Inventiones math. 50, 75–90 (1978)

Inventiones mathematicae

On K_2 of Finite Dimensional Division Algebras Over Arithmetical Fields

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§ 0. Introduction

In this paper we try to "determine" the group $K_2(D)$, where D is a finite dimensional central division algebra over either a local field or a global (number or function) field F. Here the word "determine" means to reduce the determination to problems on commutative fields (e.g., the determination of $K_2(F)$ and certain Galois cohomological computations). As far as we know, only rudimentary results on this question can be found in the literature. We mention Harris and Stasheff [8], who proved that under certain conditions the group $K_2(D)$ contains a direct factor isomorphic to $K_2(F)$ in the case of a locally compact division algebra D over a p-adic F. Another general result is due to Keating [11], who compared Quillen's localization sequence of the global field F with that of D. A special example has been studied by Alperin and Dennis [1]. They computed the group $K_2(D)$ in the case of the Hamilton skew field $D|\mathbf{R}$, and in a certain sense their techniques are the starting point for our paper.

Let us briefly describe what we do. By general results of K-theory, there is a bimultiplicative pairing $F^* \times K_1(D) \to K_2(D)$. In the case of local or global fields we have $SK_1(D)=1$ by classical theorems of Matsushima-Nakayama [13] and Wang [25]; hence $K_1(D) \simeq \mathcal{N}_{D|F} := \operatorname{Im} RN_{D|F} \subseteq F^*$ ($RN_{D|F}$ is the reduced norm of D over F). This gives a homomorphism $F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F} \to K_2(D)$, and it turns out that under a slight technical condition (which is always valid for local or global fields) this map is even a symbol; that is, it vanishes on the subgroup $Z \subseteq F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F}$ generated by elements $\alpha \otimes (1-\alpha)$ ($\alpha \in F^*$, $1-\alpha \in \mathcal{N}_{D|F}$). Hence, if $Y(D|F) := (F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F})/Z$, we have a homomorphism

$$\psi\colon\ Y(D|F) \to K_2(D).$$

By Matsumoto's theorem on the presentation of K_2 of fields [12], there is a canonical map $Y(D|F) \to K_2(F)$, which is clearly an isomorphism if $\mathcal{N}_{D|F} = F^*$ (e.g., if F is p-adic, a global function field or a totally imaginary number field [7]). If F is a number field with real places (or if $F=\mathbf{R}$), then $\mathcal{N}_{D|F} = \{\alpha \in F \mid \alpha$

0020-9910/78/0050/0075/$03.20

positive at all real places where D is ramified} by Eichler's norm theorem [7]. In this case we do not know whether or not this map has a non-trivial kernel except if F has exactly one real place, in which case the map is split injective: $K_2(F) \simeq \langle\{-1, -1\}\rangle \times Y(D|F)$ (see also the footnote to 2.7). Now, if E is any splitting field of D of finite degree, we show that we have a commutative diagram

$$\begin{array}{ccc} & & K_2(D) \\ & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle \rho_{D|E}} \\ Y(D|F) & \longrightarrow & K_2(E) \end{array}$$

where the horizontal arrow is defined by the map $Y(D|F) \to K_2(F)$ composed with the natural map $K_2(F) \to K_2(E)$, and $\rho_{D|E}$ is the map $K_2(D) \to K_2(D \otimes_F E) \xrightarrow{\sim} K_2(E)$, where the latter isomorphism is given by the natural identification. If $E|F$ is Galois, then certainly $K_2(E)$ may be replaced by $K_2(E)^{\mathrm{Gal}(E|F)}$ in the diagram (cf. 2.4).

To prove the injectivity of ψ, now one only has to construct, for each $\xi \in Y(D|F)$, a splitting field of D of finite degree such that ξ has non-trivial image in $K_2(E)$. We solve this problem completely in case F is local or global, $\operatorname{char} F \neq 0$ (3.1, 3.6). In the case of a local field of $\operatorname{char} F = 0$, residue class characteristic p, our Theorem (3.1) has two gaps: one of them depends on the solution of the question of whether or not the p-primary part of the divisible subgroup of $K_2(F)$ is trivial (which is a conjecture of Tate and proven by him recently in many cases [22]), while the second occurs only in the case char $F=0$, $p=2$, $\sqrt{-1} \notin F$. Nevertheless, these local results yield global ones via the existence theorem of Grunwald-Wang together with the theorem of Hasse-Brauer-Noether, which give the injectivity of ψ for any D for instance in the case $F = \mathbf{Q}$ (3.4, 3.5). In the global function field case, our tools are Tate's Galois cohomological description of K_2 [21] together with a classical theorem of Tsen [23], which yields the existence of a cyclotomic splitting field $E|F$ of D. We get $Y(D|F) \simeq K_2(F) \simeq K_2(E)^{\mathfrak{g}}$ and therefore the result that $K_2(F) \simeq \psi K_2(F)$ is a direct factor of $K_2(D)$. Here $\mathfrak{g}$ denotes the Galois group $\mathrm{Gal}(E|F)$.

Our results are unpleasant in the general number field case. But it seems possible that the combination of our two methods will give satisfactory results at least in the totally real case, where Tate's cohomological description is as good as in the function field case. We hope to come back to this question later.

The question of whether or not ψ is surjective looks like a difficult problem. We have a general positive result only in the case of generalized quaternion skew fields over arbitrary fields, where we can prove that ψ is surjective whenever it is defined. This together with our discussion of injectivity gives the results of § 4. (The proof of the surjectivity of ψ we give here is essentially due to R.K. Dennis, who kindly permitted us to use it.)

§ 1. Some Properties of the Reduced Norm in K-Theory

Let A be an associative ring with unit. In the Steinberg group, $St(A)$, we denote by U_A the subgroup generated by elements $c(u, v) = c^A(u, v) = [h_{12}(u), h_{13}(v)]$ with

$u, v \in A^*$, $h_{ij}(u) \in St(A)$ defined as in [14, §5]. (If x, y are elements of any group we use the notations ${}^x y := x y x^{-1}$, $[x, y] := {}^x y \cdot y^{-1}$.) One easily verifies the following relations among the generators of U_A [18]:

0) $c(u, 1-u) = 1 \quad (u, 1-u \in A^*)$,

1) $c(u, v)\, c(v, u) = 1$,

2) $c(uv, w) = c({}^u v, {}^u w)\, c(u, w)$,

3) $c(u, vw) = c(u, v)\, c({}^v u, {}^v w)$.

