A Categorification of the Fibonacci Numbers Using Representations of Quivers

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Categorification

Replace	**by**
elements	objects
equations between elements	isomorphisms between objects
sets	categories
functions	functors
equations between functions	natural isomorphisms between functors

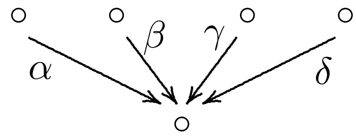
We replace	**by**
numbers	vector spaces
equality of numbers	isomorphism of vector spaces
...	...
functions	functors
...	...

The notion “categorification” was coined by L. Crane in 1998.

k field

$Q = (Q_0, Q_1)$ finite quiver
(some say directed graph or digraph; multiple arrows are allowd)
Q_0 set of vertices, Q_1 the set of arrows.

Example:
the so-called 4-subspace quiver



$M = (M_x, M_\alpha)_{x,\alpha}$ a representation of Q:
for every vertex x there is given a finite-dimensional vector space M_x,
for every arrow $\alpha\colon x \to y$ there is given a linear map $M_\alpha\colon M_x \to M_y$

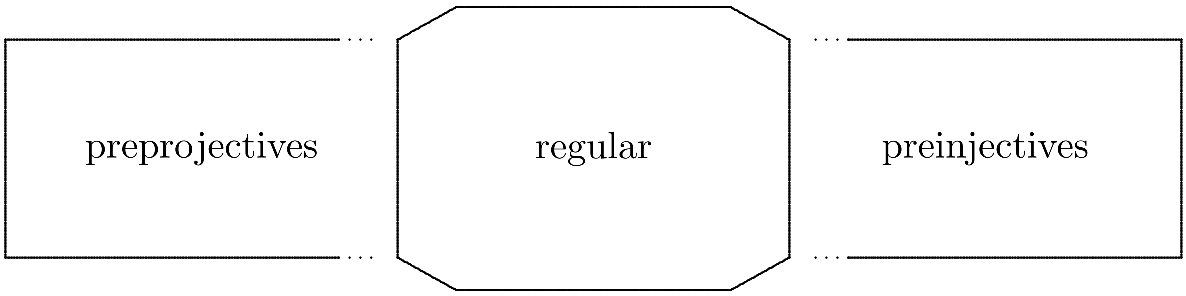
representations of Q "are" just the kQ-modules:

kQ the path algebra of Q
the vector space with basis the set of all paths (including paths of length zero),
as multiplication take the concatination of paths, whenever this is possible,
otherwise the product is set to be 0,
extend this multiplication of paths to the linear combination of paths bilinearly

mod kQ the category of kQ-modules (= representations of Q)

We assume: at every vertex x start and end only finitely many paths.
This yields an indecomosable projective module $P(x)$,
as well as an indecomposable injective module $Q(x)$.

The structure of mod kQ, for Q a connected finite quiver,
with no oriented cycles:



maps go from left to right (and arbitrarily inside the regular part).

Exceptional modules.

These are the indecomposable kQ-modules M without self-extensions (i.e.: given a kQ-module M' with submodule M and with M'/M isomorphic to M, then $M' = M \oplus M$).

The exceptional modules are indecomposable modules which are "generic": there is a neighborhood (in the Zariski topology) of isomorphic modules.

In particular, an exceptional module M is determined by its dimension vector $\mathbf{dim}\, M = (\dim M_x)_{x \in Q_0}$.

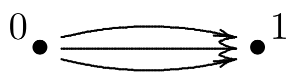
The preprojective modules and the preinjective modules are always exceptional.

Example: The Kronecker quivers.

The n-Kronecker quiver has two vertices 0, 1,
and n arrows $0 \to 1$.

For a Kronecker quiver, the preprojectives and the preinjectives are the only exceptional modules,
any other representation-infinite quiver (connected, finite, no oriented cycles) has additional exceptional modules.

Now, let Q be the 3-Kronecker quiver.



The preprojective kQ-modules P_t have dimension vector (f_{2t}, f_{2t+2})
where $f_0, f_1, \ldots$ are the Fibononacci numbers $0, 1, 1, 2, 3, 5, \ldots$.

Fibonacci Numbers (introduced by Fibonacci (Leonardo da Pisa), 1202):

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	$\cdots$
0	1	1	2	3	5	8	13	21	34	55	89	$\cdots$

they are recursively given by

$$f_0 = 0, \qquad f_1 = 1$$
$$f_{n+1} = f_{n-1} + f_n$$

We also use Fibonacci numbers with negative indices
(with the same recursion formula; Knuth calls them NegaFibonacci Numbers)

$\cdots$	f_{-6}	f_{-5}	f_{-4}	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\cdots$
$\cdots$	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	$\cdots$

Note that $f_{-n} = (-1)^{n+1} f_n$.

