Conjugacy Classes of Non-Connected Semisimple Algebraic Groups
(by Stephan Mohrdieck)

The aim of my research is to understand the structure of semisimple algebraic groups and Kac-Moody groups, in particular the conjugacy classes and the invariant theory. As is well known non-connected semisimple algebraic groups appear naturally as the stabilisers of semisimple elements of semisimple but non simply connected algebraic groups. In the case where the semisimple algebraic group is connected the invariant theory and conjugacy classes are well understood due to the work of Steinberg [St1,St2]. He showed that the quotient of the group by the action by conjugation is isomorphic to a maximal torus divided by the Weyl group. In the case of a simply connected group the quotient turns out to be an affine space and admits a cross section. In this latter case the quotient map is given by the fundamental characters. Brüchert, [Br], showed that an existence result for a cross section also holds in the case of affine Kac-Moody groups. Here new problems arise, since the fundamental representations are in general infinite dimensional. Therefore the quotient map will only be defined on a subset of the Kac-Moody group.
In the case of an algebraic group, i.e. the finite dimensional case, I obtained the following results, [Mo].

Let G be a semisimple (connected) algebraic group of rank r, T a maximal torus, $\mathcal{W} = N_G(T)/T$ the Weyl group and R the root system. Furthermore denote a basis of R by $\Delta = \{\alpha_1, ..., \alpha_r\}$. I assume the base field to be algebraically closed. Now let τ be a symmetry of the Dynkin diagram Δ and denote by Γ the finite subgroup generated by τ. Using these data one can construct a non-connected semisimple algebraic group $\tilde{G}$ as a semidirect product $\tilde{G} := G \rtimes \Gamma$. Now the unit component $G = \tilde{G}_0$ acts on the exterior component $G\tau$ by conjugation:

$$\begin{aligned} G \times G\tau &\to G \\ (g, h\tau) &\mapsto Ad(g)(h) = gh\tau g^{-1}. \end{aligned}$$

For investigating the conjugacy classes and invariant theory of this group action, the methods of Steinberg do not work any longer. For instance it is not clear what should correspond to the concept of a maximal torus.
So new methods and techniques have to be invented. Let me denote the unit component of the τ-fixed points of T by T_0^τ. I was able to show that the subgroup of $\tilde{G}$, generated by τ and T_0^τ has finite index in its normaliser in $\tilde{G}$.
Then I defined the *generalised Weylgroup* $\widetilde{\mathcal{W}} := N_G(< T_0^\tau, \tau >)/T_0^\tau$. But now one encounters the difficulty that $\widetilde{\mathcal{W}}$ is no longer a Weyl group, it is a finite abelian extension of the fixed point Weyl group $\mathcal{W}^\tau$. The characteristic of the base field should neither be two nor divide the order of τ.

I proved the following result, which looks quite similar to that of Steinberg:

Theorem 1. *(i) The inclusion map $i : T_0^\tau\tau \to G\tau$ gives rise to an isomorphism $\tilde{i}$:*

$$\tilde{i} : T_0^\tau\tau/\widetilde{\mathcal{W}} \to G\tau /\!/ G.$$

(ii) If the fixed point character lattice of the maximal torus $\chi(T)^\tau$ coincides with fixed point weight lattice $\Lambda(R)^\tau$ even more is true:
(a) The quotient is an affine space $\mathbb{A}^s$, $s = rk\, G^\tau$.
(b) The quotient map π is given by the s-tuple of some dominant characters $(X_{\lambda'_1}, ..., X_{\lambda'_s})$. Here

the λ_i' are the generators of the fixed point weight lattice $\Lambda(R)^\tau$.
(c) The quotient admits a cross section which is given by the following recipe: First choose s simple roots α_i, exactly one for each Γ-orbit on Δ. Then define:

$$\psi : \mathbb{A}^s \to G\tau$$
$$(c_1, ..., c_s) \mapsto \prod_{i=1}^{s} X_{\alpha_i}(c_i)\, n_{\alpha_i}\tau$$

Here the X_{α_i} are the root groups corresponding to the simple roots. The $n_{\alpha_i} \in N_G(T)$ are representatives of the simple reflexions in $\mathcal{W}$.

I can also give a full description of the conjugacy classes:

Theorem 2. *(i) Each reduced fibre of π has the structure of an associated bundle:*

$$\pi^{-1}(\pi(t\tau))_{red} = G \times^{C_G(t\tau)} V(t\tau),$$

where $V(t\tau)$ is the unipotent variety of $C_G(t\tau)$. (ii) In particular there are only finitely many conjugacy classes in each fibre of π. (iii) Under the same condition on the character lattice of a maximal torus of G as in (ii) of the theorem above we get that
(a) the quotient map π is flat and
(b) its fibres are normal.

References

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