Irreducible components of varieties of modules

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We prove some basic results about irreducible components of varieties of modules for an arbitrary finitely generated associative algebra. Our work generalizes results of Kac [2] and Schofield [4] on representations of quivers, but our methods are quite different, being based on deformation theory.

Let k be an algebraically closed field, and let A be a finitely generated k-algebra (associative, with 1). By $\mathrm{mod}_A^d(k)$ we denote the variety of d-dimensional A-modules.

Given irreducible components $C_1 \subseteq \mathrm{mod}_A^{d_1}(k)$ and $C_2 \subseteq \mathrm{mod}_A^{d_2}(k)$ let

$$\mathrm{ext}_A^1(C_1, C_2) = \min\{\dim \mathrm{Ext}_A^1(M_1, M_2)\}$$

where the modules M_i range over all modules in C_i.

Given irreducible components $C_i \subseteq \mathrm{mod}_A^{d_i}(k)$ we consider all modules of dimension $d = d_1 + \cdots + d_t$, which are of the form $M_1 \oplus \cdots \oplus M_t$ with the M_i in C_i, and we denote by $C_1 \oplus \cdots \oplus C_t$ the corresponding irreducible subset of $\mathrm{mod}_A^d(k)$.

Theorem. *If $C_i \subseteq \mathrm{mod}_A^{d_i}(k)$ are irreducible components ($1 \leq i \leq t$), and $d = d_1 + \cdots + d_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of $\mathrm{mod}_A^d(k)$ if and only if $\mathrm{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.*

This theorem has numerous applications.

The case of Lusztig's nilpotent variety is particularly interesting, since by work of Kashiwara and Saito [3] its irreducible components are in 1-1 correspondence with elements of the crystal basis of a quantum group. Because these varieties arise as module varieties, one can hope to use decomposition properties of modules, and homological algebra techniques, to study the irreducible components.

References

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