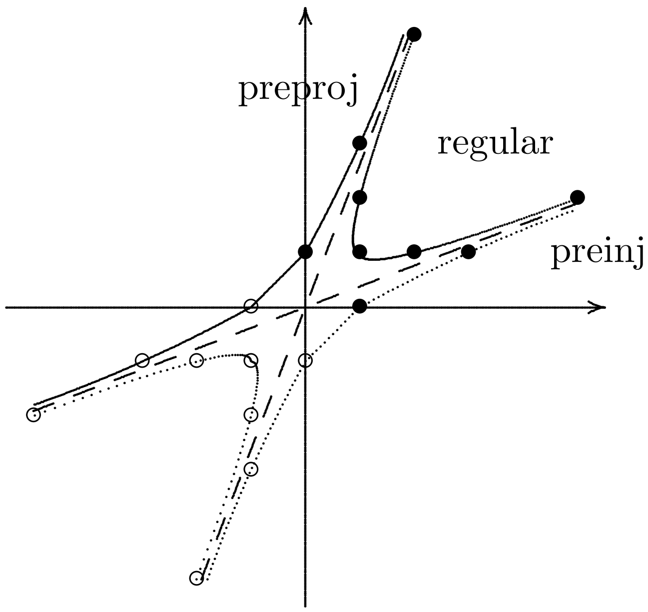
We call the pairs (f_t, f_{t+2}) **Fibonacci pairs.**

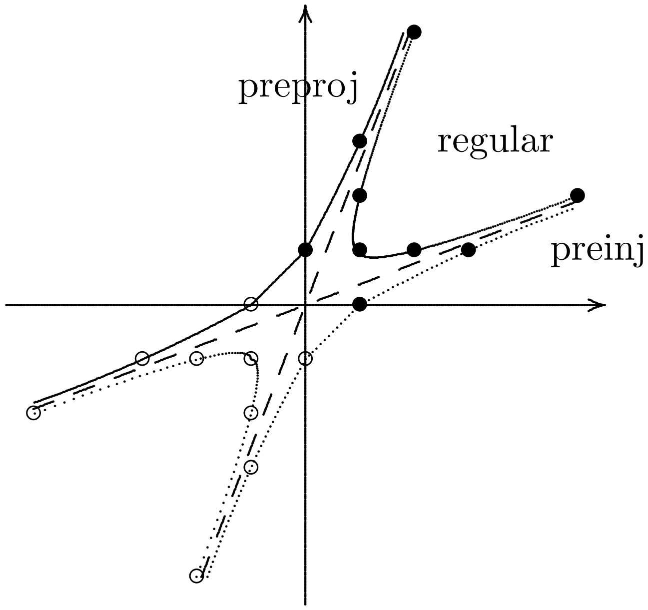
Categorification of the Fibonacci pairs:

We call an indecomposable 3-Kronecker module M a *Fibonacci module* provided $\dim \mathrm{Ext}^1(M,M) \le 2$.

There are three kinds of Fibonacci modules:

- The preprojective modules P_t with $\mathbf{dim}\, P_t = (f_{2t}, f_{2t+2})$, and $t \ge 0$.
- The preinjective modules Q_t, with $\mathbf{dim}\, Q_t = (f_{2t+2}, f_{2t})$, with $t \ge 0$.
- Families $R_t = R_t(\lambda)$ of regular modules with $\mathbf{dim}\, R_t = (f_{2t-1}, f_{2t+1})$ and $\lambda \in \mathbb{P}_2$.





Here we see the Grothendieck group $K_0(kQ) = \mathbb{Z}^2 \subset \mathbb{R}^2$,
the bullets mark the position $\mathbf{dim}\, M$ of the Fibonacci modules M.

On $K_0(kQ)$, the bilinear form $\langle -, - \rangle$ plays an important role:

$$\langle (x,y), (x'y') \rangle = xx' + yy' - 3xy', \qquad \text{since}$$

$$\langle \mathbf{dim}\, M_1, \mathbf{dim}\, M_2 \rangle = \dim \operatorname{Hom}(M_1, M_2) - \dim \operatorname{Ext}^1(M_1, M_2)$$

for all modules M_1, M_2, it is called *Euler form.*

Let q be the corresponding quadratic form $q(x,y) = \langle (x,y), (x,y) \rangle$. Then:

The Fibonacci modules are precisely
the indecomposable 3-Kronecker modules M with $|q(\mathbf{dim}\, M)| = 1$.

