The Fibre of the Bott-Samelson Resolution

Let G be an adjoint semi-simple group over an algebraically closed field and let $B \supset T$ be a Borel subgroup and a maximal torus of G. And let also W be the Weyl group of (G, T). Let us fix a element $w \in W$.

Following Contou-Carrère [CC], we consider the Bott-Samelson resolution $\hat{\Sigma}(\gamma_w)$ of the Schubert variety $\overline{\Sigma}_w$ as a variety of galleries in the building associated to the group. We first determine a cellular decomposition of this variety analogous to the Bruhat decomposition of a Schubert variety. This decomposition is indexed by the $T-$fixed points of $\hat{\Sigma}(\gamma_w)$. We call them combinatorial galleries, these are the same as the subexpressions introduced by Deodhar [Deo85].

Then we describe the trace of this decomposition on the fibre over a $T-$ fixed point. But the resolution is a $B-$equivariant morphism so we obtain the following :

Theorem. *Let $\pi : \hat{\Sigma}(\gamma_w) \to \overline{\Sigma}_w$ be the Bott-Samelson resolution of $\overline{\Sigma}_w$ associated to a reduced decomposition γ_w of w. Let x be a point of $\overline{\Sigma}_w$. Then the fibre $\pi^{-1}(x)$ admits a finite cellular decomposition, where each cell is isomorphic to a $k-$vector space. More precisely, each cell is an affine subvariety of an open affine subset of $\hat{\Sigma}(\gamma_w)$ defined by linear equations.*

Our description of the fibre agrees with, and gives a geometrical meaning to, some results of Deodhar concerning the dimension of $\pi^{-1}(x)$ [Deo90].

References

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