

# Understanding Skyrmions Using Rational Maps

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**Abstract.** We discuss an ansatz for Skyrme fields in three dimensions which uses rational maps between Riemann spheres, and produces shell-like structures of polyhedral form. Houghton, Manton and Sutcliffe showed that a single rational map gives good approximations to the minimal energy Skyrmions up to baryon number of order ten. We show how the method can be generalised by using two or more rational maps to give a double-shell or multi-shell structure. Particularly interesting examples occur at baryon numbers twelve and fourteen.

## 1. Introduction

The Skyrme model is a nonlinear theory of pions in  $\mathbb{R}^3$ , with an  $SU(2)$  valued scalar field  $U(\mathbf{x}, t)$ , the Skyrme field, satisfying the boundary condition  $U \rightarrow 1$  as  $|\mathbf{x}| \rightarrow \infty$ . Static fields obey the equation

$$\partial_i(R_i - \frac{1}{4}[R_j, [R_j, R_i]]) = 0 \quad (1)$$

where  $R_i$  is the  $su(2)$  valued current  $R_i = (\partial_i U)U^{-1}$ . Such fields are stationary points (either minima or saddle points) of the energy function

$$E = \int \left\{ -\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j][R_i, R_j]) \right\} d^3\mathbf{x}. \quad (2)$$

Associated with a Skyrme field is a topological integer, the baryon number  $B$ , defined as the degree of the map  $U: \mathbb{R}^3 \mapsto SU(2)$ . It is well defined because of the boundary condition at infinity.

Solutions of equation (1) are known for several values of  $B$ , but they can only be obtained numerically. Many of these solutions are stable, and probably represent the global minimum of the energy for given  $B$ . We shall refer to the solutions believed to be of lowest energy for each  $B$  as Skyrmions.

There is a nine-dimensional symmetry group of the equation and boundary condition. It consists of translations and rotations in  $\mathbb{R}^3$  and the  $SO(3)$  isospin transformations  $U \mapsto \mathcal{O}U\mathcal{O}^{-1}$  where  $\mathcal{O}$  is a constant element of  $SU(2)$ .

Skyrme found the spherically symmetric  $B = 1$  Skyrmion. The  $B = 2$  Skyrmion is toroidal. A substantial numerical search for Skyrmion solutions was undertaken by Braaten, Townsend and Carson [4], and minimal energy solutions

up to  $B = 5$  were found. (Their solution for  $B = 6$  was rather inaccurate.) Surprisingly, the  $B = 3$  solution has tetrahedral symmetry  $T_d$ , and the  $B = 4$  solution has cubic symmetry  $O_h$ . Battye and Sutcliffe [2] subsequently found all Skyrmons up to  $B = 8$ . (Their solution for  $B = 9$  is probably a saddle point.) The  $B = 7$  Skyrmion has icosahedral symmetry  $Y_h$ . Recently, with new methods, they have found candidate solutions up to  $B = 18$  [3].

## 2. Skyrme Fields from Rational Maps

Let us denote a point in  $\mathbb{R}^3$  by its coordinates  $(r, z)$  where  $r$  is the radial distance from the origin and  $z = \tan(\theta/2) \exp i\varphi$  specifies the direction from the origin. Houghton, Manton and Sutcliffe [6] showed that one can understand the structure of the known Skyrmons in terms of an ansatz using rational maps

$$R(z) = \frac{p(z)}{q(z)} \quad (3)$$

where  $p$  and  $q$  are polynomials, and a radial profile function  $f(r)$ . One identifies the target  $S^2$  of the rational maps with spheres of latitude on  $SU(2)$ , that is, spheres at a fixed distance from the identity element.

