A Free Boundary Problem: Contributions from Modern Analysis

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Abstract. We exemplify the role of Free Boundary Problems as an important source of ideas in modern analysis. With the help of a model problem we illustrate the use of analytical, algebraic and geometrical techniques obtaining uniqueness of weak solutions via the use of entropy inequalities, existence through nonlinear semigroup theory, and regularity using a method, called intrinsic scaling, based on interpreting a partial differential equation in a geometry dictated by its own structure.

1. Introduction

In this contribution to the mini-symposium on Free Boundary Problems (FBPs) we use a model problem to support the idea that this is an important topic in modern analysis, both because of the mathematical questions it raises and of the variety of techniques it employs to produce interesting answers.

In a brief definition we can say that a FBP is a boundary value problem defined in a domain that is not given *a priori*, thus being part of the unknown. This models a feature that is common to many physical phenomena, and it comes with no surprise that the main motivation to study FBPs lies in absolutely practical matters. But our concern here is different and somehow nonstandard. We will not use the physical motivations and the successful practical achievements to justify the study of FBPs; we'll try to illustrate their strength and beauty as a modern topic in mathematical analysis. Although their origin can be traced back to the 19th century, FBPs only flourished as mathematical problems in the late 1960's and 1970's, mainly due to the systematic approach to existence provided by the theory of variational inequalities (cf. [3]). Since then the joint effort of an enormous number of mathematicians shed light into many difficult questions and opened up several new directions of research in analysis. We present here, in a descriptive form, a model problem that shares some of the aforementioned characteristics and address three main issues to illustrate it (the existence of a weak solution, its uniqueness and regularity) using three different methods in analysis. The existence is obtained from nonlinear semigroup theory, one of the most successful modern attempts of *algebrizing* analysis. The uniqueness is based on a typically *analytical* approach to PDEs, consisting on the establishment of integral inequalities related to the concept of entropy solution and using them as a tool to prove a contraction property. The regularity, namely the continuity of the weak solution, follows from a technique, called intrinsic scaling, that provides an adequate setting to understand what we can heuristically summarize in the sentence (cf. [7]): “the equation behaves in its own *geometry* like the heat equation”. The other basic question commonly attached to a FBP, the regularity of the free boundary, remains in this case a challenging open problem.

1.1. The model problem

The model problem we have in mind describes a phase transition at fixed temperature for an homogeneous material that diffuses in a nonlinear way. The free boundary is the interface between the two phases and we assume it moves according to a Stefan type condition (cf. [11]). Consider the maximal monotone graph γ defined by

$$\gamma(s) = s + \lambda H(s) \ , \qquad H(s) = \begin{cases} 0 & \text{if} \quad s < 0 \\ [0,1] & \text{if} \quad s = 0 \\ 1 & \text{if} \quad s > 0 \end{cases} \ ,$$

where λ stands for the latent heat of the phase transition, i.e. the amount of energy it requires to take place, and H is the Heaviside graph. The two main physical features of the phenomenon, the phase transition and the nonlinear diffusion, are captured by the PDE

$$\partial_t v - \Delta_p u - \nabla \cdot V(u) = f \ , \quad v \in \gamma(u) \tag{1}$$

through the maximal monotone graph γ and the nonlinear degenerate operator $\Delta_p u := \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right)$, the so called p-Laplacian. We denote with u the temperature and v the enthalpy, with $V(u)$ a temperature dependent velocity field and with f a reaction term. For a derivation of (1) from the principles of continuum mechanics under suitable constitutive assumptions see, e.g., [12]. We attach to the equation an homogeneous Dirichlet boundary condition at the fixed boundary of the space-time domain $Q = \Omega \times (0,T)$ and an initial condition

$$u = 0 \ \text{ on } \ \partial\Omega \times (0,T) \qquad \text{and} \qquad v(0) = v_0 \ \text{ in } \ \Omega \tag{2}$$

and formulate the problem weakly performing the usual multiplication by test functions and formal integration by parts. This leads to the following

Definition 1.1. *A weak solution of* (1)-(2) *is a pair of functions*

$$(u,v) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \times L^\infty(Q)$$

such that $v \in \gamma(u)$, a.e. in Q, and, for all testing functions φ,

$$-\int\!\!\int_Q v\, \partial_t \varphi + \int\!\!\int_Q \Big(|\nabla u|^{p-2} \nabla u + V(u) \Big) \cdot \nabla \varphi = \int\!\!\int_Q f\, \varphi + \int_\Omega v_0\, \varphi(0) \, .$$

