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Interactions between
Algebraic Geometry and Noncommutative Algebra

Torsion pairs and matrix categories

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Matrix categories were introduced by Drozd in 1972 in order to provide an abstract setting for dealing with matrix problems (as considered at that time intensively in Kiev and elsewhere), but results concerning the use of matrix categories are hidden in the literature. The theme of the lecture (in particular, its start) was devoted to a surely folklore result concerning the partial reconstruction of an abelian category from a torsion pair. By definition, a torsion pair $(\mathcal{F}, \mathcal{G})$ in an abelian category $\mathcal{A}$ is a pair of two full subcategories which are closed under isomorphisms such that first of all $\mathrm{Hom}(G, F) = 0$ for all objects G in $\mathcal{G}$ and F in $\mathcal{F}$, and second any object A in $\mathcal{A}$ has a subobject tA which belongs to $\mathcal{G}$ such that A/tA belongs to $\mathcal{G}$ (note that we use the convention that the first entry of the pair $(\mathcal{F}, \mathcal{G})$ is the subcategory $\mathcal{F}$ of "torsionless" objects, the second entry the subcategory $\mathcal{G}$ of "torsion" objects; this corresponds to the vision that non-zero maps are drawn from left to right whenever this is possible).

Given two additive categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ is by definition a bilinear functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{mod}\, k$. Given such an $\mathcal{A}$-$\mathcal{B}$-bimodule $E = {}_{\mathcal{A}}E_{\mathcal{B}}$, we consider the matrix category of E as introduced by Drozd: its objects are triples (A, B, m), where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$ and $m \in E(A, B)$, and a morphism $(A, B, m) \to (A', B', m')$ is a pair (α, β), where $\alpha : A \to A'$ and $\beta : B \to B'$ are morphisms in $\mathcal{A}$, and $\mathcal{B}$ respectively, such that $m\beta = \alpha m'$.

Proposition. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in the abelian category $\mathcal{A}$. Given $A \in \mathcal{A}$, let ϵ_A be the equivalence class of the canonical exact sequence $0 \to tA \to A \to A/tA \to 0$ with tA in $\mathcal{G}$ and A/tA in $\mathcal{F}$, this is an element of the group $\mathrm{Ext}^1(A/tA, tA)$. Then $\eta(A) = (A/tA, tA, \epsilon_A)$ defines a functor η from the category $\mathcal{A}$ to the matrix category of the $\mathcal{F}$-$\mathcal{G}$-bimodule $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$ which is full and dense and its kernel is the ideal generated by all maps $\mathcal{F} \to \mathcal{G}$.*

In particular, in case $\mathcal{A}$ is a Krull-Remak-Schmidt category, then we see that the kernel of the functor η lies in the radical of $\mathcal{A}$ and therefore η provides a bijection between the isomorphism classes of indecomposable objects of $\mathcal{A}$ and of the matrix category of the bimodule $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$.

The second part of the lecture was devoted to an application concerning the module category of a cluster-tilted algebra as introduced by Buan, Marsh and Reiten. Here, one begins with a tilted algebra B, thus with a tilting module T over a finite-dimensional hereditary algebra A so that B is the endomorphism ring of

T. Note that the global dimension of B is at most 2 and we may consider the B-B-bimodule $I = \mathrm{Ext}^2(DB, B)$, where D is the k-duality. The corresponding cluster-tilted algebra B^c may be defined as the trivial extension of B by the bimodule I. The tilting module T provides a torsion pair $(\mathcal{F}, \mathcal{G})$ in the module category $\mathrm{mod}\, A$, as well as a torsion pair $(\mathcal{Y}, \mathcal{X})$ in $\mathrm{mod}\, B$, and the latter is even split (this means that any indecomposable B-module belongs either to $\mathcal{Y}$ or to $\mathcal{X}$)), such that $\mathcal{G}$ is equivalent to $\mathcal{Y}$ and $\mathcal{F}$ is equivalent to $\mathcal{X}$. The pair $(\mathcal{Y}, \mathcal{X})$ is still a torsion pair in $\mathrm{mod}\, B^c$, but usually no longer split: the indecomposable B^c-modules which are not B-modules (thus those not annihilated by I) are neither torsion nor torsionfree; it is the multiplication by I which is responsible for obtaining non-trivial extensions of torsion modules by torsionfree modules, and these are the modules we are interested in!

Now the proposition asserts that the category $\mathrm{mod}\, A/\langle \mathrm{Hom}(\mathcal{F}, \mathcal{G})\rangle$ is equivalent to the matrix category for the $\mathcal{F}$-$\mathcal{G}$-bimodule $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$. Some calculations show that for $\mathrm{mod}\, B_c$, one may invoke in a similar way the matrix category of the $\mathcal{G}$-$\mathcal{F}$-bimodule $\mathrm{Hom}(\mathcal{G}, \tau\mathcal{F})$. In this way, we see that the module categories $\mathrm{mod}\, A$ and $\mathrm{mod}\, B_c$ are related to each other via the bimodule $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$ as well as its dual $\mathrm{Hom}(\mathcal{G}, \tau\mathcal{F})$, the duality being one of the basic assertions of the Auslander-Reiten theory (note that the algebra A we start with is assumed to be hereditary). The relationship shows that for k algebraically closed and T a preprojective tilting module (so that B is a concealed and B^c a "cluster-concealed" algebra) the dimension vectors of the indecomposable B^c-modules are the absolute values of the roots of the corresponding Kac-Moody root system. The proof relies on the one hand on a separation property or the support of torsion and torsionless B-modules, and on the other hand on an old result of de la Peña and Simson: they have shown that the indecomposable objects of the matrix category of a bimodule E correspond bijectively to the positive roots of some quadratic form r_E provided the matrix category of E is directed.

In the special case when A (and therefore also B^c) is representation-finite, we show in this way that the indecomposable B^c-modules are determined by their dimension vectors (this result has been independently obtained by Geng and Peng, and it provides a proof of the Fomin-Zelevinsky denominator conjecture for cluster algebras of simply laced Dynkin type).

References

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