Cluster-additive functions on stable translation quivers

Part of the Trondheim Lecture, May 12, 2011

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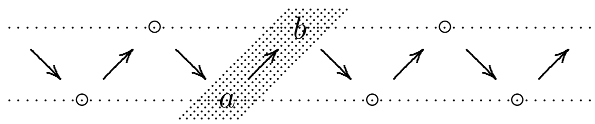
Definition: Stable translation quiver.
Example: $\mathbb{Z}\Delta$, Δ a finite directed quiver. Here of type $\mathbb{D}_5$.

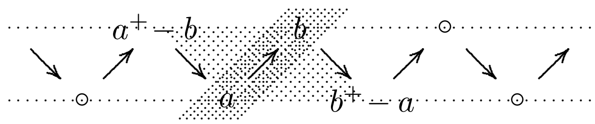
Definition: Additive function
Definition: $z = z^+ - z^-$ with $z^+z^- = 0$ and both $z^+ \geq 0, z^- \geq 0$.
Definition: Cluster-additive function.

First observation: The F-periodicity.

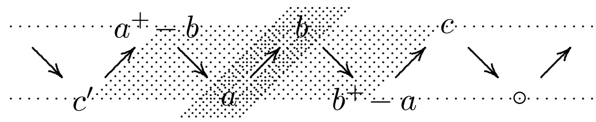
Example: The case $\mathbb{A}_2$.



Calculation on the right and on the left provide



Next step, c (to the right):



with

$$c = (b^+ - a)^+ - b.$$

Claim:

$$c = (b^+ - a)^+ - b = \begin{cases} a^- + b^- & \text{if} \quad b \leq 0 \text{ or } a \leq 0 \\ -\min(a, b) & \text{if} \quad b \geq 0 \text{ and } a \geq 0 \end{cases}$$

For the proof, one may consider various cases: If $b \leq 0$, then we have

$$c = (-a)^+ - b = a^- + b^-.$$

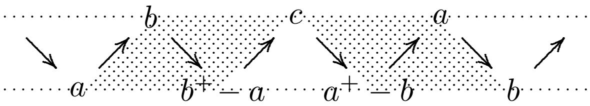
If $b \geq 0,\ a \leq 0$. Then $b^+ = b$ and $b \geq a$, thus

$$c = (b^+ - a)^+ - b = (b - a) - b = -a = a^- = a^- + b^-.$$

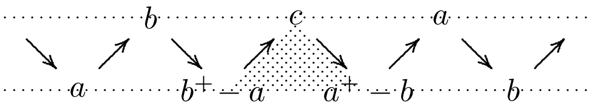
Finally, if $b \geq 0,\ a \geq 0$. Then

$$\begin{aligned} c &= (b^+ - a)^+ - b = (b-a)^+ - b \\ &= \begin{cases} (b-a) - b = -a & \text{if} \quad b \geq a \\ 0 - b = -b & \phantom{\text{if}} \quad b \leq a \end{cases} \\ &= -\min(a,b) \end{aligned}$$

We see: the answer is symmetric in a, b, thus we also get $c' = c$.
Let us consider



Claim: Also the middle mesh is cluster-additive:



Proof:

$$\begin{aligned} f(z) + f(\tau z) &= (a^+ - b) + (b^+ - a) \\ &= (a^+ - a) + (b^+ - b) \\ &= a^- + b^- \\ &= c^+. \end{aligned}$$

Namely, if $a \leq 0$ or $b \leq 0$, then $c = a^- + b^-$, in particular $c \geq 0$, thus $c^+ = c = a^- + b^-$. On the other hand, if $a \geq 0$ and $b \geq 0$, we have $c = -\min(a,b) \leq 0$, thus $c^+ = 0$ and $a^- = 0$ as well as $b^- = 0$.

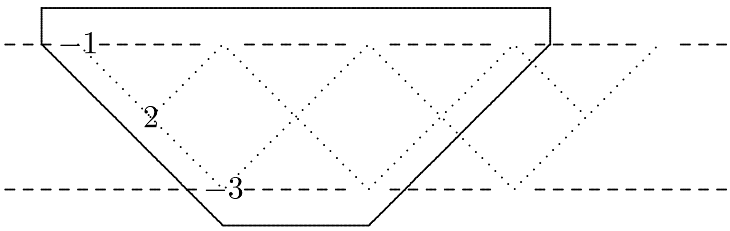
We have shown:

Any cluster-additive function f on $\mathbb{Z}\mathbb{A}_2$ is periodic, namely F-periodic with $F = [1]\tau^{-1}$.

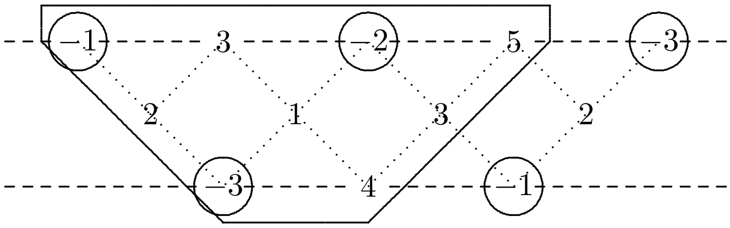
The second observation concerns the position of the negative values.

Example $\mathbb{A}_3$.

