Auslander varieties for wild algebras

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1. Quiver Grassmannians and Auslander varieties.

Let k be an algebraically closed field and Λ a finite-dimensional k-algebra. Let $\operatorname{mod}\Lambda$ be the category of left Λ-modules of finite length (we call them just modules). A *dimension vector* $\mathbf{d}$ for Λ is a function defined on the set of isomorphism classes of simple modules S with non-negative integral values $\mathbf{d}_S$. If M is a module, its dimension vector $\mathbf{dim}\, M$ attaches to the simple module S the Jordan-Hölder multiplicity $(\mathbf{dim}\, M)_S = [M : S]$.

Given a module M, let $\mathbb{G}_{\mathbf{e}}M$ be the set of all submodules of M with dimension vector $\mathbf{e}$, this is called a quiver Grassmannian, it is always a projective variety. If we denote by $\mathcal{S}M$ be the set of the submodules of M, then $\mathcal{S}M$ is the disjoint union of (finitely many) subsets $\mathbb{G}_{\mathbf{e}}M$. Note that $\mathcal{S}M$ is a lattice with respect to intersection and sum, and the subsets $\mathbb{G}_{\mathbf{e}}M$ consist of pairwise incomparable elements.

If C, Y are modules, then we consider $\operatorname{Hom}(C, Y)$ as a $\Gamma(C)$-module, where $\Gamma(C) = \operatorname{End}(C)^{\mathrm{op}}$. The easiest way to define the Auslander varieties for Λ is to say that they are just the quiver Grassmannians $\mathbb{G}_{\mathbf{e}} \operatorname{Hom}(C, Y)$. This is the fast track definition, but it conceales the relevance of the Auslander varieties.

In order to provide the motivation, we have to outline Auslander's theory of C-determination of morphisms, developed already in 1974 (see [1], and also [4]). We assume now only that Λ is an artin algebra. The aim of Auslander's theory is to describe the global directedness of the category $\operatorname{mod}\Lambda$.

Let Y be a module. Let $\bigcup_X \operatorname{Hom}(X, Y)$ be the class of all morphisms ending in Y. We define a preorder $\preceq$ on this class as follows: Given $f : X \to Y$ and $f' : X' \to Y$, we write $f \preceq f'$ provided there is a morphism $h : X \to X'$ with $f = f'h$. As usual, such a preorder defines an equivalence relation by saying that f, f' are *right equivalent* provided we have both $f \preceq f'$ and $f' \preceq f$.

The object studied by Auslander is the **set $[\to Y\rangle$ of right equivalence classes of maps ending in Y** (it should be stressed that it is a set, not only a class). Using the preorder $\preceq$, this set $[\to Y\rangle$ is a poset, even a lattice (for the joins, one uses direct sums, for the meets, one uses pullbacks). The map $0 \to Y$ is the zero element of $[\to Y\rangle$, the identity map $Y \to Y$ is its unit element.

Recall that a map $f : X \to Y$ is said to be *right minimal* provided any direct summand X' of X with $f(X') = 0$ is equal to zero. Every right equivalence class in $[\to Y\rangle$ contains a right minimal morphism.

Let $f : X \to Y$ be a morphism and C a module. Then f is said to be *right C-determined* provided the following condition is satisfied: given any morphism $f' : X' \to Y$ such that $f'\phi$ factors through f for all $\phi : C \to X'$, then f' itself factors through f. We denote by ${}^C[\to Y\rangle$ the subset of $[\to Y\rangle$ of sll right equivalence classes of right C-determined morphisms.

Here are the main assertions of Auslander:

(1) *The set $[\to Y\rangle$ is the union of the subsets ${}^C[\to Y\rangle$.* If C, C' are modules, both ${}^C[\to Y\rangle$ and ${}^{C'}[\to Y\rangle$ are contained in ${}^{C\oplus C'}[\to Y\rangle$, thus we deal with a filtered union. The essential assertion is that any morphism is right determined by some module.

(2) Let C, Y be modules. *There is a lattice isomorphism*

$$\eta_{CY} : \ {}^C[\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y)$$

defined as follows: if $f : X \to Y$, then $\eta_{CY}(f)$ is the image of $\operatorname{Hom}(C, f) : \operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$. The essential assertion is again the surjectivity of η_{CY}, thus to say that any $\Gamma(C)$-submodule of $\operatorname{Hom}(C, Y)$ is of the form $\Im \operatorname{Hom}(C, f)$. The isomorphisms η_{CY} are called the **Auslander bijections.**