Remark 1.2. *Observe that since $u = \gamma^{-1}(v)$ and γ^{-1} is continuous, we have,* a fortiori, *$u \in L^\infty(Q)$.*

Remark 1.3. *Note that in this weak or variational formulation the free boundary is (hidden or) implicit in the nonlinearity $\gamma(u)$. The price to be paid is the presence of the singularity "$\gamma'(0) = \infty$".*

We introduce some notation and the main assumptions concerning the data of the problem. The set $\Omega \in \mathbf{R}^n$ is a domain with a smooth boundary and $T > 0$ is a real number. The inverse of γ, which is non decreasing continuous function in $\mathbf{R}$, is denoted by $\varphi \equiv \gamma^{-1}$; we have $\varphi(0) = 0$. We denote with Sign_0^+ the discontinuous function corresponding to the choice $\mathrm{Sign}_0^+(0) = 0$. Concerning the convective term, that we took independent of the jumps of γ, i.e., depending only on the temperature, we assume that $V : \mathbf{R} \longrightarrow \mathbf{R}^N$ is Lipschitz, with $V(0) = \mathbf{0}$. The initial data is taken $\gamma(u_0) \ni v_0 \in L^\infty(\Omega)$ and such that

$$\exists\, M > 0 \quad : \quad \|u_0\|_{L^\infty(\Omega)} \leq M\,. \tag{3}$$

2. Uniqueness Via Entropy Inequalities

The basic idea in the proof of the uniqueness is to show that weak solutions satisfy certain integral inequalities, called *entropy inequalities* due to the fact that they are inspired in the notion of entropy solution introduced by S. Kruzkov in the context of quasilinear first-order equations. The word entropy had its roots in gas dynamics since the inequalities model, in that setting, the requirement of increasing entropy for shock waves. With the entropy inequalities at hand it is relatively easy to obtain a contraction property in L^1 and the uniqueness as an obvious corollary. This section is based on the article [9]. We start with

Definition 2.1. *An entropy solution of* (1)–(2) *is a weak solution that additionally satisfies the (entropy) inequalities*

$$\iint_Q \mathrm{Sign}_0^+(v-k)\Big\{(v-k)\,\xi_t - \big(|\nabla u|^{p-2}\nabla u + V(u)\big)\cdot\nabla\xi\Big\}$$
$$\geq -\int_\Omega (v_0-k)^+\,\xi(0) - \iint_Q \mathrm{Sign}_0^+(v-k)\,f\,\xi \tag{4}$$

for any $(k,\xi) \in \mathbf{R} \times \Big[L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap W^{1,1}\big(0,T;L^\infty(\Omega)\big)\Big]$ *such that* $\xi \geq 0$, $\xi(T) = 0$*; and any* $(k,\xi) \in \mathbf{R}_0^+ \times \Big[L^p\big(0,T;W^{1,p}(\Omega)\big) \cap W^{1,1}\big(0,T;L^\infty(\Omega)\big)\Big]$ *such that* $\xi \geq 0$*,* $\xi(T) = 0$.

The main step in the proof of the uniqueness is

Proposition 2.2. *Every weak solution of* (1)–(2) *is an entropy solution.*

Remark 2.3. *The entropy inequalities can basically be obtained by choosing as test function in the definition of weak solution the Steklov average of (an extension of) the function*

$$\Phi = H_\epsilon\Big(\varphi(v) - \varphi(k)\Big)\,\xi\,, \tag{5}$$

where H_ϵ is the approximation of the Heaviside graph defined by

$$H_\epsilon(r) = \min\left(\frac{r^+}{\varepsilon}, 1\right).$$

The main issue concerning this choice is whether Φ vanishes on the boundary of the domain or not. It is clear that if ξ vanishes on the boundary, the same happens with Φ but if that is not the case then we must have $H_\epsilon(-\varphi(k)) = 0$, i.e. $\varphi(k) \geq 0$, which follows if and only if $k \geq 0$. This explains a posteriori *the reason for choosing k and ξ in the above classes.*

