The elementary 3-Kronecker modules

Claus Michael Ringel

Abstract. The 3-Kronecker quiver has two vertices, namely a sink and a source, and 3 arrows. A regular representation of a representation-infinite quiver such as the 3-Kronecker quiver is said to be elementary provided it is non-zero and not a proper extension of two regular representations. Of course, any regular representation has a filtration whose factors are elementary, thus the elementary representations may be considered as the building blocks for obtaining all the regular representations. We are going to determine the elementary 3-Kronecker modules. It turns out that all the elementary modules are combinatorially defined.

Let k be an algebraically closed field and $Q = K(3)$ the 3-Kronecker quiver

$$1 \Rrightarrow 2$$

The *dimension vector* of a representation M of Q is the pair $(\dim M_1, \dim M_2)$.

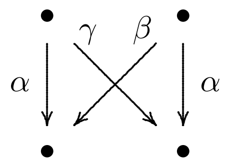
We denote by A the arrow space of Q, it is a three-dimensional vector space, thus $\Lambda = \begin{bmatrix} k & A \\ 0 & k \end{bmatrix}$ is the path algebra of Q. Note that Λ is a finite-dimensional k-algebra which is connected, hereditary and representation-infinite. The Λ-modules will be called 3-*Kronecker modules.* Of course, choosing a basis of A, the 3-Kronecker modules are just the representations of $K(3)$.

Elementary modules. In general, if Λ is a finite-dimensional k-algebra, we denote by $\operatorname{mod}\Lambda$ the category of all (finite-dimensional left) Λ-modules. We denote by τ the Auslander-Reiten translation in $\operatorname{mod}\Lambda$.

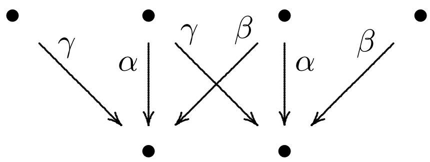
Now let Λ be the path algebra of a finite acyclic quiver. A Λ-module M is said to be *preprojective* provided there are only finitely many isomorphism classes of indecomposable modules X with $\operatorname{Hom}(X, M) \neq 0$, or, equivalently, provided $\tau^t M = 0$ for some natural number t. Dually, M is said to be *preinjective* provided there are only finitely many isomorphism classes of indecomposable modules X with $\operatorname{Hom}(M, X) \neq 0$, or, equivalently, provided $\tau^{-t} M = 0$ for some natural number t. A Λ-module M is said to be *regular* provided it has no indecomposable direct summand which is preprojective or preinjective.

A regular Λ-module M is said to be *elementary* provided there is no short exact sequence $0 \to M' \to M'' \to 0$ with M', M'' being non-zero regular modules (the definition is due to Crawley-Boevey, for basic results see Kerner and Lukas [L,KL,K]) and the appendix 1. Of course, any regular module has a filtration whose factors are elementary. If M is elementary, then all the modules $\tau^t M$ with $t \in \mathbb{Z}$ are elementary.

The aim of this note is to determine the elementary 3-Kronecker modules. Let α, β, γ be a basis of A. Let $X(\alpha, \beta, \gamma)$ and $Y(\alpha, \beta, \gamma)$ be the Λ-module defined by the following pictures:



$X(\alpha,\beta,\gamma)$



$Y(\alpha,\beta,\gamma)$

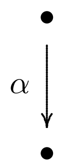
Here, we draw a corresponding coefficient quiver and require that all non-zero coefficients are equal to 1. Thus, for example $X(\alpha,\beta,\gamma) = (k^2,k^2;\alpha,\beta,\gamma)$ with $\alpha(a,b) = (a,b)$, $\beta(a,b) = (b,0)$ and $\gamma(a,b) = (0,a)$ for $a,b \in k$.

Theorem. *The dimension vectors of the elementary 3-Kronecker modules are the elements in the τ-orbits of $(1,1),(2,1),(2,2)$ and $(4,2)$.*

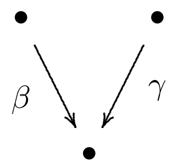
Any indecomposable representation with dimension vector in the τ-orbit of $(1,1)$ and $(2,1)$ is elementary.