Recall that the direction  $z$  corresponds to the Cartesian unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1 + |z|^2} (2 \operatorname{Re}(z), 2 \operatorname{Im}(z), 1 - |z|^2). \quad (4)$$

Similarly the value of the rational map  $R$  is associated with the unit vector

$$\hat{\mathbf{n}}_R = \frac{1}{1 + |R|^2} (2 \operatorname{Re}(R), 2 \operatorname{Im}(R), 1 - |R|^2). \quad (5)$$

The ansatz is

$$U(r, z) = \exp(iff(r) \hat{\mathbf{n}}_{R(z)} \cdot \boldsymbol{\sigma}) \quad (6)$$

where  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are Pauli matrices. For this to be well-defined at the origin,  $f(0) = k\pi$ , for some integer  $k$ . The boundary value  $U = 1$  at  $r = \infty$  requires that  $f(\infty) = 0$ . The baryon number of this field is  $B = Nk$ , where  $N = \max(\deg p, \deg q)$  is the degree of  $R$ . We only consider the case  $k = 1$  here, so  $B = N$ . Note that an  $SU(2)$  Möbius transformation of the rational map

$$R(z) \mapsto \frac{\alpha R(z) + \beta}{-\beta R(z) + \bar{\alpha}} \quad (7)$$

with  $|\alpha|^2 + |\beta|^2 = 1$  acts as an isospin transformation.

An attractive feature of the ansatz (6) is that it leads to a simple energy expression which can be minimized with respect to the rational map  $R$  and the profile function  $f$  to obtain close approximations to the true Skyrmons. A generalised rational map ansatz has also proved useful in the construction of solutions to Skyrme models with fields having values in  $SU(N)$  [7].

The energy for a Skyrme field of the form (6) is

$$E = \int \left[ f'^2 + 2(f'^2 + 1) \frac{\sin^2 f}{r^2} \left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 + \frac{\sin^4 f}{r^4} \left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \right] \frac{2i \, dz d\bar{z} r^2 dr}{(1 + |z|^2)^2}. \quad (8)$$

Now

$$\left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 \frac{2i \, dz d\bar{z}}{(1 + |z|^2)^2} \quad (9)$$

is the pull-back of the area form  $2i \, dR d\bar{R}/(1 + |R|^2)^2$  on the target sphere of the rational map; therefore its integral is  $4\pi$  times the degree  $N$ . So the energy simplifies to

$$E = 4\pi \int \left( r^2 f'^2 + 2N(f'^2 + 1) \sin^2 f + \mathcal{I} \frac{\sin^4 f}{r^2} \right) dr \quad (10)$$

where  $\mathcal{I}$  denotes the integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i \, dz d\bar{z}}{(1 + |z|^2)^2}. \quad (11)$$

Applying a Bogomolny-type argument to the expression (10), one can show that

$$E \geq 4\pi^2(2N + \sqrt{\mathcal{I}}). \quad (12)$$

This lower bound on the energy, which applies to the rational map ansatz, is higher than the Fadeev-Bogomolny bound satisfied by any Skyrme field,  $E \geq 12\pi^2 B$ . This is because the Schwarz inequality implies  $\mathcal{I} \geq N^2$ .

To minimize the energy  $E$ , one should first minimize  $\mathcal{I}$  over all maps of degree  $N$ . Then the profile function  $f$  minimizing (10) is found by solving a second order differential equation with  $N$  and  $\mathcal{I}$  as parameters. In [6] only rational maps of a given symmetric form were considered, with symmetries corresponding to a known Skyrmion. If these symmetric maps still contained a few free parameters,  $\mathcal{I}$  was minimized with respect to these, and then  $f$  was calculated. A rational map,  $R: S^2 \mapsto S^2$ , is symmetric under a subgroup  $G \subset SO(3)$  if there is a set of Möbius transformation pairs  $\{g, D_g\}$  with  $g \in G$  acting on the domain  $S^2$  and  $D_g$  acting on the target  $S^2$ , such that

$$R(g(z)) = D_g R(z). \quad (13)$$

Some rational maps also possess additional reflection or inversion symmetry. In their recent work, Battye and Sutcliffe have systematically sought the rational map that minimizes  $\mathcal{I}$ , up to  $N = 18$  [3]. This work has confirmed that the choice of maps in [6], up to  $N = 8$ , was optimal.

The zeros of the Wronskian

$$W(z) = p'(z)q(z) - q'(z)p(z) \quad (14)$$

of a rational map  $R(z)$  give interesting information about the shape of the corresponding Skyrme field. Where  $W$  is zero, the derivative  $dR/dz$  is zero, so the baryon density vanishes. The energy density is also low. The Skyrme field baryon density contours therefore look like a polyhedron with holes in the directions given by the  $2N - 2$  zeros of  $W$ .