The natural homomorphism $\phi = \phi_A$: $St(A) \to GL(A)$ induces the epimorphism $p = p_A$: $U_A \to [A^*, A^*]$ $(c(u, v) \mapsto [u, v])$, which shows that U_A is a central extension of $[A^*, A^*]$, and via ${}^x c(u, v) := c({}^x u, {}^x v)$ $(x \in A^*)$ one gets an action of A^* on U_A which fixes every element of kernel p, with respect to which p is an A^*-equivariant map.

As usual one defines $K_1(A) := \operatorname{cokernel} \phi_A$ and $K_2(A) := \operatorname{kernel} \phi_A$. Now, if α is an element of the centre of A^*, relation 3) together with the above remark on the action of A^* on U_A shows that the map $u \mapsto c(\alpha, u)$ $(u \in A^*)$ defines a homomorphism $A^* \to K_2(A)$, which clearly vanishes on $[A^*, A^*]$.

If $A = M_r(D)$ is the ring of $r \times r$-matrices with entries in a skew field D with centre F, the well-known properties of the Dieudonné determinant det: $A^* \to D^*/[D^*, D^*]$ give a canonical isomorphism of $K_1(A) = A^*/[A^*, A^*]$ with $K_1(D) = D^*/[D^*, D^*]$ which we also call det.

These considerations yield the following

1.1. **Lemma.** *Let D be a skew field with centre F, $A = M_r(D)$. Then the correspondence*

$$(\alpha, u) \mapsto c^A(\alpha \cdot 1_A, u) \qquad (\alpha \in F^*,\ u \in A^*)$$

defines a bimultiplicative pairing

$$F^* \times K_1(A) \to K_2(A).$$

(Therefore, by abuse of notation, we also use entries $u \in K_1(A)$ in expressions $c^A(\alpha \cdot 1_A, u)$.)

Now we make the general assumption that rings we consider are finite dimensional algebras over some fixed commutative field F. Let D be any division algebra over F and X a second F-algebra, not necessarily with centre F. Then clearly $D_X := D \otimes_F X$ is a free left D-module, and therefore we have a homomorphism $GL(D_X) \to GL(D)$ (induced by some choice of a D-basis of D_X) which gives a homomorphism $N_D^{D_X}$: $K_1(D_X) \to K_1(D)$ (independent of the D-basis chosen).

1.2. **Lemma.** *Let D be a central division algebra over F and let $E|F$ be a finite field extension with norm N_F^E. Then the following diagram commutes:*

$$\begin{array}{ccc} K_1(D_E) & \xrightarrow{RN_{D_E|E}} & E^* \\ \big\downarrow{\scriptstyle N_D^{D_E}} & & \big\downarrow{\scriptstyle N_F^E} \\ K_1(D) & \xrightarrow{RN_{D|F}} & F^*. \end{array}$$

Proof. The assertion is an application of the more or less well-known "tower formula of the reduced norm", which we cite without proof from [6, p. 28]:

If B is a simple F-subalgebra of a central simple F-algebra A and if E is in the centre of B then

$$t=\left(\frac{[A:F]}{[E:F][B:F]}\right)^{1/2}\in\mathbf{N}$$

and

$$RN_{A|F}(b)=N_{E|F}\cdot RN_{B|E}(b)^t \qquad (b\in B).$$

We choose $A=D\otimes_F M_{[E:F]}(F)\simeq M_{[E:F]}(D)$, $B=D_E=D\otimes_F E$ which is embedded into A by the correspondence

$$u\otimes\xi\mapsto u\otimes\mathscr{L}_\xi \qquad (u\in D,\ \xi\in E)$$

where $\mathscr{L}_\xi\in M_{[E:F]}(F)$ denotes the matrix of left multiplication by ξ on E with respect to some F-basis of E. Then $t=1$ and, if $b\in B$,

$$RN_{M_{[E:F]}(D)|F}(b)=N_{E|F}\circ RN_{D_E|E}(b).$$

Since the left-hand side is equal to $RN_{D|F}(\det b)$, the lemma follows.

The homomorphism $GL(D_X)\to GL(D)$ also induces the transfer homomorphism $\mathrm{Tr}_D^{D_X}$: $K_2(D_X)\to K_2(D)$ (independent of the chosen D-basis). We have the following

1.3. **Lemma.** *Let* $\alpha\in F^*$, $u\in D_X^*$. *Then*

i) *if* $X=M_r(F)$, *we have* $D_X\cong M_r(D)$, $K_2(D_X)=K_2(D)$ *and*

$$c^{D_X}(\alpha\cdot 1_{D_X}, u)=c^D(\alpha\cdot 1_D, \det u),$$

ii) *if* X *is arbitrary,*

$$\mathrm{Tr}_D^{D_X} c^{D_X}(\alpha\cdot 1_{D_X}, u)=c^D(\alpha, N_D^{D_X} u).$$

Proof. i) The obvious isomorphism $D_X\simeq M_r(D)$ gives $St(D_X)\simeq St(D)$, hence a natural isomorphism $K_2(D_X)\simeq K_2(D)$, which allows us to identify both groups. Now we have $u\equiv\mathrm{diag}(u_1, 1_D, \ldots, 1_D) \bmod E_r(D)=[GL_r(D), GL_r(D)]$ where $u_1\in D^*$ such that $\det u=u_1[D^*, D^*]$. Using the relations in $St(D_X)$ (as deduced in [14, §9]) and the fact that changing the entries of a commutator modulo the centre does not affect its value one easily derives the formula.

ii) Under the map

$$j\colon D_X\to D\otimes_F M_{[X:F]}(F)=M_{[X:F]}(D)=:A$$

(defined by $u\otimes x\mapsto u\otimes\mathscr{L}_x$ $(u\in D,\ x\in X)$) we have $j(\alpha)=\alpha\cdot 1_A$,

$$j(u)\equiv\mathrm{diag}(u_1, 1_D, \ldots, 1_D) \bmod E_{[X:F]}(D)=[A^*, A^*]$$

where $u_1\in D^*$ with $u_1[D^*, D^*]=\det(u\otimes\mathscr{L}_1)=N_D^{D_X}(u)$. Now the formula follows from i).

