What is known about invariant subspaces of nilpotent operators?

Survey. I

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I. The problem: Invariant subspaces of a nilpotent operator T.

k a field.

(V, T) a *nilpotent operator*:
V vector space, $T\colon V \to V$ linear map with $T^n =$ for some n.

Typical example: The pair $M(n) = (k^n, J(n))$, with Jordan block

$$J(n) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

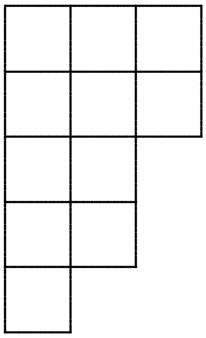
If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$ is a decreasing sequence of natural numbers (a "partition"), let

$$M(\lambda) = M(\lambda_1) \oplus \cdots \oplus M(\lambda_t).$$

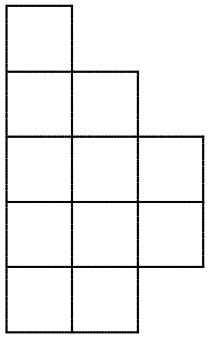
Any nilpotent operator is isomorphic to $M(\lambda)$ for a **unique** *partition λ.*

Visualization: Dealing with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$, we draw the corresponding *Young diagram.*
Our convention: the parts correspond to the **columns**,
the i-the column consists of λ_i boxes.

Example: $\lambda = (5, 4, 2)$

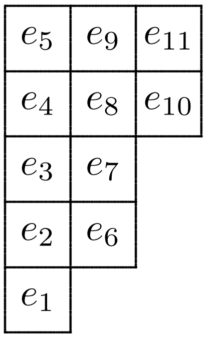


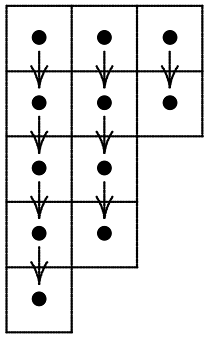
or, say



Consider the boxes as base vectors for k^n with T the shift downwards.

This yields
$J(5) \oplus J(4) \oplus J(2)$





Let $\mathcal{S}(n)$ be the class of *triples* (V, T, U), where

V is a finite-dimensional k-space V,
T a linear operator $T\colon V \to V$ with $T^n = 0$,
U a subspace of V with $T(U) \subseteq U$, write $W = V/U$ (if needed)

An *isomorphism* between (V, T, U) and (V', T', U') is an invertible linear map $f\colon V \to V'$ with $f(U) = U'$ and $fT = T'f$.

The *direct sum* of $X = (V, T, U)$ and $X' = (V', T', U')$ is the triple

$$X \oplus X' = (V \oplus V', T \oplus T', U \oplus U').$$

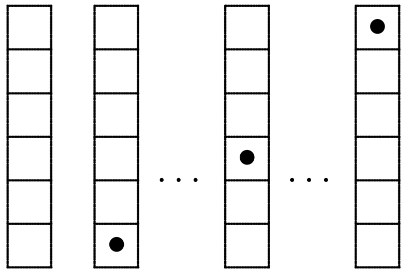
(V, T, U) is said to be *indecomposable* provided it is not zero and not isomorphic to a direct sum $X \oplus X'$ with non-zero triples X, X' (the *zero triple* is $(0, 0, 0)$).

Any triple is a direct sum of indecomposable triples and these direct summands are unique up to isomorphism (Krull-Remak-Schmidt property).

Aim. To classify the isomorphism classes of indecomposable triples.

Example 1 ("pickets"):
The triples (V,T,U) with $(V,T)=(k^n, J(n))$.

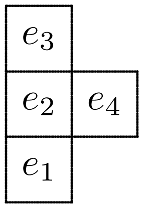
The only invariant subspaces U are the subspaces
$0, \quad k\times 0^{n-1}, \quad k^2\times 0^{n-2}, \quad \dots \quad , \quad k^n.$



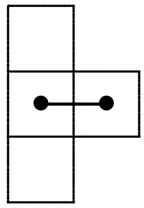
A bullet indicates a generator of (U,T).

Easy: If (V,T,U) is indecomposable and $\dim U = 1$, then (V,T) is a picket.

Example 2:



with (U,T) generated by e_2+e_4



Note: Both U, W are of dimension 2 and indecomposable (with respect to T)
It follows: (V,T,U) is an indecomposable triple.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing n.

In $S(n)$ there are two special triples: $(k^n, J(n), 0)$ and $(k^n, J(n), k^n)$.