### 3. Symmetric Rational Maps

In this section, we present the symmetric rational maps of degrees one to eight, determined in reference [6]. Table 1 gives the energy of the resulting approximate Skyrmions, and also the energy of the true Skyrmions. All numerical values for the energies are the real energies divided by  $12\pi^2 B$ , and hence close to unity. Figure 1 shows a surface of constant baryon density for most of the approximate Skyrmions. The true solutions have very similar shapes [2].

For  $B = 1$  the basic map is  $R(z) = z$ , for which the integral  $\mathcal{I} = 1$ , and (6) reduces to Skyrme's hedgehog field

$$U(r, \theta, \varphi) = \cos f + i \sin f (\sin \theta \cos \varphi \sigma_1 + \sin \theta \sin \varphi \sigma_2 + \cos \theta \sigma_3). \quad (15)$$

This is  $SO(3)$  invariant, since  $R(g(z)) = g(z)$  for any  $g \in SU(2)$ . It gives the standard exact spherically symmetric Skyrme field with its usual profile  $f(r)$ , and with energy  $E = 1.232$ .

The rational map which gives the toroidal  $B = 2$  Skyrme field is

$$R(z) = z^2. \quad (16)$$

Using this, one finds  $\mathcal{I} = \pi + 8/3$  and after determining the profile  $f(r)$  one obtains  $E = 1.208$ , an energy 3% higher than that of the true solution.

The  $B = 3$  Skyrme field has tetrahedral symmetry. A rational map with this symmetry is obtained by imposing

$$R(-z) = -R(z), \quad R(1/z) = 1/R(z) \quad (17)$$

$$R\left(\frac{iz + 1}{-iz + 1}\right) = \frac{iR(z) + 1}{-iR(z) + 1}. \quad (18)$$

This gives the degree three maps

$$R(z) = \frac{\sqrt{3}az^2 - 1}{z(z^2 - \sqrt{3}a)} \quad (19)$$

with  $a = \pm i$ . Note that  $z \mapsto (iz + 1)/(-iz + 1)$  sends  $0 \mapsto 1 \mapsto i \mapsto 0$  and hence generates the  $120^\circ$  rotation cyclically permuting the Cartesian axes. The sign of  $a$  can be changed by the  $90^\circ$  rotation  $z \mapsto iz$ . For these maps  $\mathcal{I} = 13.58$ . Solving for the profile  $f(r)$ , one finds an energy  $E = 1.184$ . The Wronskian of maps of the form (19) is proportional to  $z^4 \pm 2\sqrt{3}iz^2 + 1$ , a tetrahedral Klein polynomial [8].

The  $B = 4$  Skyrmion has cubic symmetry. The cubically symmetric rational map of degree four is the ratio of tetrahedral Klein polynomials

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (20)$$

The  $90^\circ$  rotation is a symmetry, because  $R(iz) = 1/R(z)$ . Using (20) in the ansatz (6) gives an energy  $E = 1.137$ .

The  $B = 5$  Skyrmion of minimal energy has symmetry  $D_{2d}$ , which is somewhat surprising. A nearby cubically symmetric solution exists but is a saddle point. The  $D_{2d}$ -symmetric family of rational maps is

$$R(z) = \frac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1} \quad (21)$$

with  $a$  and  $b$  real. If  $b = 0$  then  $R(z)$  has  $D_4$  symmetry, the symmetry of a square. There is cubic symmetry if, in addition,  $a = -5$ . This value ensures the  $120^\circ$  rotational symmetry (18) and the Wronskian is then proportional to  $z^8 + 14z^4 + 1$ , the face polynomial of an octahedron. When  $b = 0$ ,  $a = -5$ , the integral  $\mathcal{I} = 52.05$ . However,  $\mathcal{I}$  is minimized when  $a = 3.07$ ,  $b = 3.94$ , taking the value  $\mathcal{I} = 35.75$ . This is consistent with the structure of the  $B = 5$  Skyrmion, a polyhedron made from four pentagons and four quadrilaterals. With the optimal profile function  $f(r)$ , the energy is  $E = 1.147$ . The octahedral saddle point has  $E = 1.232$ . There is a further, much higher saddle point at  $a = b = 0$ , where the map (21) simplifies to  $R(z) = z^5$ , and gives a toroidal field.