1.4. **Lemma.** *Let D be a central finite dimensional division algebra over F, and let $E|F$ be a splitting field for D. If $E|F$ is Galois, we denote by $\mathfrak{g}=\mathrm{Gal}(E|F)$ the Galois group. Then we have a canonical homomorphism $\rho=\rho_{D|E}$: $K_2(D)\to K_2(E)$ with $\mathrm{Im}\,\rho\subseteq K_2(E)^{\mathfrak{g}}$ in the Galois case.*

If i: $F\to E$ denotes the inclusion, one has, for $\alpha\in F^*$, $u\in D^*$,

$$\rho(c^D(\alpha,u))=c^E(i\alpha,\, i\,RN_{D|F}(u))=i_*\,c^F(\alpha, RN_{D|F}(u))$$

(here i_*: $K_2(F)\to K_2(E)$ is the map induced by i).

Proof. We define ρ to be the composition of the map $K_2(D)\to K_2(D\otimes_F E)$ (induced by $u\mapsto u\otimes 1$ $(u\in D)$) with the natural identification $K_2(D\otimes_F E)\xrightarrow{\sim} K_2(E)$. In the Galois case $D\otimes_F E$ is a $\mathfrak{g}$-module via ${}^{\sigma}(u\otimes v)=u\otimes {}^{\sigma}v$ $(\sigma\in\mathfrak{g})$. Therefore, fixing an isomorphism λ: $D\otimes_F E\xrightarrow{\sim} M_n(E)$ $(n=\mathrm{ind}\,D=\sqrt{[D:F]})$, $M_n(E)$ carries two $\mathfrak{g}$-module structures: the natural one and the one induced by λ. Both agree on E and hence by the theorem of Skolem-Noether the two operations of the element $\sigma\in\mathfrak{g}$ are the same modulo a certain inner automorphism of $M_n(E)$. Now, using the fact that the elements of kernel $p_{M_n(E)}=\mathrm{kernel}\,p_E=K_2(E)$ are fixed under the operation of $M_n(E)^*$, we conclude that both operations define the same $\mathfrak{g}$-structure on $K_2(E)$.

Hence clearly λ induces a map ρ: $K_2(D)\to K_2(E)^{\mathfrak{g}}$. A second application of the Skolem-Noether theorem shows the independence from the choice of λ.

The formula follows immediately from 1.3 i) and the definition of the reduced norm.

1.5. *Remark.* If we compose the transfer $\mathrm{Tr}_D^{D_X}$: $K_2(D_X)\to K_2(D)$ of 1.3 with the canonical map i_*: $K_2(D)\to K_2(D_X)$ induced by the embedding, we get $\mathrm{Tr}_D^{D_X}\circ i_*(\xi)=\xi^d$ $(\xi\in K_2(D),\ d=[X:F])$.

On the other hand, if D is a central division algebra over F of degree $d=[D:F]$, if i_*: $K_2(F)\to K_2(D)$ denotes the map induced by the embedding and i^*: $K_2(D)\to K_2(F)$ denotes the associated transfer ($i^*=\mathrm{Tr}_F^{F_D}$ in our former notation), we also have the formula $i_*\circ i^*(\eta)=\eta^d$ $(\eta\in K_2(D)$ [8]. This means that any divisible subgroup and any torsion element of order relatively prime to d of $K_2(D)$ (resp. $K_2(F)$) is contained in the image of i_* (resp. i^*).

§ 2. The Map ψ

Now, and for the rest of the paper, if we speak about a skew field D, we will always assume that it is of finite degree over its centre F. Then we have the reduced norm $RN_{D|F}$ and it is clear that kernel $RN_{D|F}\supseteq[D^*,D^*]$. In general it is not true that equality holds, as recent examples of V.P. Platonov and P.K.J. Draxl show. It may also happen that kernel $RN_{D|F}=[D^*,D^*]$, but that there exists a finite extension $E|F$ of fields such that kernel $RN_{D_E|E}\neq[D_E^*,D_E^*]$ (where $D_E=D\otimes_F E$) [6, p. 81].

We consider the following condition on the central finite dimensional division algebra D over F:

(A) For each finite field extension $E|F$ we assume kernel $RN_{D_E|E}=[D_E^*,D_E^*]$. This condition is fulfilled if

i) F is local, or global, or D is discretely valuated with commutative residue class field [13, 25].

ii) F is arbitrary and the index $\operatorname{ind} D = \sqrt{[D:F]}$ is not divisible by a square integer $\neq 1$ [25].

(A) implies $K_1(D) \simeq \mathcal{N}_{D|F} := \operatorname{Im} RN_{D|F} \subseteq F^*$; therefore, by 1.1, we have a bimultiplicative mapping $F^* \times \mathcal{N}_{D|F} \to K_2(D)$, or what amounts to the same thing, a homomorphism $\psi_0\colon F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F} \to K_2(D)$.

We need another more technical assumption which, as we will show, holds in the case of a local or global field.

(B) Let $n = \operatorname{ind} D = \sqrt{[D:F]}$ be the index of the central finite division algebra D over F. For each $\alpha \in F^*$ such that $1-\alpha \in \mathcal{N}_{D|F}$ and for every β in some fixed algebraic closure of F with $\beta^n = \alpha$ we assume $1-\beta \in \mathcal{N}_{D_E|E}$ where $E = F(\beta)$.

2.1. **Lemma.** (B) *is true if for each finite field extension $E|F$ the reduced norm $RN_{D_E|E}$ is surjective,* e.g. *if F is of type C_2' in the sense of* [20, II–19]. *(This happens for instance if F is local or global and D is ramified only at non-real places of F, by Eichler's norm theorem.)* (B) *is also true if F is a local or global field.*

Proof. The first assertion is trivial as is the second in case $F = \mathbf{R}$. Since it is easy to see that the reduced norm of a p-adic locally compact skew field is always surjective, we only have to consider the global case. But then Eichler's norm theorem [7] tells us that

$$\mathcal{N}_{D|F} = \{\alpha \in F^* \mid \alpha > 0 \text{ for every real ramified place of } D\}.$$

Now let $(\alpha, 1-\alpha) \in F^* \times \mathcal{N}_{D|F}$ and $E = F(\beta)$, $\beta^n = \alpha$. If there exists some embedding $v\colon E \to \mathbf{R}$ such that $D_E \otimes_E \mathbf{R}$ is ramified, then of course $D \otimes_F \mathbf{R}$ is ramified which implies $v(1-\alpha) > 0$. But then $v(1-\beta) > 0$ and hence $1-\beta \in \mathcal{N}_{D_E|E}$.

