Category $\mathcal{O}$: Gradings and Translation Functors
(Catharina Stroppel)

I am interested in representation theory of semisimple complex Lie algebras, in particular in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. This category decomposes into blocks which are equivalent to the category of finite dimensional modules over a quasi-hereditary finite dimensional algebra A.
For any weight λ there is a universal object, the so-called *Verma module* with highest weight λ and denoted by $\Delta(\lambda)$. Each Verma module $\Delta(\lambda)$ has a simple head, denoted by $L(\lambda)$. All simple objects in $\mathcal{O}$ are constructed in this way.

A famous problem was to determine the multiplicities $[\Delta(\lambda) : L(\mu)]$, how often a simple module $L(\mu)$ occurs in a composition series of $\Delta(\lambda)$. This problem was made to a combinatorial problem by a conjecture ([KL, Conjecture 1.5]) of Kazhdan and Lusztig, which was proved in [BB1]. The conjecture states that the multiplicity is given by some inductively defined polynomials evaluated at 1.
The question now was, if there is also an interpretation of the exponents occurring in these Kazhdan-Lusztig-polynomials. One answer to this problem is given in the article [BGS] of Beilinson, Ginzburg and Soergel. They consider each block of category $\mathcal{O}$ as right modules over a finite dimensional algebra A, namely the endomorphism ring of a minimal projective generator. This algebra is isomorphic to an algebra of self-extensions of some module; so it inherits a natural $\mathbb{Z}$-grading. In [BGS, 3.11] is shown, that a Verma module $\Delta(\lambda)$ has a graded lift, i.e. there is a graded A-module M such that $M \cong \widetilde{\Delta(\lambda)}$ after forgetting the grading. Here $\widetilde{\Delta(\lambda)}$ denotes the right A-module corresponding to $\Delta(\lambda)$. On the other hand all simples have graded lifts concentrated in one single degree.
Since by definition the ring A is positively graded, the filtration associated to such a graded module is in fact a filtration by submodules. Moreover, the exponents of the Kazhdan-Lusztig-polynomials indicate in which layer of this filtration a certain simple module occurs ([BGS, Theorem 3.11.4]).
This observation was the motivation for me to ask, whether there are other "interesting" objects in $\mathcal{O}$ which are 'gradable' in the sense described above. And if so, whether there is a combinatorial description of their filtrations induced by the grading in terms of Kazhdan-Lusztig-polynomials.

For this reason I describe in [St1] another approach to the graded version of category $\mathcal{O}$ using Soergel's functor $\mathbb{V}$ (see [So]):
In this approach it is straight forward to define graded versions of translation functors. Inductively, this yields to graded versions of "important" modules such as Verma modules, dual Verma modules and principal series in general. For Verma modules it is just the same lift as described in [BGS].
I prove an Adjointness Theorem for these graded translation functors. I also explain what happens to the graded versions of Verma modules and simples when translated *through*, *onto* or *out of* the wall. I show the existence of a graded duality, which is a lift of the usual duality on $\mathcal{O}$. This gives the graded version of the Bernstein-Gelfand-Gelfand reciprocity (see also [BGS, Theorem 3.11.4]). The main result is a combinatorical description of the graded versions of translation functors in terms of elements in the corresponding Hecke algebra. The advantage of this approach is that it provides a way to construct lifts of principal series. The motivation to define such lifts was given by A. Joseph's article [Jo] where he describes the connection between filtered versions of principal series and primitive ideals in the universal enveloping algebra of $\mathfrak{g}$. Details of how to construct graded versions of principal series from the graded versions of translation functors and why this is useful to determine composition factors of quotients of the universal enveloping algebra of $\mathfrak{g}$ by the annihilator of some simple $\mathfrak{g}$-module can be found with some explicit examples in [St2]. Another advantage of our approach is that it can be easily generalised to the categories of Harish-Chandra bimodules with generalized trivial central character from both sides, whose power is fixed from one side; although we do not know how to deduce a graded duality.

References

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