We give now a flavor of how the proof of **proposition 2.2** carries through. We first derive the entropy inequalities for constants k such that $\varphi(k) \notin [0, \lambda]$, which is, say, the favorable case. Then we establish a "filling" lemma which says that if the entropy inequalities are valid for $k = 0$ and $k = \lambda$, then they are also valid for any $k \in [0, \lambda]$ and are left to prove the inequalities at the extreme points of this interval. These are relatively easy to handle if we can "approximate" the extreme points with k's for which the inequalities already hold; this is clearly possible in the case $(k, \xi) \in \mathbf{R} \times \Big[L^p\big(0, T; W_0^{1,p}(\Omega)\big) \cap W^{1,1}\big(0, T; L^\infty(\Omega)\big)\Big]$. But when ξ does not vanish on the boundary we must choose $k \geq 0$ and we have problems since we can not approximate $k = 0$ from the left; this also prevents us from using the filling lemma to cover the case $k \in (0, 1)$ since it requires the entropy inequality at $k = 0$. The key idea is then to deal with functions z (instead of constants k) such that $\varphi(z)$ vanish on the boundary $\partial\Omega \times (0, T)$, and to establish more general entropy inequalities for this case. Then, with the help of a strong maximum principle, we are able to choose a convenient sequence z_n such that $z_n < 0$ and $z_n \to 0$, thus approximating $k = 0$ in this way. Now z_n may be negative since it is the fact that $\varphi(z_n)$ vanishes on the boundary that guarantees that (5) is a good test function. From **proposition 2.2** we obtain

Theorem 2.4. *Let $f_1, f_2 \in L^1(Q)$ and $v_{01}, v_{02} \in L^1(\Omega)$. If v_i $(i = 1, 2)$ are the corresponding weak solutions of* (1)–(2)*, then, for a.e. $t \in (0, T)$,*

$$\int_\Omega \Big(v_1(t) - v_2(t)\Big)^+ \leq \int_\Omega \big(v_{01} - v_{02}\big)^+ + \int_0^t \int_\Omega \big(f_1 - f_2\big)^+ .$$

The proof is outlined in [9] and is a simple modification to the nonlinear case of a result in [4]. It makes essential use of the entropy inequalities to derive the contraction property. The uniqueness is an obvious corollary.

Remark 2.5. *It is very important for the uniqueness result that the convective term is independent of the jumps of γ, i.e., that it depends only on the temperature and not on the enthalpy. We now present a counterexample to the uniqueness in this more general situation. Consider the one-dimensional problem in $(t, x) \in Q = (0, 1) \times \Omega$, $\Omega = (0, 1)$*

$$v_t = u_{xx} + v_x \qquad u_{|\partial\Omega} = 0 \qquad v(0) = 1 \ .$$

A weak solution is a pair $v \in L^\infty(Q)$, $u \in L^2\big(0,1;H_0^1(\Omega)\big)$, such that $v \in \gamma(u)$, and solving the equation in the sense of distributions:

$$-\int_0^1\int_0^1 v\ \xi_t + \int_0^1\int_0^1 u_x\ \xi_x + \int_0^1\int_0^1 v\ \xi_x = \int_0^1 \xi(0) \quad , \quad \forall \xi \in \mathcal{T}\,.$$

It is obvious that the pair $(v_1,u_1) \equiv \big(1,\gamma^{-1}(1)\big)$ is a weak solution. Let's construct a different one; take the $\mathcal{C}^1$ function F defined in $[0,2]$ by

$$F(s) = \begin{cases} 1 & \text{if} \quad 0 \le s \le 1 \\ 1-(s-1)^2 & \text{if} \quad 1 < s \le 2 \end{cases}$$

and put $v_2(t,x) = F(t+x)$. We have $(v_2)_t(t,x) = F'(t+x) = (v_2)_x(t,x)$ and $v_2(0,x) = F(x) = 1$ because $x \in (0,1)$. Since $0 \le F \le 1$, we conclude that $\big(v_2,\gamma^{-1}(v_2)\big)$ is also a weak solution and it is of course different from (v_1,u_1). So there is no uniqueness for this problem.