2.2. **Theorem.** *Let D be a central division algebra of index $n = \operatorname{ind} D$ over a field F with property* (A); *let $i_*\colon K_2(F) \to K_2(D)$ be the map induced by the embedding $i\colon F \to D$, and $i^*\colon K_2(D) \to K_2(F)$ the associated transfer ($i^* = \operatorname{Tr}_F^{FD}$ in the notation of* 1.3). *Then*

i) *for each $(\alpha, \beta) \in F^* \times \mathcal{N}_{D|F}$ we have*

$$\psi_0(\alpha \otimes \beta)^n = i_*(\{\alpha, \beta\}), \qquad i^* \circ \psi_0(\alpha \otimes \beta) = \{\alpha, \beta\}^n$$

(*here we use the common notation $\{\alpha, \beta\} = c^F(\alpha, \beta)$*).

ii) *if, additionally, D has property* (B), *ψ_0 is a "symbol" i.e. it vanishes on the subgroup $Z \subseteq F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F}$ generated by elements $\alpha \otimes (1-\alpha)$ with $(\alpha, 1-\alpha) \in F^* \times \mathcal{N}_{D|F}$.*

2.3. *Definition.* If D is a central division algebra over a field F with (A), (B), we define $Y(D|F) := (F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F})/Z$. The image of $\alpha \otimes \beta$ is denoted by (α, β). Further let

$$\psi\colon Y(D|F) \to K_2(D)$$

denote the homomorphism induced by ψ_0.

2.4. **Corollary.** *If D is a central division algebra over F satisfying* (A), (B), *and if $E \supset F$ is a splitting field for D, we have a commutative diagram*

$$\begin{array}{ccc} & & K_2(D) \\ & \overset{\psi}{\nearrow} & \big\downarrow \rho_{D|E} \\ Y(D|F) & \longrightarrow & K_2(E) \end{array}$$

where the horizontal map is induced by $(\alpha, \beta) \mapsto c^E(\alpha, \beta)$.

If $E \supset F$ is Galois with group $\mathfrak{g} = \mathrm{Gal}(E|F)$ we may replace $K_2(E)$ by $K_2(E)^{\mathfrak{g}}$.

If the canonical map $Y(D|F) \to K_2(F)$ is injective then kernel ψ has exponent $n = \mathrm{ind}(D)$.

This follows from 2.2 together with Matsumoto's theorem and 1.4.

2.5. **Corollary.** *Let F be a global field.*

i) *Cokernel ψ is finite.*

ii) *If the canonical map $Y(D|F) \to K_2(F)$ is injective, then kernel ψ is finite.*

Proof. 2.2 i) shows that it is enough to prove that i_* has finite kernel and cokernel. Now both groups are of exponent n^2; therefore we only have to show that they are finitely generated. But this follows easily from Keating's comparison of the localization sequence of D with that of F ([11, Th. 1]; see also [16]) together with Quillen's result that K_2 of the integers of F (resp. of a maximal order of D) is finitely generated [17, p. 197].

Proof of 2.2. i) The definition of ψ_0 says:

$$\psi_0(\alpha \otimes \beta) = c^D(\alpha, b) \qquad \text{where } \beta = RN_{D|F}(b) \ (\alpha \in F^*, \ \beta \in \mathcal{N}_{D|F}).$$

Now $RN_{D|F}(b^n) = RN_{D|F}(\beta)$; hence

$$\psi_0(\alpha \otimes \beta)^n = c^D(\alpha, b^n) = c^D(\alpha, \beta) = i_* c^F(\alpha, \beta) = i_*(\{\alpha, \beta\}),$$
$$i^* \psi_0(\alpha \otimes \beta) = i^* c^D(\alpha, b) = c^F(\alpha, RN\, b^n) = \{\alpha, \beta\}^n.$$

ii) Let $(\alpha, 1-\alpha) \in F^* \times \mathcal{N}_{D|F}$, and let $X^n - \alpha = \prod f_i(X)^{e_i}$ be a decomposition into pairwise relatively prime monic irreducible factors $f_i(X) \in F[X]$. In an algebraic closure of F we choose, for each i, a root α_i of $f_i(X)$; let $E_i = F(\alpha_i)$. Then it is easy to see

$$1 - \alpha = \prod f_i(1)^{e_i} = \prod N_F^{E_i}(1 - \alpha_i)^{e_i}.$$

Now, since (B) is assumed, we can choose $u_i \in D_i = D \otimes_F E_i$ with $RN_{D_i|E_i}(u_i) = 1 - \alpha_i$. Using i) and the fact that the index of D_i is a divisor of n, we get $c^{D_i}(\alpha_i, u_i)^n = 1$ which yields, by 1.3:

$$\begin{aligned} 1 &= \prod \mathrm{Tr}_D^{D_i}(c^{D_i}(\alpha_i, u_i)^{e_i})^n \\ &= \prod \mathrm{Tr}_D^{D_i}(c^{D_i}(\alpha, u_i)^{e_i}) \\ &= \prod c^D(\alpha, N_D^{D_i} u_i)^{e_i} \\ &= c^D(\alpha, \prod N_D^{D_i} u_i^{e_i}). \end{aligned}$$

From 1.2 we get

$$RN_{D|F}(\prod N_D^{D_i} u_i^{e_i}) = \prod N_F^{E_i} RN_{D_i|E_i} u_i^{e_i} = 1-\alpha;$$

hence $\psi_0(\alpha, 1-\alpha)=1$, which proves 2.2.

For later applications, we need some technical facts. Let $D|F$ be as above.

2.6. **Lemma.** *If* $x, y \in \mathcal{N}_{D|F}$ *then* $(-x, x)=1$, $(x, y)^{-1}=(y, x)$.

Proof. Since $\mathcal{N}_{D|F}$ is a group we have $x^{-1}, xy \in \mathcal{N}_{D|F}$; therefore

$$(-x, x)=(1-x, x)((1-x^{-1})^{-1}, x)=(1-x^{-1}, x^{-1})=1,$$

which yields

$$1=(-xy, xy)=(-x, x)(-x, y)(y, x)(y, y)=(x, y)(y, x).$$

2.7. **Proposition.**[1] *Let* $D|F$ *be a division algebra which is ramified at exactly one real place of* F. *Then* $K_2(F)$ *is a direct sum of a subgroup isomorphic to* $Y(D|F)$ *under the natural map and the cyclic group of order two generated by* $\{-1, -1\}$.

Proof. Let $\sigma_\alpha \in \{\pm 1\}$ (resp. $|\alpha|$) denote the sign (resp. value) of $\alpha \in F$ at the real ramified place of D, such that $\alpha = \sigma_\alpha |\alpha|$.