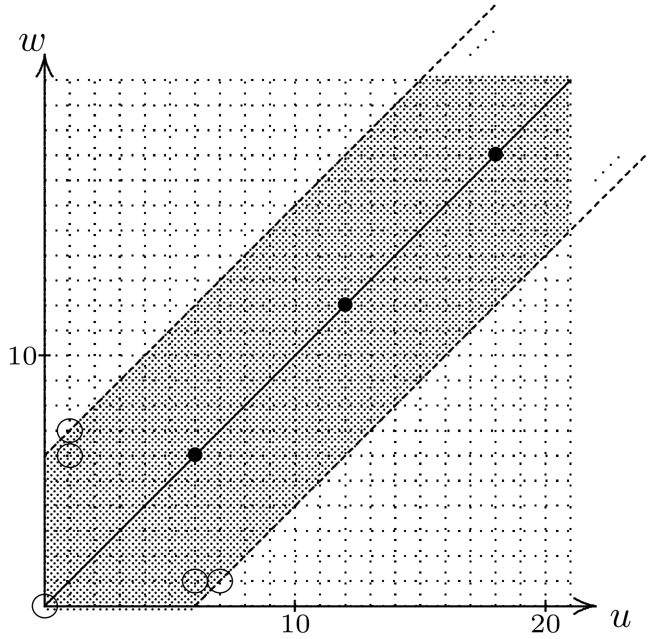
n	number of indecomposables		
1	2	$= 2 + \ \ 0$	$= 2 + 0$
2	5	$= 2 + \ \ 3$	$= 2 + \frac{3}{2} \cdot 2$
3	10	$= 2 + \ \ 8$	$= 2 + 2 \cdot 4$
4	20	$= 2 + 18$	$= 2 + 3 \cdot 6$
5	50	$= 2 + 48$	$= 2 + 6 \cdot 8$
6	∞		$= 2 + \frac{6}{0} \cdot 10$

$$2 + \frac{6}{6-n} 2(n-1)$$

The *dimension pair* of a triple (V, T, U) with $W = V/U$ is the pair $(\dim U, \dim W)$.

The case $n = 6$.

Theorem 1. *A pair (u,w) of natural numbers is the dimension pair of an indecomposable triple in $\mathcal{S}(6)$ if and only if (u,w) satisfies $|u-w| \le 6$ and is different from* $(0,0)$, $(1,6)$, $(1,7)$, $(6,1)$, $(7,1)$.



We may reformulate part of Theorem 1 as follows:

Assume (V,T,U) *is in* $\mathcal{S}(6)$ *and* $|\dim V - 2\dim U| > 6$. *Then there are non-zero subspaces* V', V'' *with* $V = V' \oplus V''$ *such that*

$$T(V') \subset V', \quad T(V'') \subset V'' \quad \text{and} \quad U = (U \cap V') \oplus (U \cap V'').$$

For (V,T,U) indecomposable in $\mathcal{S}(6)$, the dimension of U is roughly half of the dimension of V.

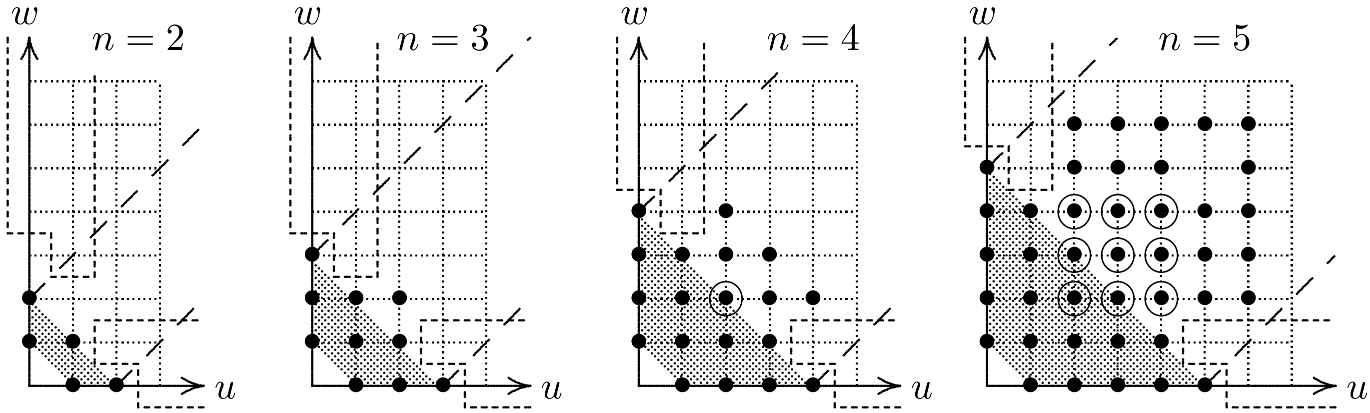
Why do we have to exclude the pairs $(1,6)$, $(1,7)$, $(6,1)$, $(7,1)$?

If (V,T,U) is indecomposable and $\dim U = 1$,
then (V,T) is indecomposable.
This shows that $(1,6)$ and $(1,7)$ do not occur.

The pairs $(6,1)$ and $(7,1)$ are excluded, using duality:
If (V,T,U) is in $\mathcal{S}(n)$, then also $(V^*, T^*, (V/U)^*)$ is in $\mathcal{S}(n)$.

For $n \leq 6$ and $(V, T, U) \in \mathcal{S}(n)$, we even have $|u - w| \leq n$,
and this bound is always optimal.

Here are the cases $n = 2, 3, 4, 5$ (always, the picket region is shaded).