3. Existence Via Nonlinear Semigroup Theory

In this section we denote the temperature with $\varphi(v)$ instead of u. Consider the stationary problem

$$\begin{cases} v \in L^\infty(\Omega), \quad \varphi(v) \in W_0^{1,p}(\Omega) \\ v - \Delta_p\,\varphi(v) - \nabla\cdot V(\varphi(v)) = h \quad \text{in} \quad \mathcal{D}'(\Omega) \quad . \end{cases} \tag{6}$$

In [9] we prove

Proposition 3.1. *Let $h \in L^\infty(\Omega)$. Then there exists a unique weak solution v of (6). Moreover $\|v\|_\infty \le \|h\|_\infty$ and, for any $h_i \in L^\infty(\Omega)$, $i=1,2$, and v_i the corresponding weak solutions,*

$$\big\|(v_1-v_2)^+\big\|_1 \le \big\|(h_1-h_2)^+\big\|_1 \quad .$$

Now define in $L^1(\Omega)$ an operator A by

$$Az = -\Delta_p\,\varphi(z) - \nabla\cdot V(\varphi(z)) \ \text{ in } \mathcal{D}'(\Omega)$$

with domain

$$D(A) = \Big\{ z \in L^\infty(\Omega) \ : \ \varphi(z) \in W_0^{1,p}(\Omega) \ \text{ and } \ Az \in L^1(\Omega) \Big\} \ .$$

It follows from **proposition 3.1** that A is T-accretive and that

$$R(I+\lambda A) \supseteq L^\infty(\Omega) \ , \quad \text{for all } \lambda > 0\,.$$

Moreover, the domain $D(A)$ of the operator A can be shown to be dense in $L^1(\Omega)$ (cf. [2]), i.e.,

$$\overline{D(A)}^{\,L^1(\Omega)} = L^1(\Omega)\,.$$

We can now use the general theory of evolution equations (cf. [5]) to obtain, for any $v_0 \in L^1(\Omega)$ and $f \in L^1(Q)$, a unique mild solution of

$$v \in C\big([0,T]; L^1(\Omega)\big) \quad : \quad \frac{\mathrm{d}v}{\mathrm{d}t} + Av = f \quad \text{on} \quad (0,T)\ , \qquad v(0) = v_0\,.$$

Finally, for data in L^∞, we can show (cf. [9, Prop. 4]) that this mild solution is also a weak solution of (1)–(2), thus obtaining

Theorem 3.2. *Given $v_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$ there exists at least one weak solution $\big(\varphi(v), v\big)$ of* (1)–(2).

Remark 3.3. *For data in L^1 we are not able to prove that the mild solution is a weak solution and so there's no existence result in this case. It remains the possibility that the mild solution coincides with another type of solution, maybe the solution in the renormalized sense. This topic is to be investigated in the future.*

Remark 3.4. *The existence was treated in* [12]*, in the more general setting of a convective term depending on the enthalpy v, using a regularization method and* a priori *estimates.*

4. Regularity Via Intrinsic Scaling

We finally consider the problem of the regularity of the weak solution, showing that the temperature u is in fact a continuous function. We only mention here the interior continuity obtained in [13], although the results hold up to the Dirichlet boundary (cf. [14]). Also, for the sake of simplicity of the arguments, the analysis is performed with the restrictions $V \equiv 0$ and $f \equiv 0$ but the techniques employed can easily be adapted to cover the general case. A more crucial restriction is the assumption, in this section, that $p > 2$.

The proof consists in showing that a sequence of uniformly bounded approximate solutions is equicontinuous, i.e., in deriving, at least implicitly, a modulus of continuity that is independent of the approximation. The approximated problem is obtained by regularization of the maximal monotone graph γ. Let $0 < \epsilon \ll 1$ and consider the function

$$\gamma_\epsilon(s) = s + \lambda H_\epsilon(s)\,,$$

where H_ϵ is a $\mathcal{C}^\infty$-approximation of the Heaviside function, such that

$$H_\epsilon(s) = 0 \quad \text{if} \quad s \le 0\ , \qquad H_\epsilon(s) = 1 \quad \text{if} \quad s \ge \epsilon\,,$$

$H'_\epsilon \ge 0$ and $H_\epsilon \to \mathrm{Sign}_0^+$ uniformly in the compact subsets of $\mathbf{R} \setminus \{0\}$, as $\epsilon \to 0$. The function γ_ϵ is bilipschitz and satisfies

$$1 \le \gamma'_\epsilon(s) \le 1 + \lambda L_\epsilon\ , \quad s \in \mathbf{R}\,, \tag{7}$$

with $L_\epsilon \equiv \mathcal{O}(\frac{1}{\epsilon})$ being the Lipschitz constant of H_ϵ. Taking also a uniformly bounded sequence of functions $u_{0\epsilon}$ that appropriately approximates the initial data, the approximated problem is defined as follows.