Then a split epimorphism $K_2(F) \to Y(D|F)$ is defined by

$$\{\alpha, \beta\} \mapsto (\alpha, |\beta|)(\sigma_\beta, |\alpha|)^{-1} \qquad (\alpha, \beta \in F^*).$$

It is clearly bimultiplicative, and the value of $\{\alpha, 1-\alpha\}$ is trivial, which is obvious in the case $1-\alpha>0$. If $1-\alpha<0$, then $\alpha \in \mathcal{N}_{D|F}$, and, by 2.6

$$(\alpha, |1-\alpha|)(-1, |\alpha|)^{-1}=(\alpha-1, \alpha)^{-1}(-1, \alpha)^{-1}=1.$$

Since $K_2(F)$ is generated by $\{-1, -1\}$ and $\{\alpha, \beta\}$, $\beta>0$, the proposition follows.

2.8. **Corollary.** *If* D *is ramified at at most one real place of* F, *kernel* ψ *has exponent* $n=n(D)$.

§ 3. Is ψ Injective?

In this paragraph we always assume F to be a local or global field, and D a central division algebra of finite degree.

As Corollary 2.4 shows, ψ is injective if for each non-trivial element $\xi \in Y(D|F)$ there exists a finite extension $E|F$ which splits D, but not ξ (that is, the image of ξ is not trivial under the map

$$Y(D|F) \to K_2(F) \to K_2(E)).$$

If F is a non-archimedean local field with $\mu(F)$ the torsion subgroup of F^*, a well-known theorem of C.C. Moore [10] says that the (unique) power norm

[1] It is possible to prove a similar result in the case that D is ramified at exactly two real places (as was pointed out by one of the authors together with R.K. Dennis), which yields results like 3.5, 4.4 at least if $4 \nmid n = \operatorname{ind} D$. But it is unclear to us what happens in the general case

residue symbol defines a split epimorphism π_F: $K_2(F) \to \mu(F)$ with $\mathscr{D}_F$: $=\operatorname{kernel} \pi_F$ the unique maximal divisible subgroup of $K_2(F)$. One also knows [4]:

$\mathscr{D}_F$ is uniquely divisible by any number prime to the residue class field characteristic p.

Together with 2.5, this implies that kernel $\psi \cap \mathscr{D}_F$ is of exponent p^v where v is the greatest integer such that p^v divides $n = \operatorname{ind} D$. Kernel $\psi \cap \mathscr{D}_F$ clearly would be trivial if $\mathscr{D}_F$ were uniquely divisible by p-powers. The latter assertion is a conjecture of Tate and has been proved recently by him in the following cases: i) $\operatorname{char} F \neq 0$, ii) $F \subset \mathbf{Q}_p(a)$, a some p-power root of unity [22, Th. 5.5].

3.1. **Theorem.** *Let D be a locally compact division algebra with centre F.*

i) *If $F = \mathbf{R}$, D the ordinary quaternion skew field, then*

$$\psi: Y(D|F) \to K_2(D)$$

is injective.

ii) *If F is non-archimedean with residue class field of characteristic p (then clearly $Y(D|F) \simeq K_2(F)$ since* (A) *holds), then kernel ψ is contained in the p-primary part of $\mathscr{D}_F$ and image ψ is a direct summand of $K_2(D)$ except possibly in case* $\operatorname{char} F = 0$, $p = 2$, $\sqrt{-1} \notin F$, *and* $2|n = \operatorname{ind} D$, *in which case* [kernel ψ: $\mathscr{D}_\psi] \leqq 2$ *holds where $\mathscr{D}_\psi$ denotes the intersection of* kernel ψ *with the* 2*-primary part of $\mathscr{D}_F$.*

Remark. i) has been proved recently by Alperin-Dennis [1]. A result similar to part ii) can be found in Harris-Stasheff [8], but without an explicit map ψ and with certain conditions on $n = \operatorname{ind} D$ in case $\operatorname{char} F = 0$.

Proof. i) One deduces from [3, p. 356] that $K_2(\mathbf{R}) = \langle\{-1, -1\}\rangle \times \mathscr{D}_{\mathbf{R}}$ where $\mathscr{D}_{\mathbf{R}}$ is a uniquely divisible group (see also [22, proof of Th. 5.5]). But it follows from 2.7 that $Y(D|F) \simeq \mathscr{D}_{\mathbf{R}}$, and 2.6 gives the assertion.

ii) Assume we can prove the following

3.2. **Lemma.** *If F is a non-archimedean local field and $n \in N$, then there exists a cyclic cyclotomic field extension $E|F$ of degree n with group $\mathfrak{g} = \operatorname{Gal}(E|F)$ such that the homomorphism*

$$\pi(i_*): K_2(F)/\mathscr{D}_F \to (K_2(E)/\mathscr{D}_E)^{\mathfrak{g}}$$

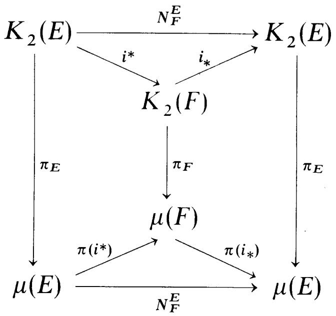
induced by the inclusion i: $F \to E$ is an isomorphism except when $\operatorname{char} F = 0$, $p = 2$, $\sqrt{-1} \notin F$ *and* $2|n$.

In the exceptional case we have, for any finite extension $E|F$ of degree n, $|\operatorname{kernel} \pi(i_*)| \geqq 2$, *and there exists a (not necessarily cyclic) cyclotomic field extension $E|F$ of degree n such that* $|\operatorname{kernel} \pi(i_*)| = 2$.

Then part ii) of 3.1 follows from 2.5 together with the remarks in the beginning of this paragraph, since a classical theorem says that D is split by any extension $E|F$ of degree $n = \operatorname{ind} D$ (cf. [19]).

Proof of 3.2. For the proof we borrow an argument from Harris-Stasheff [8, p. 343]. For any Galois extension $E|F$ with group $\mathfrak{g}$, denote by i^*: $K_2(E) \to K_2(F)$ the transfer and by i_*: $K_2(F) \to K_2(E)$ the induced map of i: $F \to E$.