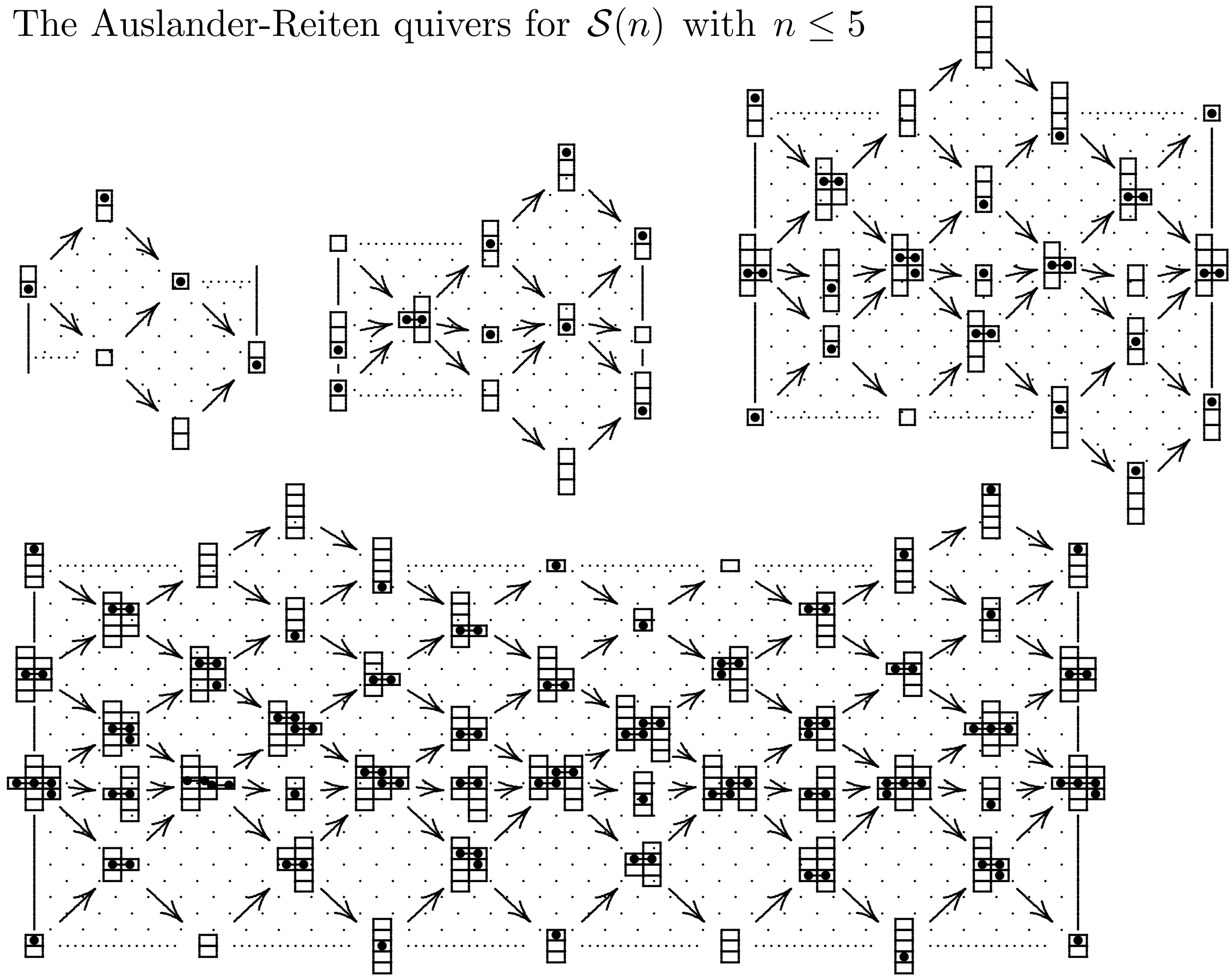


Encircled bullets: there are (precisely) two indecomposable triples.

Note: For $n \leq 5$, the pair (n, n) does not occur as dimension pair!

For $n \geq 7$, the numbers $|u - w|$ are not bounded,
but the possible dimension pairs are not yet known.

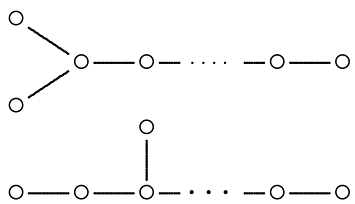
The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$



n	number of indecomposables			hidden Lie type
1	2	$= 2 + 0$	$= 2 + 0$	$\emptyset$
2	5	$= 2 + 3$	$= 2 + \frac{3}{2} \cdot 2$	A_2
3	10	$= 2 + 8$	$= 2 + \frac{6}{3} \cdot 4$	D_4
4	20	$= 2 + 18$	$= 2 + \frac{6}{2} \cdot 6$	E_6
5	50	$= 2 + 48$	$= 2 + 6 \cdot 8$	E_8
6	∞			

↑ a tree

Here is the list of the simply laced Dynkin diagrams considered in Lie theory:

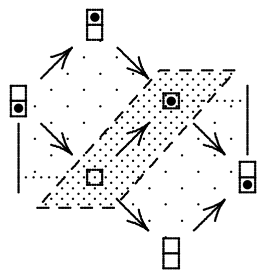


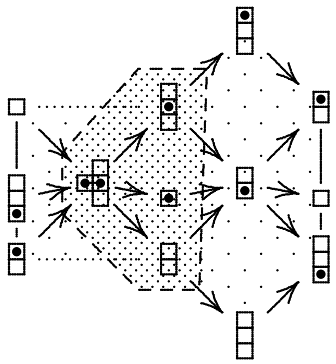
A_n

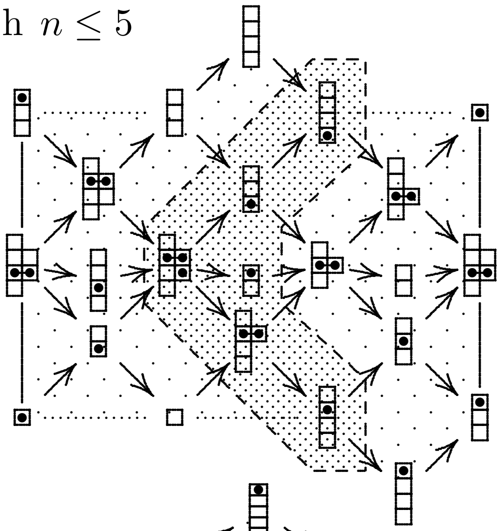
D_n

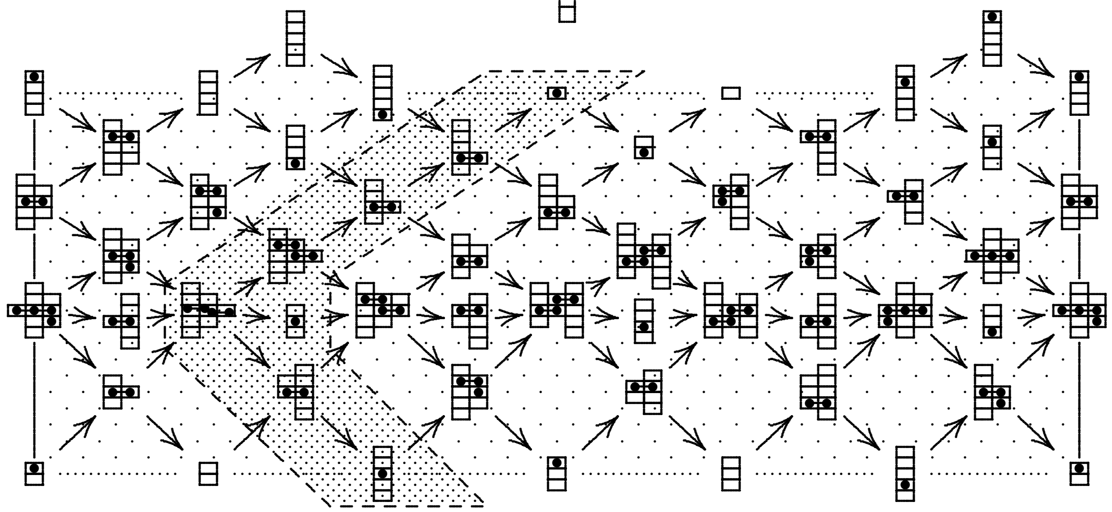
$E_m \quad (m = 6, 7, 8)$

The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$



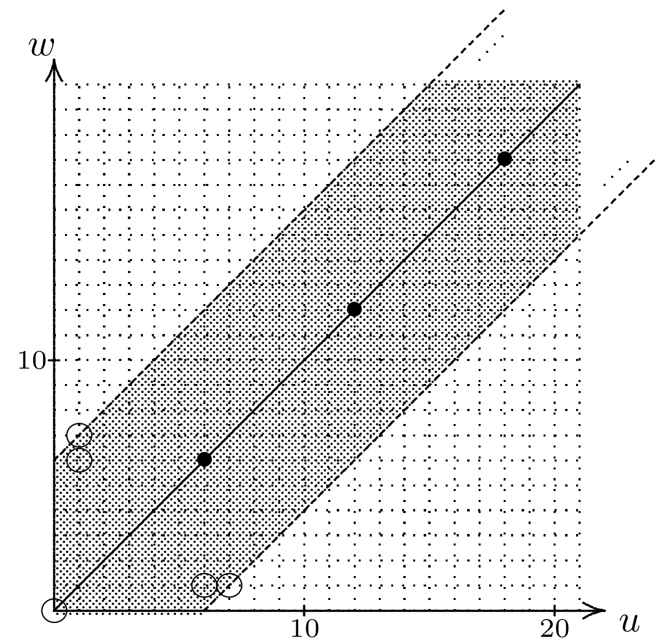






We return to $n = 6$.

Theorem 2. *If (u, w) is not in $\mathbb{N}(6,6)$, then the number of isomorphism classes of indecomposable objects in $\mathcal{S}(6)$ with dimension pair (u, w) is finite (and independent of k).*

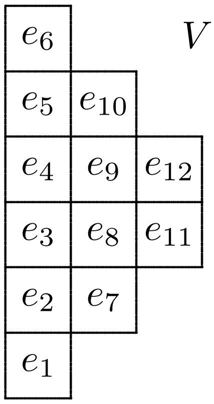


By contrast, the triples with dimension pair in $\mathbb{N}(6,6)$ depend on the field k.

Let $(u, w) = t(6, 6)$ with $t \in \mathbb{N}_1$.

There are t disjoint one-parameter families of indecomposable triples with dimension pair (u, w).

$t = 1$



$V = M(6, 4, 2)$

generators of U: $e_8 + e_{11}$

$e_4 + (1 - c)e_9 - ce_{12}$

for any $c \in k$

$t = 2$

$V = M(6, 6, 4, 4, 2, 2)$

or

$V = M(6, 6, 5, 3, 3, 1)$

Weakly homogeneous triples. An indecomposable triple (V, T, U) in $\mathcal{S}(n)$ is called *weakly homogeneous* provided $(V/U, T)$ is isomorphic to (U, T) and (V, T) is isomorphic to $(U, T) \oplus M(n)^t$ for some t. Then

$$\dim U + tn = \dim V = \dim U + \dim V/U = 2 \dim U,$$

thus $\dim(V, T, U) = t(n, n)$.