Definition 4.1. *An approximate solution of* (1)–(2) *is a function*

$$u_\epsilon \in H^1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q)$$

such that, for all testing functions φ,

$$-\iint_Q \gamma_\epsilon(u_\epsilon)\,\partial_t\varphi + \iint_Q |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla\varphi = \int_\Omega \gamma_\epsilon(u_{0\epsilon})\,\varphi(0)\,. \tag{8}$$

For each $\epsilon > 0$ this problem has a unique solution u_ϵ that satisfies a uniform estimate in L^∞ ($\|u_\epsilon\|_\infty \le M$; cf. (3)) and the sequence $(u_\epsilon, \gamma_\epsilon(u_\epsilon))$ converges to the solution of the original problem as $\epsilon \to 0$ (cf. [12]). It is also clear, from the available theory (cf. [7]), that the solution u_ϵ of the approximated problem is Hölder continuous.

The proof of the equicontinuity of (u_ϵ) is based on certain uniform local estimates of energy and logarithmic type. We will only mention here the energy estimates since that is enough to illustrate the main difficulties involved and to give a clear idea of the essential parts of the proof. Consider a cylinder in Q

$$(x_0,t_0) + Q(\tau,\rho) := K_\rho(x_0)\times(t_0-\tau,t_0)$$

and let $0 \le \zeta \le 1$ be a piecewise smooth cutoff function in $(x_0,t_0)+Q(\tau,\rho)$ such that

$$|\nabla\zeta| < \infty \qquad \text{and} \qquad \zeta(x,t) = 0\ ,\quad x \notin K_\rho(x_0)\,. \tag{9}$$

For the sake of simplicity and without loss of generality, we will be restricted to cylinders that are centered at the origin $(0,0)$, the changes being obvious in the case the center is a point (x_0,t_0). The energy estimates are obtained for the truncated functions $(u_\epsilon - k)_\pm$ choosing $\varphi = \pm(u_\epsilon - k)_\pm\zeta^p$ in (8). We only mention the negative case.

Proposition 4.2. *Let u_ϵ be a solution of the approximated problem and $k < M$. There exists a constant $C > 0$, that is independent of ϵ, such that for every cylinder $Q(\tau,\rho) \subset Q$,*

$$\sup_{-\tau<t<0}\int_{K_\rho\times\{t\}} (u_\epsilon-k)_-^2\zeta^p + \int_{-\tau}^0\int_{K_\rho}\big|\nabla(u_\epsilon-k)_-\zeta\big|^p$$
$$\le C\int_{-\tau}^0\int_{K_\rho}(u_\epsilon-k)_-^p|\nabla\zeta|^p + C\int_{K_\rho\times\{-\tau\}}(u_\epsilon-k)_-\zeta^p + C\int_{-\tau}^0\int_{K_\rho}(u_\epsilon-k)_-\zeta^{p-1}\partial_t\zeta\,.$$

Once these estimates are obtained, the problem becomes a problem in analysis and we can forget the PDE; they are essential to set forward an iterative argument consisting of showing that, for every point $(x_0,t_0) \in Q$, we can find a sequence of nested and shrinking cylinders $(x_0,t_0)+Q(\tau_n,\rho_n)$, such that, as the cylinders shrink to the point, the essential oscillation of each function θ_ϵ in the cylinders converges to zero; and this in a way that is qualitatively independent of ϵ. The iterative argument was introduced for strongly elliptic equations by DeGiorgi in [6]and later adapted by the Russian school to the parabolic case (cf. [10]). But it was essential in the argument that the equation was nondegenerate so that the integral norms appearing in the energy estimates were homogeneous.

