Invariants of free Lie superalgebras

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Let $G \subset \mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$ be a finite subgroup, where $V_+ = \langle x_1, \ldots, x_m \rangle_K$, $V_- = \langle x_{m+1}, \ldots, x_{m+k} \rangle_K$ and K is of characteristic zero. We consider the diagonal action of G on the free Lie superalgebra generated by even and odd sets $X_+ = \{x_1, \ldots, x_m\}$, $X_- = \{x_{m+1}, \ldots, x_{m+k}\}$. For $g \in G$ we denote $g = (g_+, g_-)$. It is known that any subalgebra in the free Lie superalgebra is again free (analogue of Shirshov-Witt's theorem). Suppose that the subalgebra of invariants $H = L^G$ is freely generated by $Y = \cup_{i,j=0}^{\infty} Y_{ij}$, where Y_{ij} is subset of Y with degrees i and j with respect to X_+ and X_-. Let $\mathcal{H}(Y, t_+, t_-) = \sum_{i,j=0}^{\infty} |Y_{ij}| t_+^i t_-^j$ be the generating function of Y. Then

1. $$\mathcal{H}(Y, t_+, t_-) = 1 - \prod_{g \in G} \prod_{n=1}^{\infty} \sqrt[n|G|]{\prod_{a|n} \left(1 - \mathrm{tr}(g_+^a)\, t_+^n + \mathrm{tr}(g_-^a)(-t_-)^n\right)^{\mu(a)}};$$

2. If the order of $g \in G$ decomposes into prime numbers as $p_1^{c_1} \cdots p_d^{c_d}$, then in the second product we can consider only the factors with $n = p_1^{a_1} \cdots p_d^{a_d}$, $a_i \geq 0$;

3. $g = e$ yields only the factor $\sqrt[|G|]{1 - mt_+ - kt_-}$.

As an example we consider $G = S_3$, its 2-dimensional irreducible representation, which acts on two-generated free Lie algebra.

$$\begin{aligned} \mathcal{H}(Y, t) &= 1 - \sqrt[6]{(1-2t)(1+t)^2} \prod_{d=1}^{\infty} \sqrt[2^{d+1}]{\frac{1}{1-2t^{2^d}}} \sqrt[3^{d+1}]{\frac{1+t^{3^d}}{1-2t^{3^d}}} = \\ &= t^5 + t^6 + 3t^7 + 4t^8 + 9t^9 + 15t^{10} + 30t^{11} + 50t^{12} + 98t^{13} + \ldots. \end{aligned}$$

References

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