Degeneration of modules and the construction of Prüfer modules

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Let Λ be an artin algebra (this means that Λ is a module-finite k-algebra, where k is an artinian commutative ring). Bautista-Pérez [BP] and Smalø [S] have recently shown the following: Let W, W' be Λ-modules of finite length with isomorphic tops and isomorphic first syzygy modules. If W and W' have no self-extensions, then W and W' are isomorphic. This is well-known in case k is an algebraically closed field, but it is of interest to know such a result also for example for Λ being a finite ring. Actually, for k an algebraically closed field, the usual algebraic geometry arguments allow a stronger conclusion: If W has no self-extension, then W' is a degeneration of W (in the following sense: W' belongs to the closure of the orbit of W in the corresponding module variety). The first aim of the lecture was to show a corresponding result for general Λ, using the notion of a degeneration as introduced by Riedtmann-Zwara [Z1]: the module W' is said to be a *degeneration* of W provided there is an exact sequence of finite length modules of the form: $0 \to X \to X \oplus W \to W' \to 0$ (in case k is algebraically closed, the notions coincide, as Zwara [Z2] has shown).

Proposition 1. *Let U_0, U_1 be finite length modules, and $w, w'\colon U_0 \to U_1$ monomorphisms. Denote by W, W' the cokernels of w, w', respectively. If W has no self-extensions, then W' is a degeneration of W.*

Let us describe in which way one obtains a corresponding Riedtmann-Zwara sequence. Actually, let us consider a slightly more general setting for the following *tower construction*: Start with a pair of maps $w_0, v_0\colon U_0 \to U_1$ between finite length modules, such that w_0 is a proper monomorphism with cokernel W. Forming inductively pushouts, we obtain a sequence of maps $w_i, v_i\colon U_i \to U_{i+1}$ with $i \geq 0$, such that all the maps w_i are monomorphisms with cokernel W (and such that $w_{i+1}v_i = v_{i+1}w_i$ for all i). We form the direct limit U_∞ of all the modules U_i with respect to the monomorphisms w_i (and we may assume that these maps w_i are inclusion maps), and consider also the module U_∞/U_0.

If we assume that W has no self-extensions, then U_∞/U_0 is an (infinite) direct sum of copies of W, and this implies that one of the inclusion maps w_i is a split monomorphism: thus U_{i+1} is isomorphic to $U_i \oplus W$. Now, if v_0 is also a monomorphism, say with cokernel W', then the inductive constuction of the module U_{i+1} yields an exact sequence $0 \to U_i \to U_{i+1} \to W' \to 0$. As we have seen, we can replace U_{i+1} by $U_i \oplus W$, thus we deal with a Riedtmann-Zwara sequence. This completes the proof of proposition 1.

Let us return to the general setting of dealing with a pair of maps $w_0, v_0\colon U_0 \to U_1$ between finite length modules, such that w_0 is a proper monomorphism with cokernel W. The maps $v_i\colon U_i \to U_{i+1}$ yield a map $v_\infty : U_\infty \to U_\infty$ which maps U_0 into U_1 and which induces an isomorphism $\overline{v}\colon U_\infty/U_0 \to U_\infty/U_1$. If we compose the canonical projection $U_\infty/U_0 \to U_\infty/U_1$ with the inverse of $\overline{v}$, we obtain a locally nilpotent surjective endomorphism of U_∞/U_0 with kernel W. Let us call a module M a *Prüfer module with basis* W, provided there exists a locally nilpotent surjective endomorphism of M with kernel W of finite length; thus U_∞/U_0 is a Prüfer module with basis W.

A module M is said to be of *finite type* provided it is a direct sum of copies of a finite number of indecomposable modules of finite length (thus if and only if M is both endo-finite and pure-projective). Note that for the tower construction exhibited above, the module U_∞ is of finite type if and only if the Prüfer module U_∞/U_0 is of finite type. We are interested in Prüfer modules which are not of finite type, since there is the following result:

Proposition 2. *Let M be a Prüfer module which is not of finite type, and let I be an infinite set. Then the product module M^I has an indecomposable direct summand G which is of infinite length and endo-finite.*

Recall that a module N is said to be *endo-finite* provided it is of finite length when considered as a module over the opposite of its endomorphism ring. Indecomposable infinite length modules which are endo-finite have been called *generic* modules by Crawley-Boevey [CB]. He has shown that the existence of a generic module implies that there are infinitely many isomorphism classes of indecomposable finite-length modules of some fixed endo-length d (and actually the proof shows that there are infinitely many natural numbers d such that there are infinitely many isomorphism classes of indecomposable finite-length modules of endo-length d).

Proposition 2 is based on previous investigations of Krause [K], see also [R1]: Let M be a Prüfer module, then there is a surjective locally nilpotent endomorphism f with kernel of finite length; denote by $W[n]$ the kernel of f^n. Then M^I contains the union $U = \bigoplus_n W[n]^I$. This submodule is a direct sum of copies of M, and it is a direct summand of M^I, say $M^I = U \oplus U'$. The module U' is endo-finite, thus a direct sum of copies of finitely many indecomposable endo-finite modules. In case the latter modules all are of finite length, then one can show that M is of finite type. This then completes the proof of proposition 2.

We want to use the tower construction in order to obtain a wealth of Prüfer modules. For this, one needs submodules $U_0 \subset U_1$ with additional homomorphisms (or even embeddings) $U_0 \to U_1$, and of special interest seems to be the take-off part of the category of all Λ-modules of finite length (as introduced in [R2]).

References

[BP] R. Bautista and E. Pérez: On modules and complexes without self-extensions. Communications in Algebra (to appear).

[CB] W. Crawley-Boevey: Modules of finite length over their endomorphism ring. In: Representations of algebras and related topics (eds. Brenner, Tachikawa). London Math. Soc. Lec. Note Series 168. (1992), 127–184.

[K] H. Krause: Generic modules over artin algebras. Proc. London Math. Soc. 76 (1998), 276–306.

[R1] C.M. Ringel: A construction of endofinite modules. In: Advances in Algebra and Model Theory. Gordon-Breach. (ed. M. Droste, R. Göbel). London (1997). 387–399.

[R2] C.M. Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005). 726–748.

[S] S. Smalø: Homological conjectures and degenrations of modules. Lectures. Mar del Plata 2006.

[Z1] G. Zwara: A degeneration-like order for modules. Archiv der Mathematik 71 (1998). 437–444.

[Z2] G. Zwara: Degenerations of finite-dimensional modules are given by extensions. Compositio Mathematica 121 (2000). 205–218.