Then one easily verifies $i_* \circ i^* = N_F^E$ where $N_F^E(\xi) = \sum_{\sigma \in \mathfrak{g}} {}^\sigma\xi$ $(\xi \in K_2(E))$. Now one can derive from [14, Remark 15.9] that the map π_E: $K_2(E) \to \mu(E)$ can be chosen $\mathfrak{g}$-equivariant; therefore we have the following commutative diagram



where the map N_F^E: $\mu(E) \to \mu(E)$ is induced by the usual norm map $E^* \to F^*$ composed with the embedding i: $F \to E$ and where $\pi(i^*)$ (resp. $\pi(i_*)$) is induced by i^* (resp. i_*). In [14, p. 177] it is proved that i^* is surjective, hence $\pi(i^*)$ is surjective.

Now, if $\mu(F)$ has trivial p-primary part (which is always the case in $\operatorname{char} F > 0$) or if n is prime to p, then we may take for E the unique unramified extension of F of degree n. For the non-p-part of the group $\mu(F)$ it follows easily from the well-known properties of the tame symbol [14, p. 98] that $\pi(i_*)$ defines an isomorphism. In the case $(n, p) = 1$ we get injectivity on the p-primary part from 1.5, which yields $\pi(i^*) \circ \pi(i_*) = [E:F]$ id.

Therefore, now we can assume that F contains the p-th roots of unity and that $n = p^\nu$. Let λ denote the greatest integer such that F contains the p^λ-th roots of unity, and let ζ be a primitive p^λ-th root of unity. Then we show that $E = F(\sqrt[p]{\zeta})$ has the property wanted in case $n = p$ (from which one concludes the general case by induction). For the non-p-part of the groups the injectivity of $\pi(i_*)$ is obvious by $\pi(i^*) \circ \pi(i_*) = [E:F]$ id. Therefore we only have to compare the p-parts of $\mu(F)$ and $\pi(i_*)\,\mu(F)$. But for these groups the isomorphism follows from the formula $N_F^E(\sqrt[p]{\zeta}) = (-1)^{p-1}\zeta$ – except in the case $\operatorname{char} F = 0$, $p = 2$, $\sqrt{-1} \notin F$, in which case $\pi(i_*)$ has a kernel of order 2 and $E|F$ is not necessarily cyclic. Since -1 is in the image of $\pi(i^*)$ but cannot be a norm of a 2^λ-th root of unit in an extension of even degree, it must be in the kernel of $\pi(i_*)$ for any extension $E|F$ of even degree. Hence the lemma is proved.

3.3. *Remark.* i) The lemma shows that, in the exceptional case, we cannot decide (with our method) whether or not our map ψ is injective or has a kernel of order 2.

ii) It is possible to identify the non-trivial element of the presumed kernel of ψ in the exceptional case: Let $\delta \in F^*$ be an element of quadratic defect $4\,\mathfrak{o}$ in the sense of [15, §6J] ($\mathfrak{o}$ being the ring of integers of F) or, equivalently, a distinguished unit of F in the sense of [14, p. 173]. Then,

$$\operatorname{kernel} \psi = \{1\} \Leftrightarrow \psi(\{-1, 1+\delta\}) \neq 1.$$

This can be deduced from [15, 63.13, proof].

Let us now study the case of a global field F. Our first result uses C. Moore's theorem on uniqueness of reciprocity laws (in fact we need only the "classical" part of it), the Grunwald-Wang existence theorem of class field theory and the theorem of Hasse-Brauer-Noether to get a description of kernel ψ which yields injectivity results at least in the case $F=\mathbf{Q}$. The second approach uses Tate's cohomological description of $K_2(F)$ from which we get injectivity in the function field case.

We first state Moore's Theorem. For each prime v of F let F_v be the completion of F at v, m the order of $\mu(F)$ and – for non-complexe v – m_v the order of $\mu(F_v)$. Let π_v: $K_2(F_v)\to\mu(F_v)$ denote the m_v-th norm residue symbol (cf. [14, § 15]) of F_v. Composing with the natural map $K_2(F)\to K_2(F_v)$ and taking the direct sum over all non-complex v gives a map

$$\pi\colon K_2(F)\to\coprod_v \mu(F_v).$$

If $\zeta_v\in\mu(F_v)$, then clearly $\zeta_v^{m_v/m}\in\mu(F)$; therefore the definition $(\zeta_v)\mapsto\coprod_v \zeta_v^{m_v/m}$ gives a map

$$\rho\colon \coprod_v \mu(F_v)\to\mu(F)$$

and Moore's theorem says that the following sequence is exact:

$$K_2(F)\xrightarrow{\ \pi\ }\coprod_v \mu(F_v)\xrightarrow{\ \rho\ }\mu(F)\longrightarrow 1$$

(cf. [10], a nice proof of this can be found in [5]).

One knows from Garland (number field case) and Tate (function field case) that kernel π is finite (for literature, cf. [2]).

For any finite dimensional central division algebra D over F we define

$$C_D=\operatorname{kernel}\pi \qquad \text{if } \operatorname{char} F>0$$

$$C_D=\{\xi\in K_2(F)\,|\,\pi_v(\xi)=1 \text{ for all } v \text{ except those with } v|2,$$
$$\sqrt{-1}\notin F_v,\ 2|n_v=\operatorname{ind}(D\otimes_F F_v)>0;\ \text{and } \pi_v(\xi)^2=1 \text{ for all } v\}$$
$$\text{if } \operatorname{char} F=0.$$

Clearly C_D is an extension of kernel π by a finite elementary 2-group, and $C_D=\operatorname{kernel}\pi$ in the number field case for instance when $2\nmid n=\operatorname{ind} D$.

3.4. **Theorem.** *Let D be a finite dimensional central division algebra over the global field F. Then the kernel of the map ψ: $Y(D|F)\to K_2(D)$ is contained in the inverse image of C_D under the natural map $Y(D|F)\to K_2(F)$.*

3.5. **Corollary.** *If $F=\mathbf{Q}$ or, more generally, if F is a global field with* kernel $\pi=\{1\}$ *and, additionally, in the number field case, with at most one dyadic place v such that $\sqrt{-1}\notin F_v$ and with at most one real place, then ψ is injective for all D.*

This follows from the theorem and from 2.7, since, under the given assumption, Moore's theorem yields $C_D\subseteq\langle\{-1,-1\}\rangle$, hence kernel $\psi=1$. Examples of number fields, for which the conditions of the corollary hold, are the imaginary quadratic fields of discriminant $d>-35$, $d\not\equiv 1$ (8). (In case $d\equiv 1$ (8), we clearly have two dyadic v with $\sqrt{-1} \notin F_v$, and in case $d=-35$, $|\text{kernel}\,\pi|=2$, cf. [3], appendix, Prop. 3 and remark.)