Recall that the pairs (f_t, f_{t+2}) are said to be Fibonacci pairs.
Recursion formula for Fibonacci pairs, written in 3 different ways:

(1) $$f_{t+2} = 3f_t - f_{t-2}.$$

Proof: $f_{t+2} - 3f_t + f_{t-2}$ is just the sum of $f_{t+2} - f_{t+1} - f_t$, of $f_{t+1} - f_t - f_{t-1}$ and of $-(f_t - f_{t-1} - f_{t-2})$, thus equal to zero.

The same recurrence formula works not only for going up, but also for going down:

(2) $$f_{t-2} = 3f_t - f_{t+2},$$

And we can write:

(3) $$f_{t-2} + f_{t+2} = 3f_t.$$

Categorification of the recursion formula for Fibonacci pairs:

For (1) and (2), there are the reflection functors σ_+ and σ_-

Let $M = (M_1, M_2, \alpha, \beta, \gamma)$ be a 3-Kronecker module
let $(\sigma_+ M)_2 = M_1$, let $(\sigma_+ M)_1$ be the kernel of the map $(\alpha, \beta, \gamma)\colon M_1^3 \to M_2$;
if M is indecomposable and not P_0, then $\dim(\text{kernel}) = 3\dim M_1 - \dim M_2$.

$(\sigma_- M)_1 = M_2$ and $\sigma_+ M)_2$ is the cokernel of the map $(\alpha, \beta, \gamma)^t\colon M_1 \to M_2^3$;
if M is indecomposable and not Q_0, then $\dim(\text{cok}) = 3\dim M_2 - \dim M_1$.

(3) is categorified by the Auslander-Reiten sequences

$$0 \to P_{t-1} \to (P_t)^3 \to P_{t+1} \to 0$$

$$0 \to Q_{t+1} \to (Q_t)^3 \to Q_{t-1} \to 0$$

for $t \geq 1$ and

$$0 \to R_{t-1}(\lambda) \to E_t(\lambda) \to R_{t+1}(\lambda) \to 0$$

with an indecomposable module $E_t(\lambda)$ having dimension vector $3\,\mathbf{dim}\,R_t(\lambda)$.

Categorification of the recursion formula for Fibonacci numbers:

There are exact sequences of the following form:

$$0 \to P_{t-1} \to P_t \to R_t \to 0,$$

$$0 \to P_{t-1} \to R_t \to R_{t-1} \to 0.$$

Let us write the dimension vectors instead of the modules:

$$0 \to \begin{bmatrix} f_{2t-2} \\ f_{2t} \end{bmatrix} \to \begin{bmatrix} f_{2t} \\ f_{2t+2} \end{bmatrix} \to \begin{bmatrix} f_{2t-1} \\ f_{2t+1} \end{bmatrix} \to 0,$$

$$0 \to \begin{bmatrix} f_{2t-2} \\ f_{2t} \end{bmatrix} \to \begin{bmatrix} f_{2t-1} \\ f_{2t+1} \end{bmatrix} \to \begin{bmatrix} f_{2t-3} \\ f_{2t-1} \end{bmatrix} \to 0.$$

Iteration:

P_t has a filtration

$$P_0 \subset P_1 \subset \cdots \subset P_{t-1} \subset P_t$$

with factors

$$P_i/P_{i-1} = R_i.$$

This corresponds to the summation formula $f_{2t} = \sum_{i=1}^{t} f_{2i-1}$.

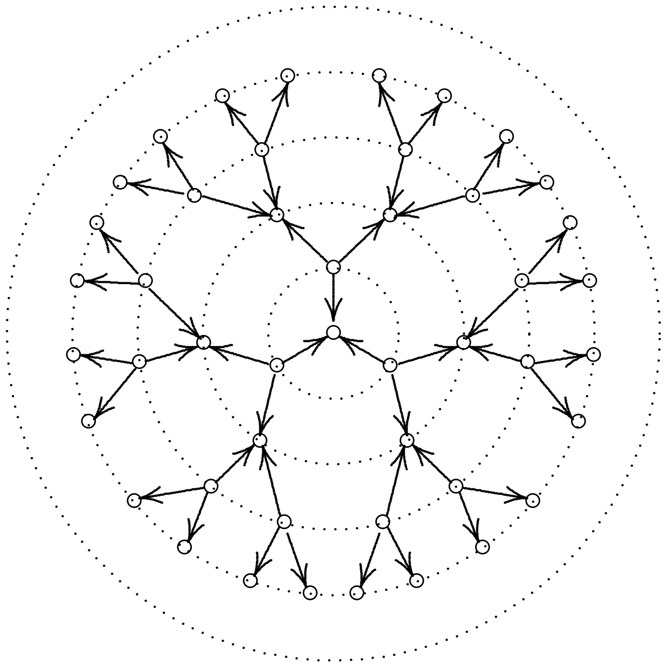
The corresponding filtration for the modules R_t splits nearly everywhere: R_t has a submodule R'_t with