The isomorphism η_{CY} allows to shift properties from $\mathcal{S}\operatorname{Hom}(C, Y)$ to ${}^C[\to Y\rangle$. Many properties of submodule lattices are known, all can be transfered via η_{CY} to ${}^C[\to Y\rangle$. It is a modular lattice (thus ${}^C[\to Y\rangle$ is a modular lattice): The modules M we are dealing with have finite length, we denote the length of M by $|M|$. The Jordan-Hölder theorem asserts that all composition series have the same length and given two composition series, there is a bijection between the composition factors. Via the transfer, we have a corresponding Jordan-Hölder theorem for ${}^C[\to Y\rangle$: given a right C-determined map f ending in Y, we can define its C-length $|f|_C = |\operatorname{Hom}(C, Y)| - |\eta_{CY}(f)|$. The C-length of f can also be defined directly, looking at suitable factorizations of f. Given a factorization $f = f'h$, where f, f' are right C-determined maps ending in Y with $|f|_C = |f'|_C + 1$, then $\eta_{CY}(f) \subset \eta_{CY}(f')$ and the factor $\eta_{CY}(f')/\eta_{CY}(f)$ is a simple $\Gamma(C)$-module. Thus, the Jordan-Hölder theorem for ${}^C[\to Y\rangle$ allows to attach to any right C-determined map its C-dimension vector.

Let us return to the case where Λ is a finite-dimensional k-algebra and k is an algebraically closed field. If C, Y are modules, we use the Auslander bijection $\eta_{CY} : \ {}^C[\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y)$. Given a dimension vector $\mathbf{e}$ for $\Gamma(C)$, the elements of the Auslander variety $\mathbb{G}_{\mathbf{e}}\operatorname{Hom}(C, Y)$. correspond under η_{CY} to the right equivalence classes of maps ending in Y with C-dimension vector $\mathbf{e}$.

2. (Controlled) wildness

According to Drozd, any finite dimensional k-algebra is either tame or wild (and most algebras are wild). It has been conjectured that wild algebras are actually controlled wild (as defined below). A proof of this conjecture has been announced by Drozd [2] in 2007, but apparently it has not yet been published.

Let rad be the radical of $\operatorname{mod}\Lambda$, this is the ideal generated by all non-invertible maps between indecomposable modules. If $\mathcal{U}$ is a collection of objects of $\operatorname{mod}\Lambda$, we denote by $\operatorname{add}\mathcal{U}$ the closure under direct sums and direct summands. For every pair X, Y of modules, $\operatorname{Hom}(X, \mathcal{U}, Y)$ denotes the subgroup of $\operatorname{Hom}(X, Y)$ given by the maps $X \to Y$ which factor through a module in $\operatorname{add}\mathcal{U}$.

The algebra Λ is said to be *controlled wild* provided for any finite-dimensional k-algebra Γ (or, equivalently, just for the algebra $\Gamma = k[T_1, T_2, T_3]/(T_1, T_2.T_3)^2$) there is an exact embedding functor $F : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ and a full subcategory $\mathcal{U}$

of $\operatorname{mod}\Lambda$ (called the *control class*) such that for all Γ-modules X, Y, the subgroup $\operatorname{Hom}(FX, \mathcal{U}, FY)$ is contained in $\operatorname{rad}(FX, FY)$ and

$$\operatorname{Hom}(FX, FY) = F\operatorname{Hom}(X, Y) \oplus \operatorname{Hom}(FX, \mathcal{U}, FY).$$

3. Quiver Grassmannians

A recent paper of Reineke [3] asserts: *Every projective variety is a quiver Grassmannian $\mathbb{G}_{\mathbf{e}}M$ for a module M with endomorphism ring k.*

Let us outline a construction. Let $\mathcal{V}$ be a projective variety, say a closed subset of the projective space $\mathbb{P}^n$, defined by the vanishing of homogeneous polynomials $f_1, \dots, f_m$ of degree 2. Let Δ be the quiver with 3 vertices a, b, c, with $n+1$ arrows $b \to a$ labeled $x_0, \dots, x_n$ as well as $n+1$ arrows $c \to b$, also labeled $x_0, \dots, x_n$. The path algebra of Δ with all possible relations $x_i x_j = x_j x_i$ is called the Beilinson algebra B. Let Λ be the factor algebra of B taking the elements $f_1, \dots, f_m$ as additional relations (considered as linear combinations of paths of length 2). Let I be the indecomposable injective B-module corresponding to the vertex a, and take $\mathbf{e} = (1, 1, 1)$. Now $\mathbb{G}_{\mathbf{e}}I$ is the set of all serial submodules of I of length 3 (a module is serial, provided it has a unique composition series). There is an obvious identification of this set $\mathbb{G}_{\mathbf{e}}I$ with $\mathbb{P}^n$. Let M be the indecomposable injective Λ-module corresponding to the vertex a. Then M is a submodule of I. Also, a submodule W of I is a submodule of M if and only if W is a Λ-module. Thus the serial submodules W of M of length 3 correspond just to the elements of $\mathcal{V}$. One may say that this construction is really tautological.