Proof of 3.4. Let $\xi \in K_2(F)-C_D$. By definition of C_D and by 3.2, there is a place v_0 of F such that $\pi_{v_0}(\xi)\neq 1$ and such that there exists a finite cyclic extension E^{v_0} of F_{v_0} which splits $D_{v_0}:=D\otimes_F F_{v_0}$, with ξ having non-trivial image in $K_2(E^{v_0})$.

Choose for each of the finite ramified places $v \neq v_0$ of $D|F$ a finite cyclic extension E^v of F_v which splits $D_v = D\otimes_F F_v$, then, by the existence theorem of Grunwald-Wang [25], there exists a finite cyclic extension $E|F$ which has just the E^{v_0}, E^v as completions at this finite set of places, But then, by the Hasse-Brauer-Noether theorem [9], E splits D and, by construction, ξ has non-trivial image in $K_2(E)$.

This proves the theorem.

In the function field case, we get a sharper result by using the Galois-cohomological description of $K_2(F)$ due to Tate ([2, 21]). This allows us to compare $K_2(F)$ with $K_2(E)$, where E is a cyclotomic extension of F, and since a classical result of Tsen [23] tells us that any central division algebra of finite degree may be split by such an extension, our result follows.

Let us state what we need. (Main reference: [21].)

We fix some separable hull F_{sep} of our global function field F, and for any fixed prime $\ell \neq \operatorname{char} F$ we let $\mu_{\ell^v}\subseteq F_{\text{sep}}$ denote the group of ℓ^v-th roots of unity. Let $\mathbf{Z}_\ell(1)=\varprojlim \mu_{\ell^v}$. $\mathbf{Z}_\ell(1)$ is a $\mathbf{Z}_\ell$-module of rank 1 and it has also a G_F-module structure, where $G_F=\operatorname{Gal}(F_{\text{sep}}|F)$ is the Galois group of F. We need the G_F-module

$$W_\ell^{(2)} := \mathbf{Q}_\ell/\mathbf{Z}_\ell \otimes \mathbf{Z}_\ell(1)^{\otimes 2},$$

where $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ carries the trivial G_F-module structure.

Now, if $W^{(2)}=\coprod_\ell W_\ell^{(2)}$ (the sum taken over all primes $\ell \neq \operatorname{char} F$), there exists a natural isomorphism

$$g_F\colon\ H^1(F, W^{(2)}) \xrightarrow{\sim} K_2(F),$$

which is functorial in F (cf. [21], Th. 6.6), and the restriction to $H^1(F, W_\ell^{(2)})$ gives a map onto $K_2(F)\{\ell\}$ (=the ℓ-primary part of $K_2(F)$). (Note that $H^1(F, W^{(2)})\simeq \coprod_\ell H^1(F, W_\ell^{(2)})$ by [20, p. I–9]).

The following fact is more or less well-known:

3.6. **Proposition.** *Let $E|F$ be a finite cyclic cyclotomic field extension of the global function field F with Galois group $\mathfrak{g}=\operatorname{Gal}(E|F)$. Then the restriction map*

$$H^1(F, W^{(2)}) \to H^1(E, W^{(2)})^{\mathfrak{g}}$$

is an isomorphism onto.

Proof. From the "profinite" version of the Hochschild-Serre spectral sequence we get the following exact sequence [20, p. I–14]:

$$0\to H^1(\mathfrak{g}, (W^{(2)})^{G_E}) \to H^1(F, W^{(2)}) \to H^1(E, W^{(2)})^{\mathfrak{g}} \to H^2(\mathfrak{g}, (W^{(2)})^{G_E})$$

(where G_E denotes the Galois group $\operatorname{Gal}(F_{\text{sep}}|E)$).

The cohomology of finite cyclic groups $\mathfrak{g}$ and finite $\mathfrak{g}$-modules M yields:

$$|H^1(\mathfrak{g}, M)| = |H^2(\mathfrak{g}, M)| \quad \text{and} \quad H^2(\mathfrak{g}, M) \simeq M^{\mathfrak{g}}/N_{\mathfrak{g}} M,$$

where $N_{\mathfrak{g}} = \sum_{g \in \mathfrak{g}} g$.

Hence it is enough to show, for each prime number $\ell \neq \operatorname{char} F$, and for $M = (W_\ell^{(2)})^{G_E}$, that $M^{\mathfrak{g}} \simeq N_{\mathfrak{g}} M$.

But since $(W_\ell^{(2)})^{G_E}$ is the ℓ-primary part of the multiplicative group of the unique "constant" quadratic extension of F, with $\mathfrak{g}$-module structure defined by $x \mapsto \sigma^2 x$, $x \in M$, $\sigma \in \mathfrak{g}$, the last assertion follows from the fact that the norm map of a finite extension of a finite field is always onto, q.e.d.

Therefore we conclude:

3.7. **Theorem.** *Let D be a finite dimensional central division algebra over the global function field F. Then the map*

$$\psi\colon K_2(F) \to K_2(D)$$

is injective, and its image is a direct factor of $K_2(D)$.

We only have to choose, by Tsen, a finite (cyclic) cyclotomic splitting field E of D, and then the first assertion follows from 3.6 together with the properties of g_F and the second from 2.4. From 2.5 we get now:

3.8. **Corollary.** *$K_2(D) \cong K_2(F) \times$ a finite group.*

§ 4. Quaternions

The question of whether or not ψ is surjective seems to be a difficult algebraic problem. We know of no example of a finite dimensional division algebra with ψ defined and not surjective. But in the case of (generalized) quaternion skew fields we can – and will now – prove an affirmative result.

Let D be a quaternion skew field with centre F. Recall the definition of the group U_D generated by elements $c(u, v)$, $u, v \in D^*$, from § 1. As one knows, U_D is defined by relations 0)–3) given in § 1 (this result is even true for arbitrary skew fields [18], but we will need only the fact that $K_2(D)$ consists of finite products $\Pi\, c(u_i, v_i)$, with $u_i, v_i \in D^*$, $\Pi\, [u_i, v_i] = 1$, which can be easily verified).