The  $B = 6$  Skyrmion has symmetry  $D_{4d}$ . The rational maps

$$R(z) = \frac{z^4 + a}{z^2(az^4 + 1)} \quad (22)$$

have this symmetry, and the minimal energy occurs at  $a = 0.16$ , giving  $E = 1.137$ . The Skyrme field has a polyhedral shape consisting of a ring of eight pentagons capped by squares above and below.

In a sense, the  $B = 7$  case is similar to the case  $B = 6$ , but the Skyrmion has a dodecahedral shape. A dodecahedron is a ring of ten pentagons capped by pentagons above and below. Among the degree seven rational maps with  $D_{5d}$  symmetry

$$R(z) = \frac{z^5 - a}{z^2(az^5 + 1)}, \quad (23)$$

the one with icosahedral symmetry has  $a = -1/7$  (not  $a = -3$  as stated in [6]). The Wronskian is then proportional to  $z(z^{10} + 11z^5 - 1)$ , the face polynomial of a dodecahedron. In another orientation, tetrahedral symmetry  $T$  is manifest. There is a one-parameter family of maps with the symmetries (17) and (18),

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)} \quad (24)$$

where  $b$  is complex. For real  $b$ , the symmetry extends to  $T_h$  and for  $b$  imaginary it extends to  $T_d$ . When  $b = 0$  there is cubic symmetry  $O_h$ , and when  $b = \pm 7/\sqrt{5}$  there is icosahedral symmetry  $Y_h$ . Using (24) in our ansatz, one finds the minimal energy at  $b = \pm 7/\sqrt{5}$ , which gives a dodecahedral Skyrme field, with energy  $E = 1.107$ . This is particularly close to the energy of the true solution. There is a saddle point at  $b = 0$  with a cubic shape.

The  $B = 8$  Skyrminion has symmetry  $D_{6d}$ , as do the rational maps

$$R(z) = \frac{z^6 - a}{z^2(az^6 + 1)}. \quad (25)$$

This time the minimal energy is  $E = 1.118$  when  $a = 0.14$ . The shape is now a ring of 12 pentagons capped by hexagons above and below.

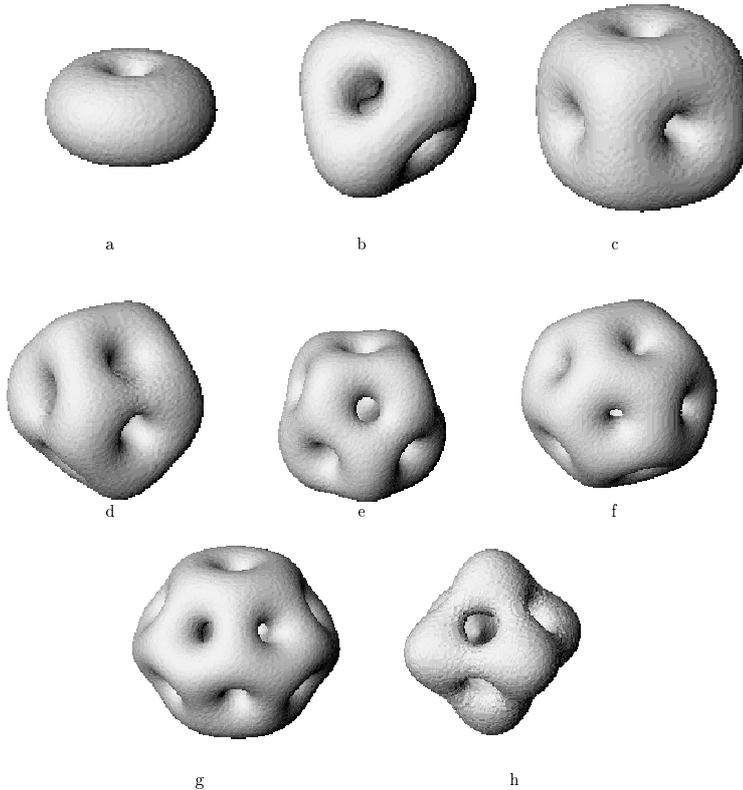


FIGURE 1. Surfaces of constant baryon density for the following approximate Skyrminions, constructed using rational maps: a)  $B = 2$  torus; b)  $B = 3$  tetrahedron; c)  $B = 4$  cube; d)  $B = 5$  with  $D_{2d}$  symmetry; e)  $B = 6$  with  $D_{4d}$  symmetry; f)  $B = 7$  dodecahedron; g)  $B = 8$  with  $D_{6d}$  symmetry; h)  $B = 5$  octahedron (saddle point).