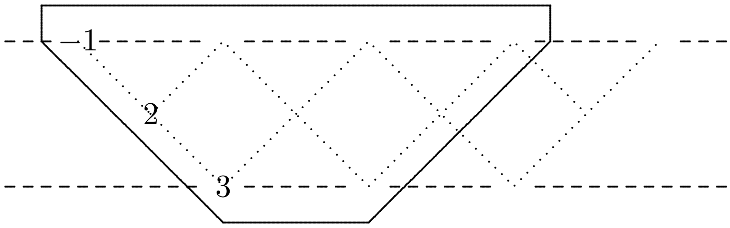
Example 1:



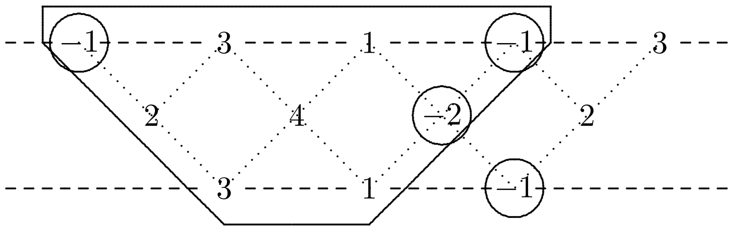
We obtain



Example 2:



We obtain



Inside the fundamental domain for F, We obtain a partial tilting set.

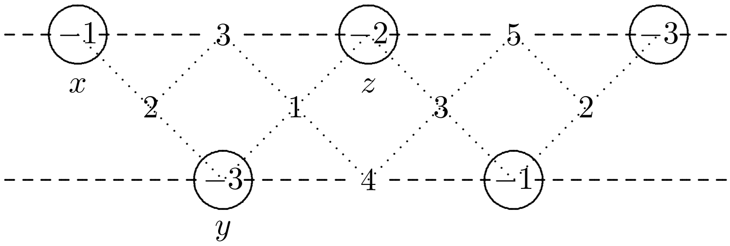
Third observation. The values on the partial tilting set of all vertices x with $f(x) < 0$ seem to determine f.

Precise formula, there is tilting set $\mathcal{T}$ such that:

$$f = \sum_{x \in \mathcal{T}} f(x)^- h_x.$$

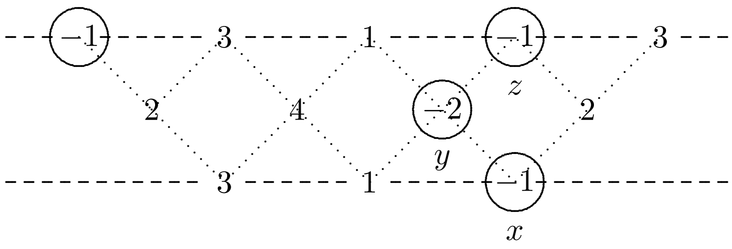
where h_x is the **cluster-hammock function** for the vertex x.

Here, in the first example:



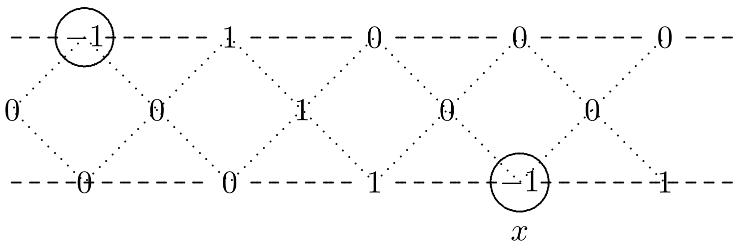
$$f = h_x + 3h_y + 2h_z$$

Second example:

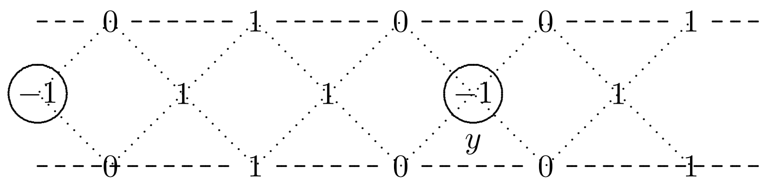


$$f = h_x + 2h_y + h_z$$

What is h_x?



And h_y :



Cluster-addition of almost positive roots.

Let $\mathcal{T} = \{x_1, \ldots, x_n\}$ be a tilting set, and consider the function

$$(h_{x_1}, \ldots, h_{x_n})\colon \Gamma_0 \to \mathbb{Z}^n = K_0(B).$$

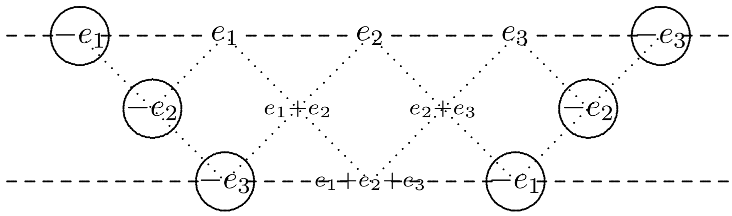
Then this function is "cluster-additive" in the sense that all its component functions are cluster-additive, or also in the sense that

$$h(z) + h(\tau z) = \sum_y m_{YZ} h(y)^+,$$

where $(a_1, \ldots, a_n)^+ = (a_1^+, \ldots, a_n^+)$.

Note that the values of h are the dimension vectors of the indecomposable B-modules (and then positive) or else the negative of one of the basis vectors.

For example:



For every tilting set $\mathcal{T} = \{x_1, \ldots, x_n\}$, there is the restriction function

$$r_{\mathcal{T}} \colon \operatorname{cadd} \Gamma \to \mathbb{Z}^n \qquad \text{defined by} \quad r_{\mathcal{T}}(f) = (f(x_1), \ldots, f(x_n))$$

which is a bijection (in the cases where the conjecture is true).