$$R'_t = P_0 \oplus P_1 \oplus \cdots \oplus P_{t-1} \qquad \text{and} \qquad R_t/R'_t = R_0.$$

This corresponds to the summation formula $f_{2t+1} = 1 + \sum_{i=1}^{t} f_{2i}$.

The universal covering.

The universal covering of the 3-Kronecker quiver Q is the 3-regular tree T (the tree such that any vertex has precisely 3 neighbours).



The fundamental group of Q is the free (non-abelian) group Γ in 3 generators.

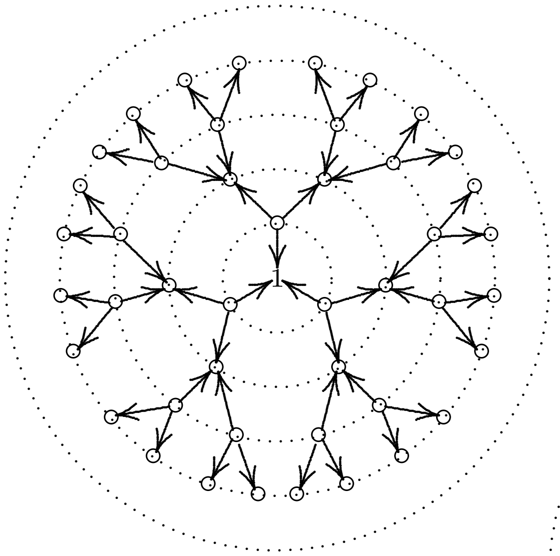
The Γ-graded representations of Q are the representations of T,

Claim: *The exceptional* 3*-Kronecker modules are gradable.*

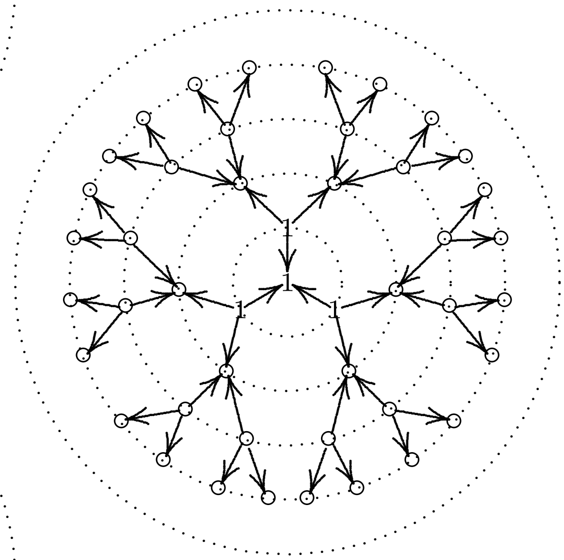
More generally: *For every Fibonacci pair, there are gradable modules with this dimension vector.*

