [25]. This turns out to be the more suitable gauge to carry out all the intermediate integrals and obtain combinatorial expressions. This approach has provided a combinatorial expression for the two primitive Vassiliev invariants at order four. The temporal gauge has been also treated in [11, 32]. Previous studies of the configuration space integrals in the limit of flat knots [15] have also led to combinatorial expressions for Vassiliev invariants of order two and three.

The starting point of the analysis in the temporal gauge is the same as in the light-cone gauge. The gauge-fixing condition is the same but now n is a unit vector of the form $n^\mu = (1, 0, 0)$. As before, the propagator presents unphysical poles, and a prescription to regulate it is needed. In this case a prescription-independent analysis is done splitting the propagators in two terms. It leads to the concept of kernel, as introduced in [25]. The kernels are quantities which depend on the knot projection chosen and therefore are not knot invariants. However, at a given order i a kernel differs from an invariant of type i by terms that vanish in signed sums of order i. The kernel contains the part of a Vassiliev invariant which is the last in becoming zero when performing signed sums, in other words, a kernel vanishes in signed sums of order $i+1$ but does not in signed sums of order i. Kernels plus the structure of the perturbative series expansion seem to contain enough information to reconstruct the full Vassiliev invariants [25]. The general expression for the kernels can be written in a universal form much in the spirit of the universal form which shares some resemblence with Kontsevich integral.

Using the kernels and taking into account general properties of the perturbative series expansion one can reconstruct the complete perturbative coefficients obtaining combinatorial formulas. Vassiliev invariants up to order four were expressed in terms of these quantities and the crossing signatures in reference [25]. Here, we collect only the formula for the primitive Vassiliev invariant at second order. It has the following expression:

$$\alpha_{21}(K) = \alpha_{21}(U) + \langle \bigoplus, \bar{G}(\mathcal{K}) \rangle \,, \tag{9}$$

where $\alpha_{21}(U)$ stands for the value of α_{21} for the unknot and $\bar{G}(\mathcal{K}) = G(\mathcal{K}) - G(\alpha(\mathcal{K}))$, where $\alpha(\mathcal{K})$ denotes the ascending diagram of the knot projection $\mathcal{K}$. $G(\mathcal{K})$ is the Gauss diagram corresponding to $\mathcal{K}$. The inner product used in (9) consists of the sum over all the embeddings of the diagram $\bigoplus$ into $G(\mathcal{K})$, each one weighted by a factor, $\varepsilon_1\varepsilon_2$, where ε_1 and ε_2 are the signatures of the chords of $G(\mathcal{K})$ involved in the embedding.

The analysis presented in [25] up to order four should be generalized to arbitrary order, trying to obtain a general expression similar to the one existing in the light-cone corresponding to the Kontsevich integral. The resulting formula would allow to fill the last box on the left column of table I. Though this study seems promising, the problems inherent to the proper treatment of gauge theories in non-covariant gauges constitute an important barrier. Much work has to be done to understand the subtle issues related to the use of non-covariant gauges. The kernels plus the properties of the perturbative series expansion are probably enough to compute the explicit form of a given invariant but certainly it does not provide a systematic way of deriving the general universal formula.

The relation between knot theory and Chern-Simons gauge theory does not end here. Most likely additional boxes to table I are waiting to be discovered. Quantum field theory is a very rich framework which is enlarged when regarded from the point of view of string theory. Recent work [29] indicates that new important connections can be established that could lead to entirely new approaches to the theory of knot invariants.

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Departamento de Física de Partículas,
Universidade de Santiago de Compostela,
E-15706 Santiago de Compostela, Spain
E-mail address: `labasti@fpaxp1.usc.es`