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Fibonacci numbers and representations of quivers.

It is well-known that the entries of the dimension vectors of some distinguished 3-Kronecker modules are the Fibonacci numbers; the 3-Kronecker modules are triples of matrices with the same shape, thus representations M of the quiver Q with two vertices a, b and three arrows $a \to b$. Using covering theory, one may consider instead of Q its universal covering $\widetilde{Q}$, this is the 3-regular tree with bipartite orientation. Thus, instead of M we now deal with a representations $\widetilde{M}$ of $\widetilde{Q}$, but this means that the two vector spaces given by M are written as direct sums of a large number of much smaller vector spaces M_i. In terms of dimensions, we exhibit in this way partition formulae for the Fibonacci numbers. The dimensions of the vector spaces M_i can be arranged to form two triangles, one for the even-index Finbonacci numbers, the other for the odd-index ones, one may compare the triangles with the Pascal triangle of binomial coefficients. These distributions of numbers are additive functions on valued translation quivers, but also they may be constructed inductively by using some hook formula. The hook formula shows that the numbers along the inclined lines can be obtained by evaluating monic integral polynomials (but the actual coefficients of these polynomials are not known). One column of the first triangle has been identified by Hirschhorn (2008) as the number of Delannoy paths which do not cross horizontally the main diagonal. It seems astonishing that no other row or column had hitherto been recorded in Sloane's Encyclopedia of Integer Sequence.

There are intriguing relations between the two triangles: for example, differences along suitable arrows in one triangle are numbers which occur in the other triangle. Also, the left hand side and the right hand side of the odd-index triangle determine each other. All these results were found by looking at certain exact sequences involving Fibonacci modules, but they can be verified also recursively. There is another interpretation of the numbers in the triangles, namely as dimensions of some (subspace) representations of the 3-regular tree now with a unique sink and no sources. Using this result, one sees that any vector space M_i itself is indecomposable with respect to an internally defined set of subspaces. This shows that the Fibonacci modules are not only indecomposable, but also locally indecomposable. In addition, the exact sequences mentioned above can be used in order to write the Fibonacci modules inductively as tree modules and surprisingly, only one-dimensional Ext-groups have to be considered.