Next, we exhibit the graded versions of the modules P_t with $0 \leq t \leq 5$

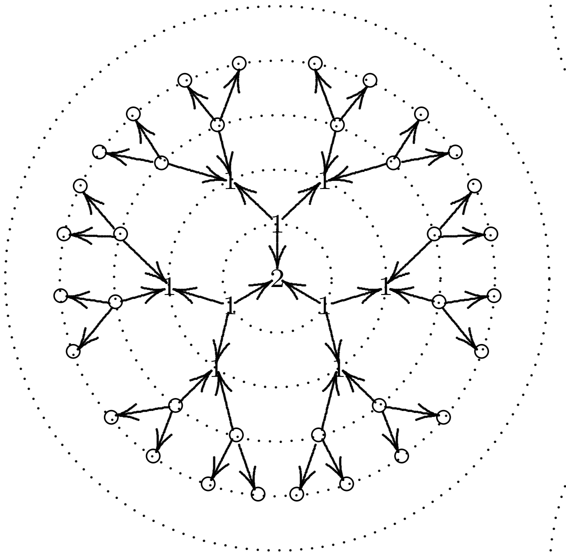
P_0



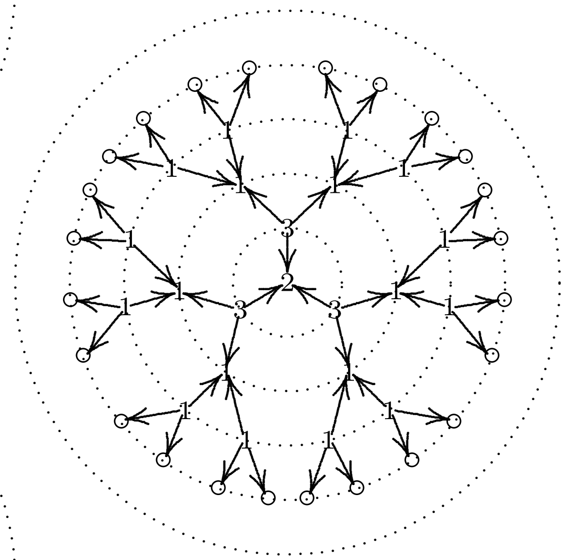
P_1



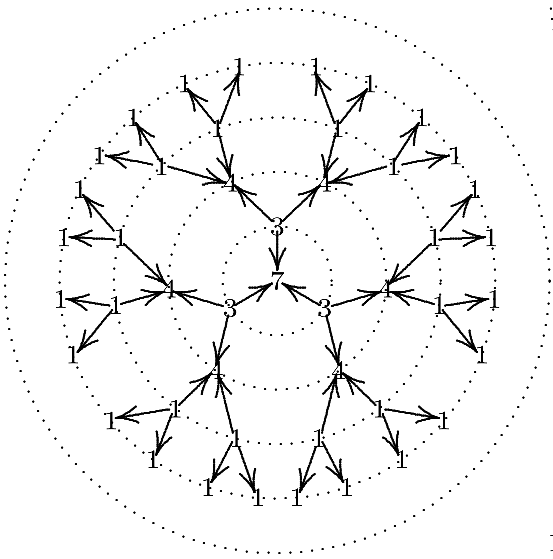
P_2



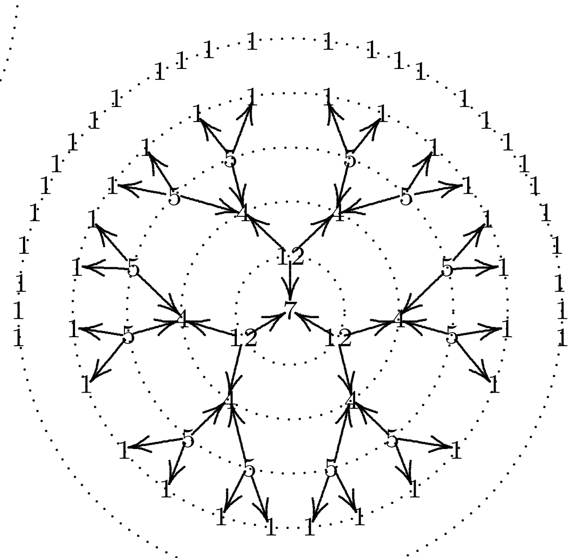
P_3



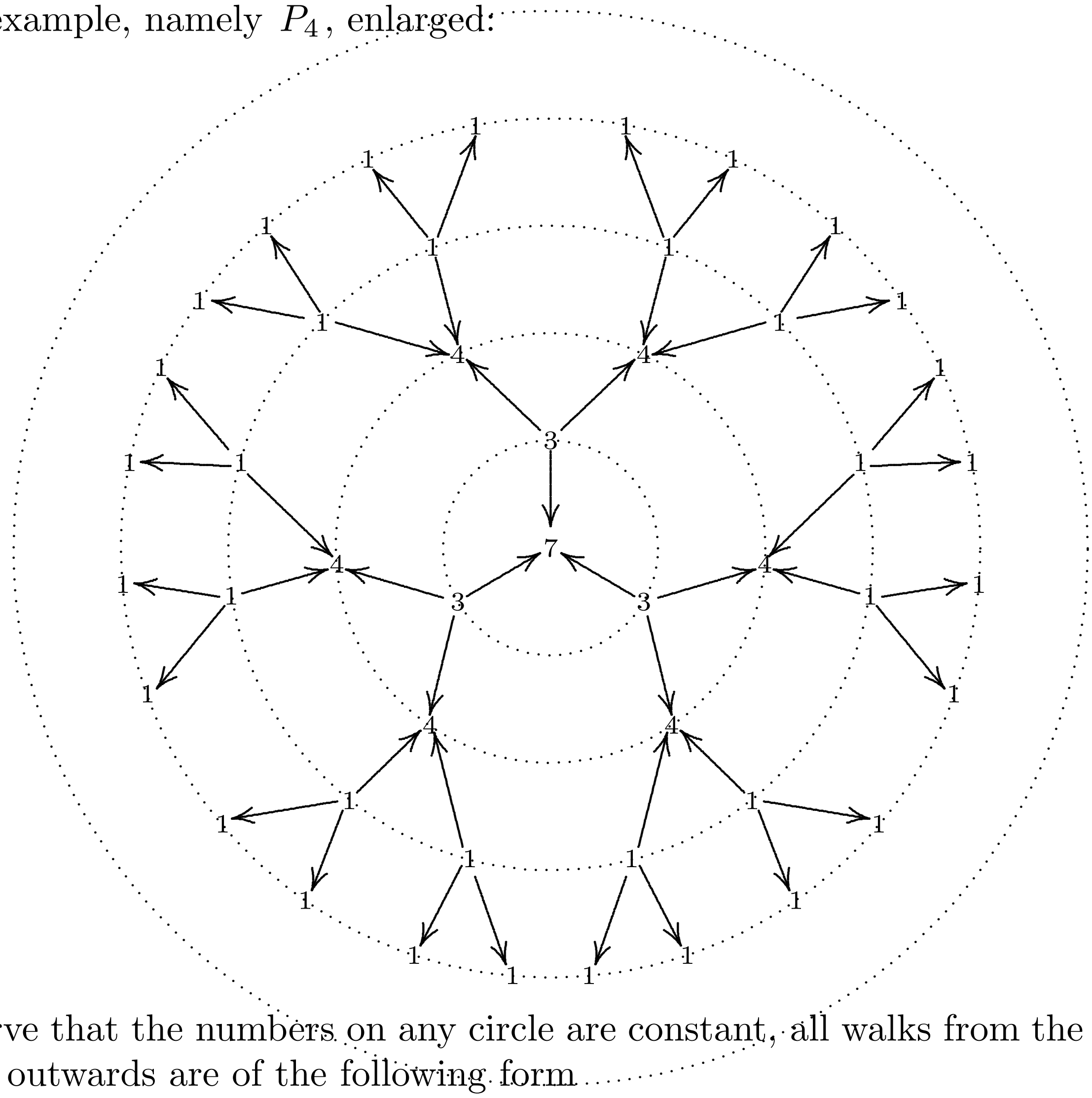
P_4



P_5



One example, namely P_4, enlarged:



Observe that the numbers on any circle are constant, all walks from the center going outwards are of the following form

$$7 \longleftarrow 3 \longrightarrow 4 \longleftarrow 1 \longrightarrow 1$$

The push-down functor for the covering (= the forgetful functor for the grading) yields the preprojective 3-Kronecker module P_4 with $\mathbf{dim}\, P_4 = (21, 55)$.