B	$\mathcal{I}$	APPROX	TRUE	SYM
1	1.00	1.232	1.232	$O(3)$
2	5.81	1.208	1.171	$O(2) \times \mathbb{Z}_2$
3	13.58	1.184	1.143	$T_d$
4	20.65	1.137	1.116	$O_h$
5	35.75	1.147	1.116	$D_{2d}$
6	50.76	1.137	1.109	$D_{4d}$
7	60.87	1.107	1.099	$Y_h$
8	85.63	1.118	1.100	$D_{6d}$
5*	52.05	1.232	1.138	$O_h$

**Table 1** : The energies of approximate Skyrmions generated from rational maps, and of true Skyrmions. The table gives the value of the angular integral  $\mathcal{I}$ , and the associated Skyrme field energy (APPROX), together with the energy of the true solution (TRUE), as determined in references [6, 2], and the symmetry (SYM) of the solution. A \* denotes a saddle point configuration.

#### 4. Multi-Shell Rational Maps

The minimal energy solution of the Skyrme equation (1) with infinite baryon number is a three-dimensional cubic crystal. It is obtained by relaxing a face-centred cubic array of Skyrmions [5]. For finite, increasing  $B$ , the single-shell polyhedral structures we have discussed so far are therefore unlikely to remain the minimal energy solutions. Skyrmions will probably look more like part of the crystal. An approximate construction of Skyrmions as part of the crystal was carried out by Baskerville, for some special values of  $B$ , but the resulting energies were rather high [1]. Here we try a rational map ansatz with a two-shell structure. This is easily generalised to a multi-shell structure. The connection with the crystal will emerge below.

The simplest version of this generalised ansatz is (6) itself, with the profile function  $f(r)$  having boundary values  $f(0) = 2\pi$ ,  $f(\infty) = 0$ . However, this does not give a low energy. More promising is to use two different rational maps,  $R_1(z)$  of degree  $N_1$  for the inner shell, and  $R_2(z)$  of degree  $N_2$  for the outer shell. Let  $r_0 > 0$  denote the radius where the inner and outer shells join. The ansatz is now

$$U(r, z) = \begin{cases} \exp(if_1(r) \hat{\mathbf{n}}_{R_1(z)} \cdot \boldsymbol{\sigma}) & 0 \leq r \leq r_0, \\ \exp(if_2(r) \hat{\mathbf{n}}_{R_2(z)} \cdot \boldsymbol{\sigma}) & r_0 \leq r, \end{cases} \quad (26)$$

where the profiles  $f_1(r)$  and  $f_2(r)$  have boundary values  $f_1(0) = 2\pi$ ,  $f_1(r_0) = f_2(r_0) = \pi$ ,  $f_2(\infty) = 0$ . The field is continuous at  $r = r_0$ , but derivatives jump there. Note that  $U = 1$  at the centre.

The baryon number of the Skyrme field (26) is easily seen to be  $B = N_1 + N_2$ . Its energy is the obvious generalisation of (10),

$$E = 4\pi \int_0^{r_0} \left( r^2 f_1'^2 + 2N_1(f_1'^2 + 1) \sin^2 f_1 + \mathcal{I}_1 \frac{\sin^4 f_1}{r^2} \right) dr + 4\pi \int_{r_0}^{\infty} \left( r^2 f_2'^2 + 2N_2(f_2'^2 + 1) \sin^2 f_2 + \mathcal{I}_2 \frac{\sin^4 f_2}{r^2} \right) dr. \quad (27)$$

There is considerably more choice than before in how to minimise this. One should consider all pairs  $N_1, N_2$  whose sum is  $B$ , then find rational maps that minimise  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , and finally find the profiles  $f_1(r)$  and  $f_2(r)$ , allowing  $r_0$  to be variable. We have not carried out such a systematic analysis. Instead, we have considered those pairs of maps  $R_1(z)$  and  $R_2(z)$  which have a high degree of symmetry, and which appear to fit well together. Our aim is to obtain a field with a low, but possibly not optimal, value of the energy (27). We have then relaxed this field numerically to find a true solution of the Skyrme equation, usually with the same symmetry. We have used this approach for baryon numbers  $B = 12, 13$  and  $14$ . In each of these cases there is the possibility of a cubically symmetric solution, rather similar to part of the Skyrme crystal. We describe these in turn.