For $n \leq 5$, there are no weakly homogeneous triples. Return to $n = 6$.

Theorem 3. *For any $t \in \mathbb{N}_1$, there are t pairwise disjoint one-parameter families of weakly homogeneous triples in $\mathcal{S}(6)$ with dimension pair $t(6, 6)$, each being indexed by $k \setminus \{0, 1\}$.*
If k is algebraically closed, then there are only finitely many additional isomorphism classes indecomposable triples in $\mathcal{S}(6)$ with dimension pair $t(6, 6)$ (and these triples are defined independently of k).

If (V, T, U) is weakly homogeneous, then $U = M(4, 2)^r \oplus M(5, 3, 3, 1)^s$ for some r, s (and $t = r + 2s$).

For $r > 0, s > 0$, there are **two** one-parameter families of weakly homogeneous triples with $U = M(4, 2)^r \oplus M(5, 3, 3, 1)^s$. Later we will see how to distinguish these two families.

Recall: (V,T,U) indecomposable in $\mathcal{S}(6)$. Then $|\dim U - \frac{1}{2}\dim V| \leq 3$.
This means: The dimension of U is **roughly** half of the dimension of V.

If (V,T,U) is weakly homogeneous, then we even have: $\dim U = \frac{1}{2}\dim V$.

The structure theorem for weakly homogeneous triples asserts:
$V = M(4,2)^r \oplus M(5,3,3,1)^s \oplus M(6)^{r+2s}$. This implies:

$$\begin{aligned} \dim \operatorname{Ker} T &= \tfrac{1}{4}\dim V \\ \dim \operatorname{Ker} T^3 &= \tfrac{2}{3}\dim V \\ \dim \operatorname{Ker} T^5 &= \tfrac{11}{12}\dim V \end{aligned}$$

$$\begin{aligned} \tfrac{11}{24}\dim V &\leq \dim \operatorname{Ker} T^2 \leq \tfrac{1}{2}\dim V \\ \tfrac{19}{24}\dim V &\leq \dim \operatorname{Ker} T^4 \leq \tfrac{5}{6}\dim V \end{aligned}$$

$$\dim \operatorname{Ker} T^4 / \operatorname{Ker} T^2 = \tfrac{1}{3}\dim V$$

Again, for indecomposable triples which are not weakly homogeneous, these (in)equalities are **roughly** true: they hold up to small differences ...

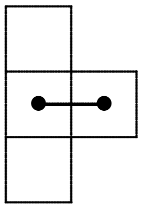
Graded triples. A grading of the triple (V, T, U) is a direct decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$ such that $T(V_i) \subseteq V_{i-1}$ and $U = \bigoplus (U \cap V_i)$.

Theorem 4. *For $n = 6$, any triple in $\mathcal{S}(6)$ can be graded. For indecomposable triples, the grading is unique up to shift.*

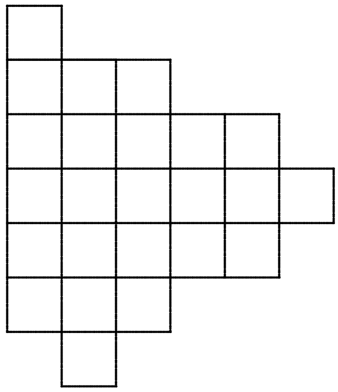
Interpretation. Write $(V, T) = M(\lambda)$ where λ is a partition.
Visualize $M(\lambda)$ using the Young diagram of λ, present the parts as columns.
A grading of (V, T, U) means **to adjust the columns** conveniently.

Examples:

The indecomposable triple with dim pair $(2, 2)$



The columns of a weakly homogeneous triple with $U = M(5, 3, 3, 1)$ are adjusted as follows:



The grading theorem is the essential result!
It provides a lot of new invariants: We can refine

$$u = \sum u_i, \quad w = \sum w_i,$$

where

$$u_i = \dim U \cap V_i, \quad w_i = \dim V_i/(U \cap V_i).$$

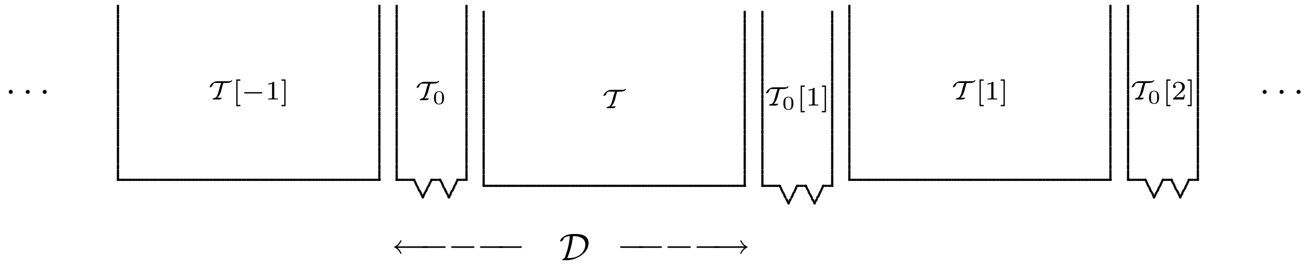
A graded triple is a system of vector spaces and linear maps as follows:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{T} & U_0 & \xleftarrow{T} & U_1 & \xleftarrow{T} & U_2 & \xleftarrow{T} & \cdots \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
\cdots & \xleftarrow[T]{} & V_0 & \xleftarrow[T]{} & V_1 & \xleftarrow[T]{} & V_2 & \xleftarrow[T]{} & \cdots
\end{array}$$

The squares are commutative (and $T^6 = 0$).