The graded pieces of the vector space $(P_4)_0$ of dimension 21 are three vector spaces of dimension 3 and twelve vector spaces of dimension 1, and $(P_4)_1$ is decomposed into one subspace of dimension 7, six subspaces of dimension 3, and twenty-four subspaces of dimension 1.

A partion formula for even index Fibonacci numbers (Fahr-R., 2008)

There are natural numbers $a_t[m]$ (defined inductively) with

$$f_{4t} = \quad 3\sum_{m\geq 0} 2^{2m}\cdot a_t[2m{+}1],$$
$$f_{4t+2} = a_t[0] \ + \ 3\sum_{m\geq 1} 2^{2m-1}\cdot a_t[2m].$$

Here is the table for $t \leq 6$.

t	$a_t[0]$	$a_t[1]$	$a_t[2]$	$a_t[3]$	$a_t[4]$	$a_t[5]$	$a_t[6]$	$a_t[7]$	$a_t[8]$	$a_t[9]$	$\cdots$	f_{4t}	f_{4t+2}
0	1											0	1
1	2	1	1									3	8
2	7	3	4	1	1							21	55
3	29	12	18	5	6	1	1					144	377
4	130	53	85	25	33	7	8	1	1			987	2584
5	611	247	414	126	177	42	52	9	10	1	$\cdots$	6765	17711
6	2965	1192	2062	642	943	239	313	63	75	11	$\cdots$	46368	121393