4.1. **Proposition.** *Let D be a quaternion skew field over an arbitrary field F. Then each element with trivial reduced norm is a commutator, and $K_2(D)$ is generated by elements $c(u, v)$ with $[u, v] = 1$.*

Proof. The first assertion is a well-known easy application of Hilbert's Satz 90 and the theorem of Skolem-Noether, if $\operatorname{char} F \neq 2$. In general the proof if as follows: For each $u \in \operatorname{kernel} RN_{D|F}$ there is a $v \in D^*$ such that v and uv have the same minimal polynomial, since the only condition for this is $RS_{D|F}(1-u)v = 0$ where $RS_{D|F}$ is the reduced trace of D over F. Hence the assertion follows from the theorem of Skolem-Noether. The proof of the second fact which we will give now is due to R.K. Dennis.

For $\xi, \eta \in U_D$, let us write $\xi \equiv \eta$ if $\xi\eta^{-1}$ is contained in the subgroup generated by $c(u, v)$ with $[u, v]=1$. We first show:

If $x, y \in D^*$, $[x, y] \neq 1$, then the set

$$Z_{x,y} := \{z \in D^* \mid \exists\, w \in D^*:\ c(x, y) \equiv c(z, w)\} \cup \{0\}$$

contains a linear F-subspace of D of dimension at least 3. Since $[x, y] \neq 1$, we have $\dim(F1+Fy+Fyx)=3$, and, if $\lambda_i \in F^*$ $(i=1, 2, 3)$, $w=\lambda_2 y+\lambda_3 yx$, we get, by relations 2) and 3):

$$\begin{aligned} c(\lambda_1 x + xw, w) &= {}^{x}c(\lambda_1 1 + w, w)\, c(x, w) \\ &\equiv c(x, w) = c(x, y)^{y}\, c(x, \lambda_2 1 + \lambda_3 x) \equiv c(x, y). \end{aligned}$$

It follows that, for any two pairs (x_i, y_i) $(i=1, 2)$ with $[x_i, y_i] \neq 1$ there is $z \in Z_{y_1, y_2}$, $z \neq 0$, such that

$$c(x_1, y_1)\, c(x_2, y_2) \equiv c(w_1, z)\, c(z, w_2)$$

for suitable $w_1, w_2 \in D^*$. But, by 1), 2):

$$c(w_1, z)\, c(z, w_2) = {}^{w_2}c(w_2^{-1} w_1, z),$$

and now an easy induction on the length of the product shows

$$\Pi\, c(x_i, y_i) \equiv 1$$

for all $x_i, y_i \in D^*$ such that $\Pi[x_i, y_i]=1$. This proves 4.1. We also need:

4.2. **Proposition.** *Let E be a quadratic extension of an arbitrary field F. Then $K_2(E)$ is generated by elements $\{\alpha, b\}$ with $\alpha \in F^*$, $b \in E^*$.*

Proof. Let $u, v \in E^*$. If u, v are linearly dependent, there exists $\alpha \in F^*$ such that $u = -\alpha v$, hence $\{u, v\} = \{\alpha, v\}$.

If u, v are linearly independent, we find $\alpha, \beta \in F^*$ such that $1=\alpha u+\beta v$. But then

$$1 = \{\alpha u, \beta v\} = \{\alpha, \beta\}\, \{\alpha, v\}\, \{u, \beta\}\, \{u, v\},$$

hence

$$\{u, v\} = \{\beta, \alpha\}\, \{\alpha, v\}^{-1}\, \{\beta, u\}$$

and the proposition follows.

4.1 and 4.2 imply

4.3. **Theorem.** *Let D be a quaternion skew field over an arbitrary field F. Then the map*

$$\psi_0\colon F^* \otimes_{\mathbf{Z}} \mathcal{N}_{D|F} \to K_2(D)$$

is surjective.

Proof. By 4.1, 4.2 we only have to show: $c^D(\alpha, b) \in \operatorname{Im} \psi_0$ for $\alpha \in F^*$, $b \in D^*$, since each pair $u, v \in D^*$ with $[u, v] = 1$ is contained in some quadratic extension E of F. But clearly, by definition of ψ_0,

$$\psi_0(\alpha \otimes RN_{D|F} b) = c^D(\alpha, b), \quad \text{q.e.d.}$$

4.4. **Corollary.** i) *If F is a C_2'-field in the sense of* [20], *the map ψ: $K_2(F) \to K_2(D)$ is surjective.*

ii) *If F is local or global of characteristic $\neq 0$, ψ: $K_2(F) \to K_2(D)$ is a surjective isomorphism.*

iii) *If $F = \mathbf{R}$, D the ordinary quaternions, ψ: $Y(D|F) \to K_2(D)$ is a surjective isomorphism.*

iv) *If F is local non-archimedian of characteristic $= 0$, ψ: $K_2(F) \to K_2(D)$ is a surjective isomorphism except possibly in the dyadic case. In the dyadic case we have the following: If $\mathscr{D}_F$ is the maximal divisible subgroup of $K_2(F)$, the group* kernel $\psi \cap \mathscr{D}_F$ *is trivial or possibly non-trivial of exponent* 2, *depending on whether or not F is contained in $\mathbf{Q}_2(a)$, a some 2-power root of unity. The group* kernel ψ/kernel $\psi \cap \mathscr{D}_F$ *is trivial or possibly of order* 2 *depending on whether or not $\sqrt{-1} \in F$.*

v) *If $F = \mathbf{Q}$, ψ: $Y(D|\mathbf{Q}) \to K_2(D)$ is a surjective isomorphism. In particular $K_2(D)$ can be described as an abstract abelian group by generators (α, β) with $\alpha \in \mathbf{Q}^*$, $\beta \in \mathbf{Q}^*_>$ (= positive numbers) or $\beta \in \mathbf{Q}^*$, according as D is ramified at infinity or not, and defining relations*

$$(\alpha, 1-\alpha) = 1 \quad (\alpha \in \mathbf{Q}^*,\ 1-\alpha \in \mathbf{Q}^*_> \text{ or } 1-\alpha \in \mathbf{Q}^*),$$

$$(\alpha\alpha', \beta) = (\alpha, \beta)(\alpha', \beta),$$

$$(\alpha, \beta\beta') = (\alpha, \beta)(\alpha, \beta').$$

vi) *If $F = \mathbf{Q}(\sqrt{d})$ is the imaginary quadratic field of discriminant $d > -35$, $d \not\equiv 1(8)$, then ψ: $K_2(F) \to K_2(D)$ is a surjective isomorphism.*

All assertions follow from 4.3 together with the results of § 3.

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Received June 19, 1978