This is not the case in **proposition 4.2**: the presence of the powers p and 1 jeopardizes the homogeneity in the energy estimates and the recursive process (leading to the conclusion sought) itself. The power p clearly comes from the p-Laplacian and we can say that the nonlinear diffusion at the physical level produces a degeneracy at the PDE level which in turn leaves its trace at the analytical level in the form of such a power in the integral norms. The power 1 is, in the same spirit, the trace of the phase transition and the singularity in the PDE. Since this is not so obvious let's reproduce the part of the estimate responsible for the appearance of the power 1, which is the one that involves the time derivative. The crux of the matter is to estimate uniformly the regularization of the maximal monotone graph:

$$-\int_{-\tau}^{t}\int_{K_\rho}\partial_t\big[\gamma_\epsilon(u_\epsilon)\big]\,\Big((u_\epsilon-k)_-\zeta^p\Big)=\int_{K_\rho}\int_{-\tau}^{t}\partial_t\bigg(\int_0^{(u_\epsilon-k)_-}\gamma_\epsilon'(k-s)s\,\mathrm{d}s\bigg)\zeta^p$$
$$\geq\frac{1}{2}\int_{K_\rho\times\{t\}}(u_\epsilon-k)_-^2\zeta^p-C\int_{K_\rho\times\{-\tau\}}(u_\epsilon-k)_-\zeta^p-C\int_{-\tau}^{t}\int_{K_\rho}(u_\epsilon-k)_-\zeta^{p-1}\partial_t\zeta\,,$$

where C is a constant depending only on p, λ and M, the uniform bound in L^∞ for u_ϵ. The inequality is justified, recalling (7), by

$$\int_0^{(u_\epsilon-k)_-}\gamma_\epsilon'(k-s)s\,\mathrm{d}s\geq\int_0^{(u_\epsilon-k)_-}s\,\mathrm{d}s=\frac{1}{2}(u_\epsilon-k)_-^2$$

and

$$\int_0^{(u_\epsilon-k)_-}\gamma_\epsilon'(k-s)s\,\mathrm{d}s\leq(u_\epsilon-k)_-\int_0^{(u_\epsilon-k)_-}\gamma_\epsilon'(k-s)\,\mathrm{d}s$$
$$=(u_\epsilon-k)_-\Big[\gamma_\epsilon(k)-\gamma_\epsilon(u_\epsilon)\Big]\leq 2(M+\lambda)\,(u_\epsilon-k)_-\,.$$

The key idea to overcome the difficulty presented by the inhomogeneity was introduced by DiDenedetto (cf. [7] and [8] for an account of the theory) in the nonsingular case ($\gamma(s)\equiv s$) and consists essentially in looking at the equation in its own geometry, i.e., in a geometry dictated by its degenerate structure. This amounts to rescale the standard parabolic cylinders by a factor depending on the oscillation of the solution. This procedure, which can be called accommodation of the degeneracy, allows one to recover the homogeneity in the energy estimates written over these rescaled cylinders and carry on with the proof. We can say heuristically that the equation behaves in its own geometry like the heat equation. In the present singular-degenerate case, no rescaling permits the compatibility of the three powers involved so we use the geometry of the nonsingular case to deal with the degeneracy and pay the price of a dependence on the oscillation in the various constants that are determined along the proof. Owing to this fact we are no longer able to exhibit a modulus of continuity but only to define it implicitly independently of the regularization. This is enough to obtain the equicontinuity of the approximations and a continuous solution for the original problem; but the Hölder continuity, that holds in the nonsingular case, is lost. Let's briefly describe the procedure. From now on we will drop the ϵ in u_ϵ. Consider a point $(x_0, t_0) \in Q$ and, by translation and to simplify, assume $(x_0, t_0) = (0, 0)$. Consider $R > 0$ such that $Q(R^{p-1}, 2R) \subset Q$, define

$$\omega := \text{ess osc}_{Q(R^{p-1},2R)} \ u$$

and construct the cylinder

$$Q(a_0 R^p, R) \ , \quad \text{with} \quad a_0 = \left(\frac{\omega}{A}\right)^{2-p}$$

where the number A is to be chosen of the form

$$A = 2^{s_3} \ , \qquad \text{with} \qquad s_3 > C\ \omega^{-\alpha} \ , \quad \alpha = \frac{2(p+1)(N+p)}{p} \, . \tag{10}$$