For example, for $t = 3$, we obtain the following two equalities:

$$144 = f_{12} = 3\cdot 12 + 12\cdot 5 + 48\cdot 1$$
$$377 = f_{14} = 29 + 6\cdot 18 + 24\cdot 6 + 96\cdot 1.$$

The filtrations for the graded modules ([-] denotes shift of the grading)

There are exact sequences of the following form:

$$0 \to P_{t-1}[1] \to P_t[0] \to R_t[0] \to 0,$$

$$0 \to P_{t-1}[1] \to R_t[0] \to R_{t-1}[1] \to 0.$$

This yields a filtration of $P_t[0]$

$$P_0[t] \;\subset\; P_1[t-1] \;\subset\; \cdots \;\subset P_{t-1}[1] \;\subset P_t[0]$$

with factors

$$P_i[t-i]/P_{i-1}[t-i+1] = R_i[t-i].$$

For the modules $R_t[0]$ we obtain a submodule $R'_t[0]$ with

$$R'_t[0] = P_0[t] \oplus P_1[t-1] \oplus \cdots \oplus P_{t-1}[1] \qquad \text{and} \qquad R_t[0]/R'_t[0] = R_0[t].$$

Let us return to the n-Kronecker algebras, in general.

They are considered as fundamental objects of non-commutative algebra, of non-commutative geometry.

There is the distinction:

$n \leq 1$	finite type
$n = 2$	tame type
$n \geq 3$	wild type

And there is the following table:

	moduli space of indecomposables of length 2	coefficients of the exceptional modules	
$n = 2$	$\mathbb{P}_1$	$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$	$\mathbb{N}$
$n = 3$	$\mathbb{P}_2$	$0,\ 1,\ 3,\ 8,\ 21,\ 55,\ \ldots$	Fibonacci
$n = 4$	$\mathbb{P}_3$	$0,\ 1,\ 4,\ 15,\ 56,\ \ldots$	(A001353) Sloane
...			

An Application

Exceptional modules of quivers.

Let Q be a finite quiver without oriented cycles.

Recall: Exceptional modules are indecomposable and have no self-extensions,
they are "generic",
using matrices for presenting the linear maps M_α, we may change the coefficients slightly without changing the isomorphism class,
in particular: we can assume that all matrix entries are non-zero.

However, also the opposite is true: We can change nearly all coefficients without changing the isomorphism class:

Theorem (R. 1998) *Exceptional modules are tree modules.*

An indecomposable module of dimension n is a *tree module* if it can be exhibited by matrices using altogether precisely $n-1$ non-zero coefficients.

(1) For any indecomposable module of dimension n, at least $n-1$ non-zero coefficents are needed.

(2) If we can use precisely $n-1$ non-zero coefficients, then we can assume that these coefficients are equal to 1.

Thus: *Tree modules can be exhibited by* 0-1-*matrices.*

Schofield induction: If M is an exceptional kQ-mdoule and not simple, then there is a non-trivial exact sequence

$$0 \to M_1^a \to M \to M_2^b \to 0,$$

where M_1, M_2 are exceptional,
and orthogonal: $\mathrm{Hom}(M_1, M_2) = 0 = \mathrm{Hom}(M_2, M_1)$.
Moreover, $[a, b] = \mathbf{dim}\, N$, where N is a non-simple exceptional representation of the e-Kronecker quiver, with $e = \dim \mathrm{Ext}^1(M_2, M_1)$.

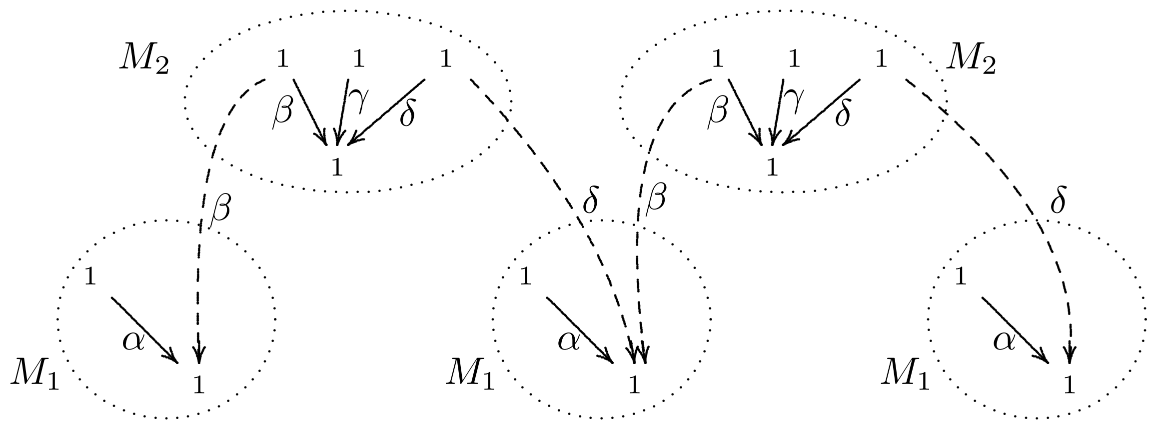
There are 3 exceptional modules which play a role: $M_1,\ M_2,\ N$ (M_1, M_2 are again kQ-modules, N is an e-Kronecker module, all are of smaller dimension than M).