An indecomposable representation with dimension vector $(2,2)$ or $(4,2)$ is elementary if and only if it is of the form $X(\alpha,\beta,\gamma)$ or $Y(\alpha,\beta,\gamma)$, respectively for some basis α,β,γ of A.

The indecomposable representations with dimension vectors in the τ-orbits of $(1,1)$ and $(2,1)$ have been studied in several papers. They are the even index Fibonacci modules, see [FR2,FR3,R4]. If M is indecomposable and $\mathbf{dim}\,M = (1,1)$ or $(2,1)$, then there is a basis α,β,γ of A such that $M = B(\alpha)$ or $M = V(\beta,\gamma)$, respectively, defined as follows:



$B(\alpha)$



$V(\beta,\gamma)$

Note that $B(\alpha)$ is the unique indecomposable 3-Kronecker module of length 2 which is annihilated by β and γ, whereas $V(\beta,\gamma)$ is the unique indecomposable 3-Kronecker module of length 3 with simple socle which is annihilated by α.

The indecomposable modules with dimension vector $(1,1)$ are called *bristles* in [R3]. The indecomposable representations with dimension vector $(2,1)$ have been considered in [BR]: there, it has been shown that any arrow α of a quiver gives rise to an Auslander-Reiten sequence with indecomposable middle term say $M(\alpha)$; in this way, we obtain the sequence:

$$0 \to V(\beta,\gamma) \to M(\alpha) \to \tau^- V(\beta,\gamma) \to 0.$$

The study of the τ-orbits of the indecomposable 3-Kronecker modules with dimension vectors $(1,1)$ and $(2,1)$ in the papers [FR2,FR3,R4,R5] uses the universal covering $\widetilde{K}(3)$ of the Kronecker quiver $K(3)$. The quiver $\widetilde{K}(3)$ is the 3-regular tree with bipartite orientation. Since the 3-Kronecker modules $B(\alpha)$ and $V(\beta,\gamma)$ are cover-exceptional (they are push-downs of exceptional representations of $\widetilde{K}(3)$), it follows that all the modules in the τ-orbits of $B(\alpha)$ and $V(\beta,\gamma)$ are cover-exceptional, and therefore tree modules in the sense of [R2].

In general, one should modify the definition of a tree module as follows: Let Q be any quiver. For any pair of vertices x, y of Q, let $A(x, y)$ be the corresponding arrow space, this is the vector space with basis the arrows $x \to y$. If $\alpha(1), \dots, \alpha(t)$ are the arrows $x \to y$ and $\beta = \sum a_i \alpha(i)$ with all $a_i \in k$ is an element of $A(x, y)$, we may consider for any representation $M = (M_x, M_\alpha)_{x \in Q_0, \alpha \in Q_1}$ the linear combination $M_\beta = \sum a_i M_{\alpha(i)}$. Given a basis $\mathcal{B}(x, y)$ of the arrow space $A(x, y)$, for all vertices x, y of Q as well as a basis $\mathcal{B}(M, x)$ of the vector space M_x, for all vertices x of Q, we may write the linear maps M_b with $b \in \mathcal{B}(x, y)$ as matrices with respect to the bases $\mathcal{B}(M, x), \mathcal{B}(M, y)$. Looking at these matrices, we obtain a coefficient quiver $\Gamma(\mathcal{B}(x, y), \mathcal{B}(M, x))$ as in [R2]. A representation M of the path algebra kQ should be called a *tree module* provided M is indecomposable and there are bases $\mathcal{B}(x, y)$ of the arrow spaces $A(x, y)$ and $\mathcal{B}(M, x)$ of the vector spaces M_x such that $\Gamma(\mathcal{B}(x, y), \mathcal{B}(M, x))$ is a tree. Of course, in case Q has no multiple arrows, this coincides with the definition given in [R2]. But in general, we now allow base changes in the arrow spaces. Note that such base changes in the arrow spaces do not effect the kQ-module M, but only its realization as the representation of a quiver. There is the following interesting consequence: Any indecomposable representation of the 2-Kronecker quiver is a tree module, see Appendix 2. Using this modified definition, we see immediately that *all the indecomposable modules with dimension vector in the τ-orbits of $(1, 1)$ and $(2, 1)$ are tree modules.* On the other hand, the modules $X(\alpha, \beta, \gamma)$ (and also $Y(\alpha, \beta, \gamma)$) are not tree modules, see Lemma 4.2.