### **$B = 12$**

There are various attractive choices for  $N_1$  and  $N_2$ . Choosing  $N_1 = N_2 = 6$ , with the rational map (22), gives a rather low symmetry. More interesting is  $N_1 = 3$  and  $N_2 = 9$ , where there are tetrahedrally symmetric maps. However, the most successful choice is  $N_1 = 5$  and  $N_2 = 7$ . One could use the optimal single-shell maps given earlier (for  $B = 5, 7$ ), but they have low combined symmetry. Better is to combine the maps with cubic symmetry mentioned earlier

$$R_5(z) = \frac{z(z^4 - 5)}{-5z^4 + 1}, \quad R_7(z) = \frac{-7z^4 - 1}{z^3(z^4 + 7)}. \quad (28)$$

These maps both have the tetrahedral symmetries (17) and (18), as well as the  $90^\circ$  rotation symmetry  $R(iz) = iR(z)$ .

We have calculated the optimal profile functions and optimal  $r_0$  for this pair of maps, obtaining an energy  $E = 1.30$ . Then, with this as a starting point, we have numerically relaxed the field to obtain a cubically symmetric, smooth solution of the Skyrme equation with energy  $E = 1.15$ . Its shape is shown in figure 2a). Note that the figure does not exhibit a two-shell structure. After relaxation, the inner and outer shells coalesce. We shall return to this below.

Battye and Sutcliffe have also studied the  $B = 12$  Skyrmion [3]. They have found the optimal single-shell rational map to use in (6). This map has only tetrahedral symmetry  $T_d$ , and gives an energy  $E = 1.102$ . They also relax their field to seek a true solution. This has the same tetrahedral symmetry, and energy  $E = 1.086$ . The cubically symmetric solution is not the true Skyrmion, but probably a saddle point.

**$B = 13$** 

To construct a cubically symmetric  $B = 13$  Skyrme field one might try a three-shell structure, with baryon numbers  $1 + 7 + 5$  from the centre outward, combining the map  $R_1(z) = z$  with the maps in (28). However, this cannot easily be implemented, because of the large size and energy of the initial configuration. Instead, we have constructed the solution, which resembles part of the Skyrme crystal, by relaxing a configuration made from a single Skyrmion and twelve nearest neighbours. It is shown in figure 2b), and has energy  $E = 1.09$ .

Battye and Sutcliffe have also investigated the  $B = 13$  case, using the single-shell ansatz. The rational map minimizing  $\mathcal{I}$  has  $D_{4d}$  symmetry. However, there is a cubically symmetric map with a slightly larger value of  $\mathcal{I}$ , and using this gives a field which looks almost identical to figure 2b). We conclude that there is a cubically symmetric  $B = 13$  solution, with  $U = -1$  at the centre, which can be found starting in several ways. However, it appears that the solution with  $D_{4d}$  symmetry is the true Skyrmion.

 **$B = 14$** 

Again we seek a cubically symmetric solution. We do this by taking a two-shell ansatz, using the dodecahedral rational maps of degree 7. The inner shell map is (24) with  $b = 7/\sqrt{5}$ , the outer shell uses the same map with  $b = -7/\sqrt{5}$ . Together, these maps (at different radii) only possess  $T_h$  symmetry, but if they can be made to coalesce, then there is cubic symmetry, because a  $90^\circ$  rotation transforms one into the other. Optimizing the profile functions gives an energy  $E = 1.39$ . Further relaxation produces a solution with  $T_h$  symmetry, and nearly cubically symmetric, with  $E = 1.14$ . Its form is shown in figure 2c). This again looks like part of the Skyrme crystal; this time what one would obtain by taking the six nearest neighbours and eight next-nearest neighbours surrounding a hole in the crystal, and relaxing the field.