These three exceptional modules determine M completely.
The tree structure of M can be obtained directly from their tree structures.

Example. Consider the 4-subspace quiver $\widetilde{\mathbb{D}}_4$

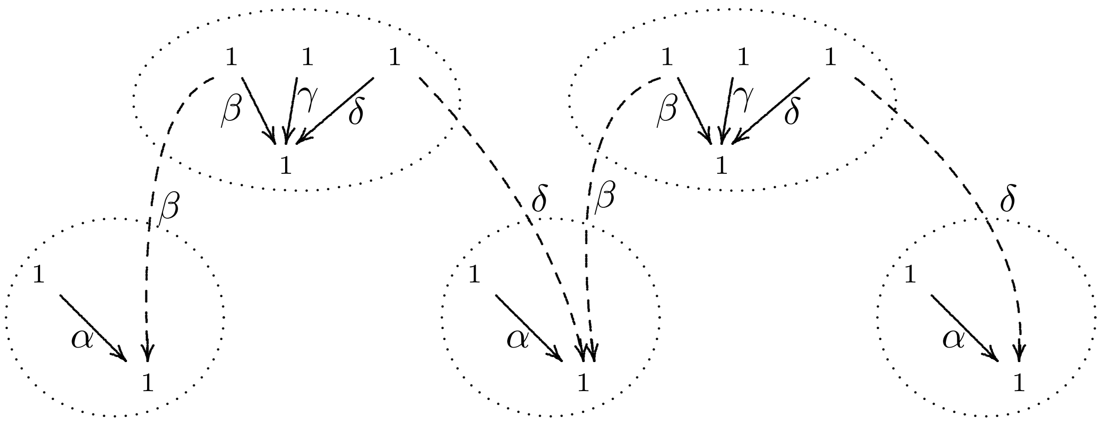
There is an exceptional representation M with dimension vector $\begin{smallmatrix}3&&2&&2&&2\\&&&5&&&\end{smallmatrix}$,

we can take M_1 with dimension vector $\begin{smallmatrix}1&&0&&0&&0\\&&&1&&&\end{smallmatrix}$, and M_2 with $\begin{smallmatrix}0&&1&&1&&1\\&&&1&&&\end{smallmatrix}$, then $e = 2$ and $a = 3$, $b = 2$. Altogether we obtain the following tree:



We obtain the following matrices

$$\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$



Schofield induction: For M exceptional, not simple, there is an exact sequence

$$0 \to M_1^a \to M \to M_2^b \to 0,$$

with M_1, M_2 exceptional and orthogonal, and
$[a, b] = \mathbf{dim}\, N$, where N is a non-simple exceptional representation of the e-Kronecker quiver, $e = \dim \operatorname{Ext}^1(M_2, M_1)$.

We need the tree structure of the exceptional e-Kronecker modules, for arbitary e.

Using covering theory, we obtain an exceptional representation $\widetilde{N}$ of the n-regular tree such that $\widetilde{N}_\alpha = (k \to k)$ for some arrow $\alpha\colon x \to y$, where one of x, y is a leaf.

This is the most trivial case of Schofield induction:

The representation $\widetilde{N}$ is obtained from a representation $\widetilde{N}'$ of a subquiver Q' of T as follows: There is a vertex x with $\dim \widetilde{N}'_x = 1$ and we use a new arrow α starting or ending in x and prolong the representation $\widetilde{N}'$, using the identity map $k \to k$.

Conclusion.

Exceptional modules are inductively constructed
using smaller orthogonal exceptional modules M_1, M_2 with $\mathrm{Ext}^1(M_2, M_1) = k$,
and using (if necessary) covering theory.