Here are some remarks on the history: The 2-page paper by Reineke attracted a lot of interest, see for example blogs by L. Le Bruyn and by J. Baez. The construction given above was presented by M. Van den Bergh in Le Bruyn's blog, but actually, it is much older: it has been used before by B. Huisgen-Zimmermann (1998) and L. Hille (1996) dealing with related problems.

There are controlled wild algebras Λ such that not every projective variety can be realized as a quiver Grassmannian of a Λ-module.

As an example, take $\Lambda = k[T_1, T_2, T_3]/(T_1, T_2, T_3)^2$. One can show that $\mathbb{G}_i M$ is rationally connected, for every module M and any $0 \le i \le \dim M$.

4. Auslander varieties

Theorem. *Let Λ be a finite-dimensional k-algebra which is controlled wild. Let $\mathcal{V}$ be any projective variety. Then there are Λ-modules C, Y and a dimension vector $\mathbf{e}$ for $\Gamma(C)$ such that $\mathbb{G}_{\mathbf{e}}\operatorname{Hom}(C, Y)$ is of the form $\mathcal{V}$.*

Outline of proof. Let $\mathcal{V}$ be a projective variety. There is a finite-dimensional algebra Γ, a Γ-module M and a dimension vector $\mathbf{g}$ for Γ such that $\mathbb{G}_{\mathbf{g}}M$ is of the form $\mathcal{V}$, as we have seen in section 3. Since Λ is controlled wild, there is a controlled embedding $F : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$, say with control class $\mathcal{U}$. Let $G = F({}_\Gamma\Gamma)$ and $Y = F(M)$. There is $U \in \operatorname{add}\mathcal{U}$ such that $\operatorname{Hom}(G, U, G) = \operatorname{Hom}(G, \mathcal{U}, G)$ and $\operatorname{Hom}(G, U, Y) = \operatorname{Hom}(G, \mathcal{U}, Y)$. Let $C = G \oplus U$ and $R = \operatorname{End}(C)^{\mathrm{op}}$. Let e_G be the projection of C onto G with kernel U and $e = e_U$ the projection of C onto U with kernel G, both e_G, e_U considered as elements of R. Note that

$$R = F(\mathrm{Hom}(\Gamma,\Gamma)) \oplus \mathrm{Hom}(G \oplus U, \mathcal{U}, G \oplus U) \oplus \mathrm{Hom}(G, U) \oplus \mathrm{Hom}(U, G) \oplus \mathrm{Hom}(U, U),$$

and

$$ReR = \mathrm{Hom}(G \oplus U, \mathcal{U}, G \oplus U) \oplus \mathrm{Hom}(G, U) \oplus \mathrm{Hom}(U, G) \oplus \mathrm{Hom}(U, U).$$

It follows that the map $\gamma \mapsto F(\gamma) \in e_G R e_G$ yields an isomorphism $\Gamma \to R/ReR$. Consider the R-module

$$N = \mathrm{Hom}(G \oplus U, Y) = \mathrm{Hom}(\Gamma, M) \oplus \mathrm{Hom}(G \oplus 0, U, Y) \oplus \mathrm{Hom}(0 \oplus U, Y).$$

If we multiply N with the element $e = e_U \in R$, we obtain $eN = \mathrm{Hom}(0 \oplus U, Y)$, thus

$$ReN = R\,\mathrm{Hom}(0 \oplus U, Y) = \mathrm{Hom}(G \oplus 0, U, Y) \oplus \mathrm{Hom}(0 \oplus U, Y).$$

This shows that N/ReN is canonically isomorphic to $F\,\mathrm{Hom}(\Gamma, M)$ as an R-module. Of course, these modules are annihilated by e, thus they are R/ReR-modules and as we know $R/ReR = \Gamma$, thus $\mathcal{S}_\Gamma(N/ReN)$ can be identified with $\mathcal{S}_R M$.

Let $\mathbf{c} = \mathbf{dim}\, ReN$. If $\mathbf{g}$ is a dimension vector and W belongs to $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$, then $W \supseteq ReN$, and W/ReN is an element of $\mathbb{G}_{\mathbf{g}}(N/ReN)$. As a consequence, the varieties $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$ and $\mathbb{G}_{\mathbf{g}}(N/ReN) = \mathbb{G}_{\mathbf{g}}M = \mathcal{V}$ can be identified.

References

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