The optimal single-shell structure with  $B = 14$  is quite different [3]. The rational map has only  $D_2$  symmetry, and gives an energy  $E = 1.103$ . The field also differs because  $U = -1$  at the centre. Relaxation of the solution will give a lower energy, but this has not yet been done.

**5. Interpretation of the Two-Shell Ansatz**

For certain profile functions  $f_1$  and  $f_2$ , the two-shell rational map ansatz describes a Skyrmion of baryon number  $N_1$  inside a Skyrmion of baryon number  $N_2$ , approximately. However, as the field relaxes, it changes its character. Consider a radial line ( $z$  fixed), and the field values  $U$  at the points along it where  $f_1 = 3\pi/2$  and where  $f_2 = \pi/2$ . If these values are close, then the field between can be relaxed to be approximately constant, which makes the energy low in this direction. Conversely, if the field values are antipodal (on  $SU(2)$ ), then the field gradient between them, and hence the energy, is large in this direction. In fact the winding

of the field along this radial line indicates that there is a  $B = 1$  Skyrmion in this direction.

Now, antipodal field values occur on this line if  $R_1(z) = R_2(z)$ . (The rational map values are the same, but  $\sin f_1 = -1$ ,  $\sin f_2 = 1$ .) Thus the two-shell rational map ansatz produces a configuration which can be interpreted as a superposition of  $B = 1$  Skyrmions located at  $r = r_0$  and in those directions  $z$  which solve the equation

$$R_1(z) = R_2(z). \quad (29)$$

Writing  $R_1(z) = \frac{p_1(z)}{q_1(z)}$  and  $R_2(z) = \frac{p_2(z)}{q_2(z)}$ , this becomes

$$p_1(z)q_2(z) - p_2(z)q_1(z) = 0 \quad (30)$$

which is a polynomial equation of degree  $N_1 + N_2$ , with  $N_1 + N_2$  solutions. So the number of Skyrmions one finds by solving (30) is precisely the total baryon number. The relative orientation of these Skyrmions has not yet been determined.

Equation (30) has a particularly symmetric form for maps we have been considering. For  $B = 12$  it reduces to

$$z^{12} - 33z^8 - 33z^4 + 1 = 0, \quad (31)$$

the Klein polynomial for the edges of a cube. For  $B = 14$  it reduces to

$$z(z^4 - 1)(z^8 + 14z^4 + 1) = 0, \quad (32)$$

the product of the Klein polynomials for the faces and vertices of a cube (one root is  $z = \infty$ ). This is what one anticipates based on the analogy with the Skyrme crystal.

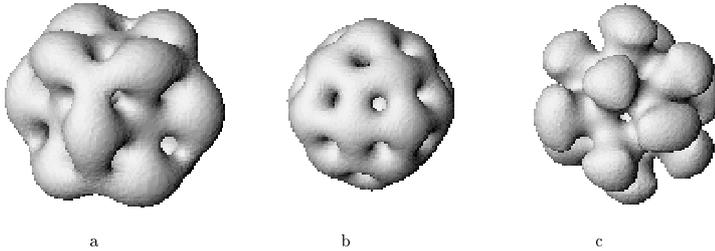


FIGURE 2. Surfaces of constant baryon density for the following solutions: a)  $B = 12$  with cubic symmetry; b)  $B = 13$  with cubic symmetry; c)  $B = 14$  two-shell with near cubic symmetry.

## 6. Conclusions

We have discussed an ansatz for Skyrme fields, based on rational maps, which allows the construction of good approximations to several Skyrmions. We have also discussed a two-shell rational map ansatz as an approach to construct multi-shell

Skyrmion solutions. These ansätze are good starting points to construct solutions with certain symmetries. We have studied other examples of the two-shell rational map ansatz than the ones described here and in most cases the configuration relaxes to a single shell solution. Two-shell solutions were only found for  $B = 14$ , as we have shown, and also when using a rational map of degree 17, together with another of lower degree. To construct two-shell solutions with a relatively low energy, the outside shell must be large enough to contain a smaller shell inside it. Although the solutions we have found using the two-shell ansatz are not minimal energy Skyrmions, their energies are not much greater, and we believe that for higher baryon numbers the minimal energy solutions will exhibit a multi-shell structure.

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