An analogue of a reductive algebraic monoid, whose unit group is a Kac-Moody group

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The minimal Kac-Moody group G_m is a group analogue of a semisimple, simply connected algebraic group. V. Kac and D. Peterson defined and investigated in [K,P] the algebra of strongly regular functions $\mathbb{C}[G_m]$ of a symmetrizable minimal Kac-Moody group. This algebra has many properties in common with the coordinate ring of of a semisimple, simply connected algebraic group. It is an integrally closed domain, even a unique factorization domain. It admits a Peter and Weyl theorem, i.e.,

$$\mathbb{C}[G_m] \;\cong\; \bigoplus_{\Lambda\in P^+} L^*(\Lambda)\otimes L(\Lambda)$$

as $G_m \times G_m$-modules. But the following things, which hold in the non-classical case, are different:

• Assigning to every element of G_m its point evaluation, G_m embeds in the set of $\mathbb{C}$-valued points of $\mathbb{C}[G_m]$, which we denote by $\operatorname{Specm}\mathbb{C}[G_m]$. But this map isn't surjective.

• There exists no comultiplication of $\mathbb{C}[G_m]$, dual to the multiplication of G_m. The situation is not too bad, left and right multiplications with elements of G_m induce comorphisms. A more serious difference, the inverse map of G_m doesn't induce a comorphism. In particular there is no natural group structure, even no natural monoid structure on $\operatorname{Specm}\mathbb{C}[G_m]$.

Kac and Peterson posed the problem to determine $\operatorname{Specm}\mathbb{C}[G_m]$, or at least the part

$$\bigcup_{m\in\mathbb{N}}\;\bigcup_{\beta_1,\ldots,\beta_m\in\Delta_{re}} \overline{U_{\beta_1}\cdots U_{\beta_m}T}^{Specm} \quad .$$

(Here the elements of G have been identified with the corresponding $\mathbb{C}$-valued points. The U_β's denote root groups, and T denotes a maximal torus. These are given by the construction of the Kac-Moody group.)

Looijenga constructed an adjoint quotient for Kac-Moody groups as a map, for which we write $\chi: G_m \to \overline{T}/\mathcal{W}$. This adjoint quotient has been investigated by P. Slodowy. Compared with its classical counterpart it has bad properties. There are missing conjugacy classes in its fibres. P. Slodowy conjectured, that it can be extended onto a certain part of $\operatorname{Specm}\mathbb{C}[G]$. He discussed this with D. Peterson. Some time later D. Peterson claimed, that this part should coincide with the part of $\operatorname{Specm}\mathbb{C}[G]$ of above. Furthermore it should have a monoid structure. He also claimed certain structural properties. For a full account with references, compare the introduction of [M 1]. But D. Peterson didn't give any proof. In my PhD-work [M 1], I proved the claims of D. Peterson. I also obtained many other results, some after my PhD, giving a complete picture of the monoid:

In a way similar to the reconstruction process of the Tannaka-Krein duality, a suitable category of representations of a Lie algebra, together with a suitable category of duals, determine a monoid with coordinate ring, [M 1], [M 2]. This monoid is the biggest monoid acting reasonably on the representations, compatible with the duals.

Recall that a module of the Kac-Moody algebra is called admissible, if it is integrable, and the set of weights are contained in a fixed weight lattice P. The category $\mathcal{O}_{adm}$ of admissible modules of $\mathcal{O}$ generalizes the category of finite dimensional representations of a semisimple Lie algebra, keeping the complete reducibility theorem.

The monoid $\widehat{G_m}$ corresponding to $\mathcal{O}_{adm}$ and the restricted duals has been determined and investigated in [M 1]. This monoid contains the minimal Kac-Moody group G_m as Zariski open, dense unit group. It has similar structural properties as a reductive algebraic monoid. The closure $\overline{T}$ of the maximal torus T gives a Weyl group equivariant affine torus embedding of non-finite type. The idempotents of $\overline{T}$ are in bijection with the faces of the Tits cone. Every idempotent of $\widehat{G_m}$ is G_m-conjugate to some idempotent of $\overline{T}$. There are Bruhat and Birkhoff decompositions, the Weyl group $\mathcal{W} = N/T$ replaced by the Weyl monoid $\widehat{\mathcal{W}} := \overline{N}/T$.

The coordinate ring $\mathbb{C}[\widehat{G_m}]$ admits a Peter and Weyl theorem. It is isomorphic to the algebra of strongly regular functions $\mathbb{C}[G_m]$ by the restriction map. In this way, the algebra of strongly regular functions is really the coordinate ring of a monoid. The image of the embedding of $\widehat{G_m}$ into $\mathrm{Specm}\,\mathbb{C}[G_m]$ coincides with the part of $\mathrm{Specm}\,\mathbb{C}[G_m]$ mentioned above. The Lie algebra of $\widehat{G_m}$ is the Kac-Moody algebra $\mathbf{g}$.

The monoid $\widehat{G_f}$ corresponding to $\mathcal{O}_{adm}$ and the full duals will be determined and investigated in [M 2]. It has similar structural properties as $\widehat{G_m}$, but the minimal Kac-Moody group G_m is replaced by the formal Kac-Moody group G_f. Its coordinate ring admits a Peter and Weyl theorem. Its Lie algebra is the formal Kac-Moody algebra $\mathbf{g}_f$.

These monoids are purely non-classical phenomena. In the classical case they reduce to a semisimple, simply connected algebraic group. Such monoids are the correct objects for generalizing results of classical invariant theory.

The category of admissible representations of a Kac-Moody algebra also generalizes the category of finite dimensional representations of a semisimple Lie algebra. Presumably the Kac-Moody group corresponds to this category and the admissible duals.

References

[K,P] V. G. Kac, D. H. Peterson, Regular functions on certain infinite-dimensional groups, in Arithmetic and Geometry, Progress in Math. **36**, Birkhäuser, Boston, 1983, 141-166

[M 1] C. Mokler, A monoid completion of a Kac-Moody group, submitted to the Journal of Representation Theory of the AMS. An new version, split into two parts, in which the Tannaka-Krein reconstruction is included, is in preparation.

[M 2] C. Mokler, The monoid associated to the category of admissible modules of $\mathcal{O}$ of a Kac-Moody group, and the category of full duals, in preparation.