Note that for $p = 2$, i.e. in the nondegenerate case, $a_0 = 1$ and these are the standard parabolic cylinders. We will assume, without loss of generality, that $\omega < 1$ and also that

$$\frac{1}{a_0} = \left(\frac{\omega}{A}\right)^{p-2} > R$$

which implies that $Q(a_0 R^p, R) \subset Q(R^{p-1}, 2R)$ and the relation

$$\text{ess osc}_{Q(a_0 R^p,R)} \ u \le \omega \tag{11}$$

which will be the starting point of the iteration process. We now consider subcylinders of $Q(a_0 R^p, R)$ of the form

$$(0, t^*) + Q(dR^p, R) \ , \quad \text{with} \quad d = \left(\frac{\omega}{2}\right)^{2-p}$$

that are contained in $Q(a_0 R^p, R)$, since $A > 2$ and if

$$\left(2^{p-2} - A^{p-2}\right) \frac{R^p}{\omega^{p-2}} < t^* < 0 \, .$$

The proof follows from the analysis of two complementary cases and the achievement of the same type of conclusion for both. We can briefly describe them in the following way: in the first case, we assume that there is a cylinder of the type $(0, t^*) + Q(dR^p, R)$ where θ is essentially away from its infimum. We show that going down to a smaller cylinder the oscillation decreases by a small factor that we can exhibit and that depends on the oscillation. If that cylinder can not be found then θ is essentially away from its supremum in all cylinders of that type and we can add up this information to reach the same conclusion as in the previous case. We summarize:

Lemma 4.3. *There exists a constant $\sigma = \sigma(\omega) \in (0, 1)$, that depends only on the data and ω, such that*

$$\text{ess osc}_{Q\left(d(\frac{R}{8})^p, \frac{R}{8}\right)} \ u \ \le \ \sigma(\omega)\ \omega \, .$$

Note the dependence of σ on the oscillation which is responsible for the loss of the Hölder continuity. Still, with this result at hand, we can define recursively two sequences of positive real numbers $(\omega_n)_n$ and $(R_n)_n$ and obtain

Proposition 4.4. *The sequences $(\omega_n)_n$ and $(R_n)_n$ are decreasing sequences that converge to zero. Moreover, for all $n = 0, 1, 2, \ldots$,*

$$Q_{n+1} \subset Q_n \qquad \textit{and} \qquad \operatorname{ess\,osc}_{Q_n} u \leq \omega_n \,.$$

An immediate consequence is that we can choose a continuous representative for each u_ϵ out of its equivalence class and implicitly obtain an interior modulus of continuity, i.e., for each $K \subset Q$, a continuous and nondecreasing function $F_K \colon \mathbf{R}^+ \to \mathbf{R}^+$, depending only on the data and K, such that

$$\left| u_\epsilon(x,t) - u_\epsilon(x',t') \right| \leq F_K\Big(|x - x'| + |t - t'|^{\frac{1}{p}} \Big) \,.$$

Since this modulus of continuity is independent of ϵ, we find that u is locally continuous as a consequence of Ascoli's theorem.

Theorem 4.5. *The function u in the definition of weak solution for* (1)–(2) *is locally continuous.*

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References

[1] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., **183** (1983), 311–341.

[2] Ph. Bénilan and P. Wittbold, *On mild and weak solutions of elliptic-parabolic problems*, Adv. Diff. Eqs., **1** (1996), 1053–1073.

[3] H. Brézis and F. Browder, *Partial differential equations in the 20th century*, Adv. Math., **135** (1998), no. 1, 76–144.

[4] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rat. Mech. Anal., **147** (1999), 269–361.

[5] M. Crandall and T. Liggett, *Generation of semi-groups of nonlinear transformations in general Banach spaces*, Amer. J. Math., **93** (1971), 265–298.

[6] E. DeGiorgi, *Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat., Ser. 3, **3** (1957), 25–43.

[7] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[8] E. DiBenedetto, *Parabolic equations with multiple singularities*, in: Proceedings of Equadiff 9, Brno, 1997.

[9] N. Igbida and J. M. Urbano, *Uniqueness for nonlinear degenerate problems,* Preprint, (2000).

[10] O. Ladyzenskaja, V. Solonnikov and N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, A.M.S. Transl. Monog. **23**, Providence, R.I., 1968.

[11] A. Meirmanov, *The Stefan Problem*, Walter de Gruyter, Berlin, 1992.

[12] J. M. Urbano, *A free boundary problem with convection for the p-Laplacian,* Rend. Mat. Appl. (VII), **17** (1997), 1–19.

[13] J. M. Urbano, *Continuous solutions for a degenerate free boundary problem*, Ann. Mat. Pura Appl., to appear.

[14] J. M. Urbano, *A singular-degenerate parabolic problem: Regularily up to the Dirichlet boundary*, in: Free Boundary Problems: Theory and Applications I, GAKUTO International Series Mathematical Sciences and Applications, **13** (2000), 399–410.

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