$\mathcal{S}(\widetilde{6})$ denotes the category of graded triples (V, T, U) with $T^6 = 0$.

The category $\mathcal{S}(\widetilde{6})$ can be described in the following way:



$\mathcal{D}$ is a fundamental domain for $\mathcal{S}(\widetilde{6})$ under the shift σ.

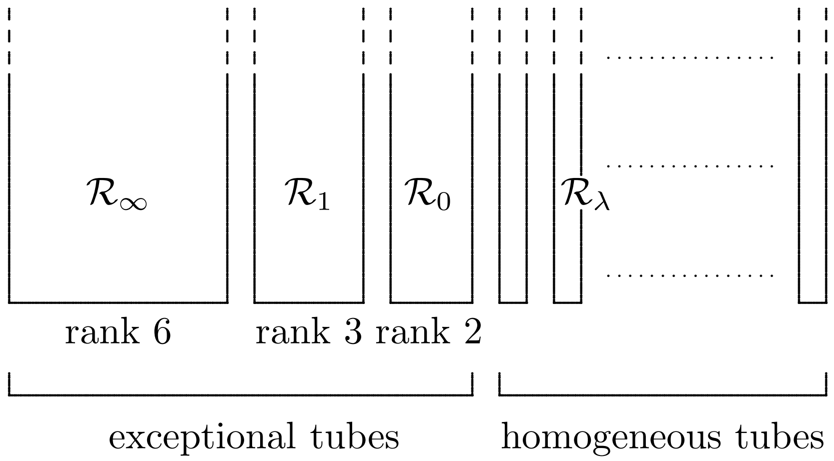
Since $\mathcal{S}(\widetilde{6})$ is “locally bounded”, it follows that $\mathcal{S}(6) = \mathcal{S}(\widetilde{6})/\sigma$ (= the grading theorem).

Of essential importance is the central part $\mathcal{T}$,
there are **countably many stable tubular families** $\mathcal{T}_\gamma$ indexed by $\gamma \in \mathbb{Q}^+$,
each $\mathcal{T}_\gamma$ is **a** $\mathbb{P}_1(k)$**-family of tubes of type** $(6,3,2)$**.**

$n = 6$:

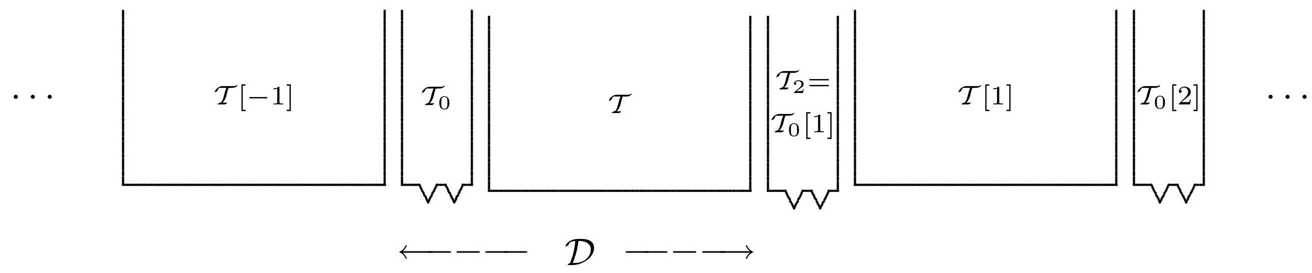
There are countable many stable tubular families in $\mathcal{T}$, all of type $(6,3,2)$.

A tubular family of type $(6,3,2)$ has the following form:



The index set for the tubular families we are interested in, will always be $\mathbb{P}_1(k)$.

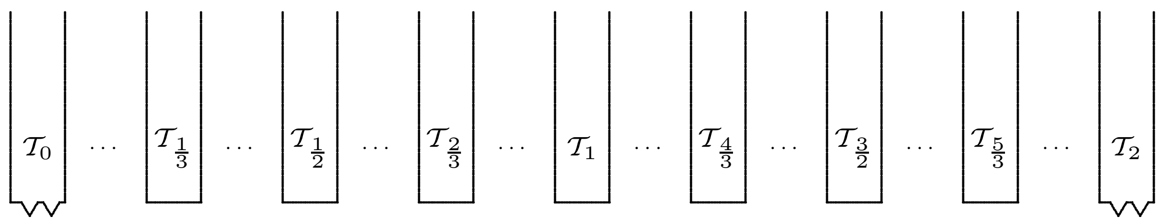
Here is the category $\mathcal{S}(\widetilde{6})$, and $\mathcal{D}$ is a fundamental domain for the shift σ.



Any object in $\mathcal{S}(\widetilde{6})$ has a “slope” $\gamma \in \mathbb{Q}$, the shift σ increases the slope by 2.

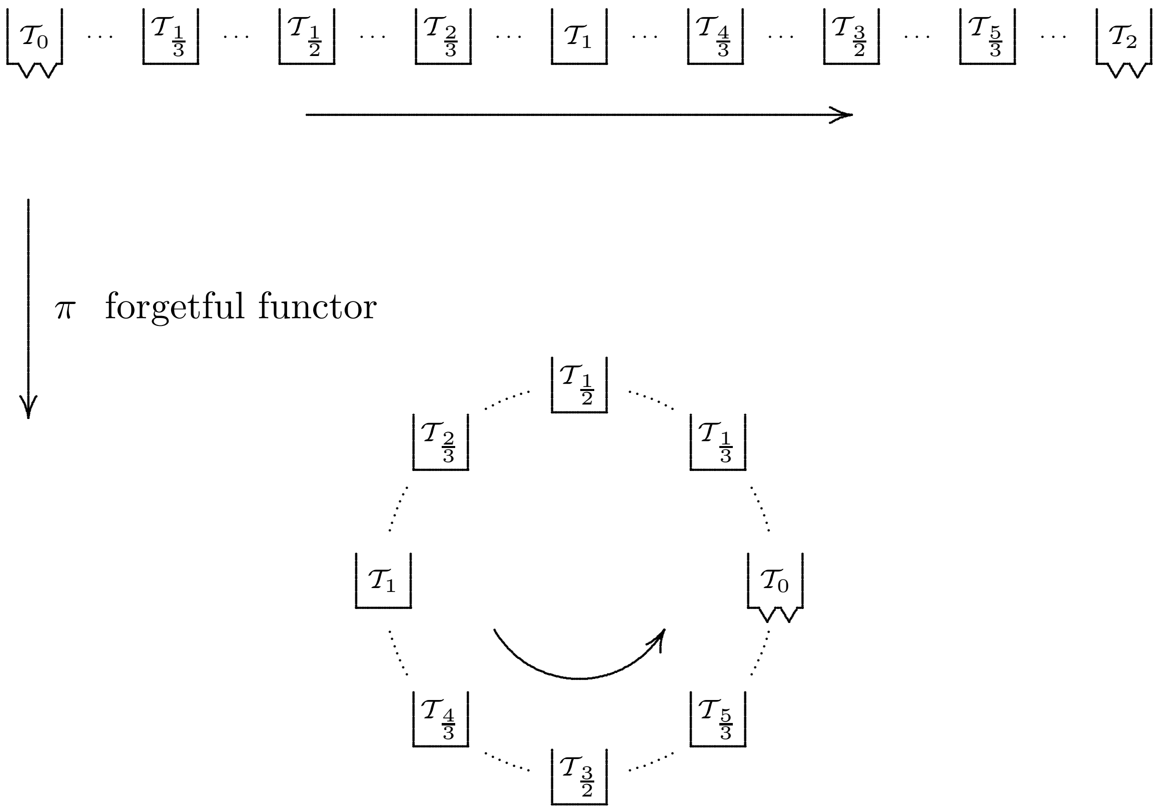
The objects with slope γ form the subcategory $\mathcal{T}_\gamma$.

The part containing the objects with slope in $\mathbb{Q}^+ \cap [0, 2]$ looks as follows:



The graded triples with a fixed slope form a tubular family of type $(6, 3, 2)$.

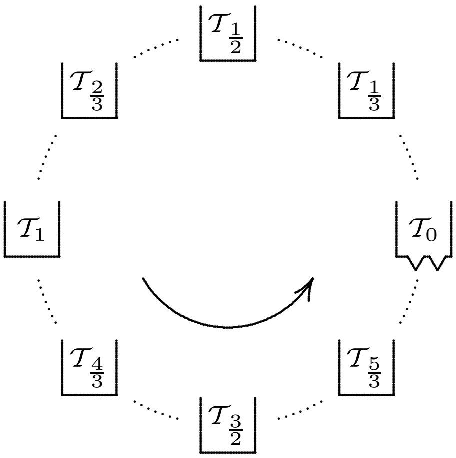
Forgetting the grading



To repeat: The classification of the indecomposable triples in $\mathcal{S}(6)$ (in case k is algebraically closed):

There are two projective-injective triples, with dimension pair $(0,6)$ and $(6,0)$.

For the remaining triples, one needs three invariants .



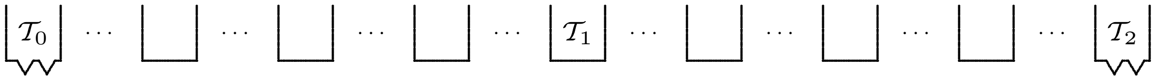
First invariant: The slope,
a rational number $0 \le \gamma < 2$

Second invariant: The spectral parameter c,
an element of $\mathbb{P}_1(k) = k \cup \{\infty\}$

Third invariant: A vertex x in a tube.
If $c \notin \{0,1,\infty\}$, then $x \in \mathbb{N}$
If $c \in \{0,1,\infty\}$, then $x = (i,m)$, $m \in \mathbb{N}$
$1 \le i \le 2$ for $c = 0$
$1 \le i \le 3$ for $c = 1$
$1 \le i \le 6$ for $c = \infty$

Recall: Almost all indecomposable triples (V, T, U) with fixed dimension pair are **weakly homogeneous** (i.e. $U \simeq V/U,\ V \simeq U \oplus M(6)^t$), and then $U = M(4,2)^r \oplus M(5,3,3,1)^s$ for some pair r, s.

Let $X = (V, T, U)$ be weakly homogeneous with slope γ.



For $\gamma = 0$ $U = M(4,2)^r$

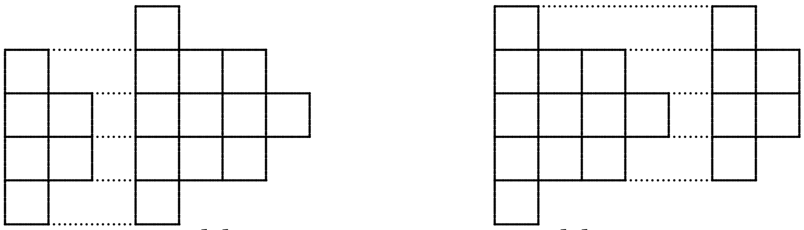
For $\gamma = 1$ $U = M(5,3,3,1)^s$

For $\gamma = 2$ $U = M(4,2)^r$

For $0 < \gamma < 1$ there is an exact sequence
$$0 \to X' \to X \to X'' \to 0$$
slope 0 slope 1

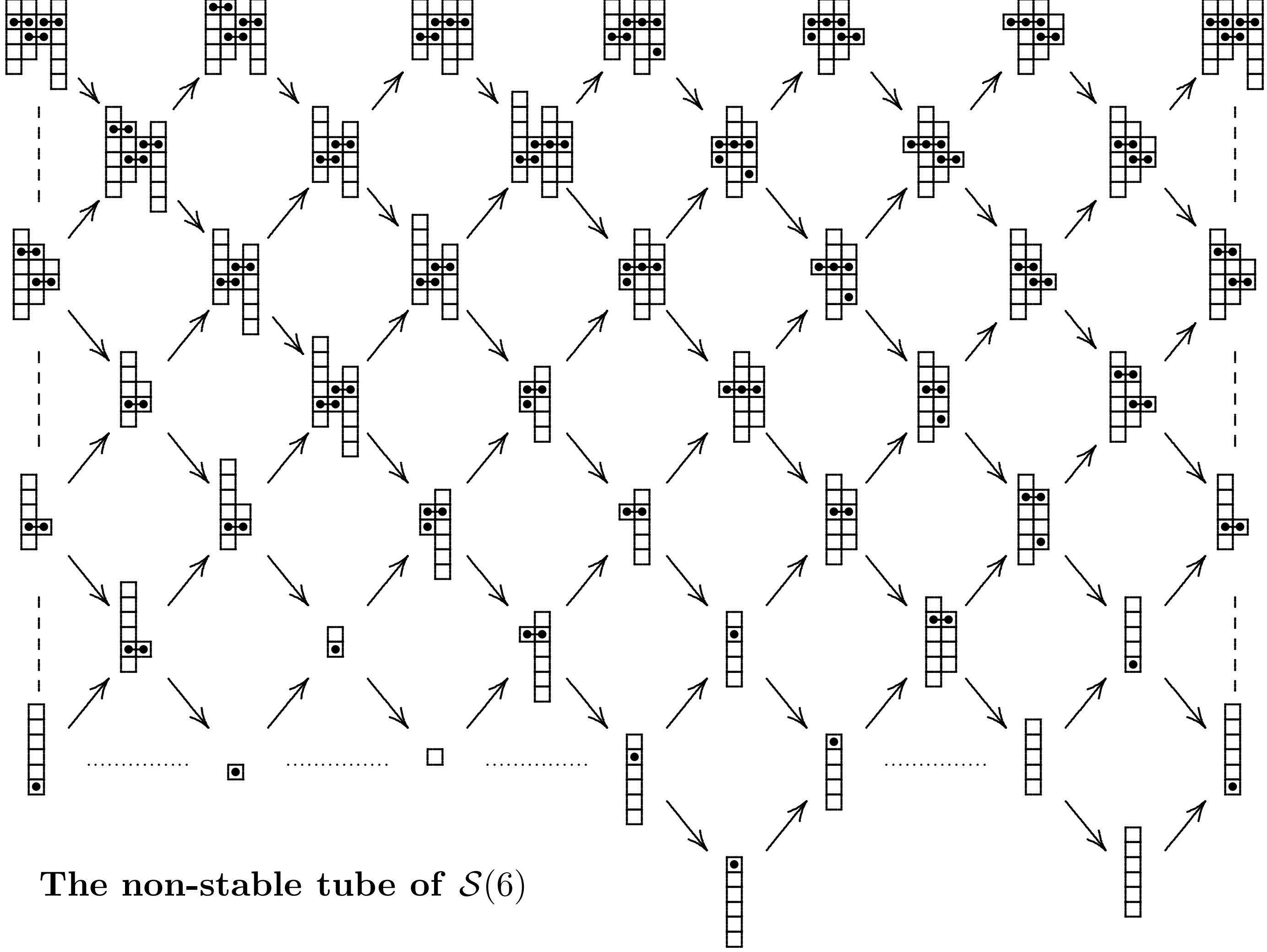
For $1 < \gamma < 2$ there is an exact sequence
$$0 \to X' \to X \to X'' \to 0$$
slope 1 slope 2

Adjustment of the columns



even odd

odd even



The non-stable tube of $\mathcal{S}(6)$

The non-homogeneous stable tubes for $\gamma = 0$.