Whereas the modules in the τ-orbits of the elementary 3-Kronecker modules with dimension vectors $(1, 1)$ and $(2, 1)$ are quite well understood, a similar study of those in the τ-orbits of modules with dimension vectors $(2, 2)$ and $(4, 2)$ is missing. It seems that any module M in these τ-orbits has a coefficient quiver with a unique cycle. A first structure theorem for these modules is exhibited in section 5.

We say that an element $(x, y) \in K_0(\Lambda) = \mathbb{Z}^2$ is *non-negative* provided $x, y \geq 0$. The non-negative elements in $K_0(\Lambda)$ are just the possible dimension vectors of Λ-modules. Note that $K_0(\Lambda)$ is endowed with the quadratic form q defined by $q(x, y) = x^2 + y^2 - 3xy$ (see for example [R1]). A dimension vector $\mathbf{d}$ is said to be *regular* provided $q(\mathbf{d}) < 0$. There are precisely two τ-orbits of dimension vectors $\mathbf{d}$ with $q(\mathbf{d}) = -1$, namely the τ-orbits of $(1, 1)$ and $(2, 1)$. Similarly, there are precisely two τ-orbits of dimension vectors $\mathbf{d}$ with $q(\mathbf{d}) = -4$, namely the τ-orbits of $(2, 2)$ and $(4, 2)$. The remaining regular dimension vectors $\mathbf{d}$ satisfy $q(\mathbf{d}) \leq -5$.

Corollary. *Let $\Lambda = kK(3)$ and (x, y) a dimension vector. There exists an elementary module M with dimension vector (x, y) if and only if $q(x, y)$ is equal to -1 or -4.*

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1. The BGP-shift σ.

Let σ denote the BGP-shift of $K_0(\Lambda) = \mathbb{Z}^2$ given by $\sigma(x, y) = (3x - y, x)$, and let $\tau = \sigma^2$.

We denote by σ, σ^- the *BGP-shift functors* for $\operatorname{mod}\Lambda$ (they correspond to the reflection functors of Bernstein-Gelfand-Ponomarev in [BGP], but take into account that the opposite of the 3-Kronecker quiver is again the 3-Kronecker quiver). If $M = (M_1, M_2; \alpha, \beta, \gamma)$ is a representation of Q, we denote by $(\sigma M)_1$ the kernel of the map $[\,\alpha\ \beta\ \gamma\,] : M_1^3 \to M_2$ and put $(\sigma M)_2 = M_1$; the maps $\alpha, \beta, \gamma : (\sigma M)_1 \to (\sigma M)_2$ are given by the corresponding projections. Similarly, $(\sigma^- M)_2$ is the cokernel of the map $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} : M_1 \to M_2^3$ and we put $(\sigma^- M)_1 = M_2$; now the maps $\alpha, \beta, \gamma : (\sigma^- M)_1 \to (\sigma^- M)_2$ are just the corresponding restrictions. Note that σ^2 is just the Auslander-Reiten translation τ (we should stress that this relies on the fact that we deal with a quiver without cyclic walks of odd length, see [G]).

Remark. The functors σ and σ^- depend on the choice of the basis α, β, γ of A, thus we should write $\sigma = \sigma_{\alpha,\beta,\gamma}$ and $\sigma^- = \sigma^-_{\alpha,\beta,\gamma}$.

If N is an indecomposable representation of $K(3)$ different from $S(2)$, then $\mathbf{dim}\,\sigma N = \sigma\,\mathbf{dim}\,N$; similarly, if N is indecomposable and different from $S(1)$, then $\mathbf{dim}\,\sigma^- N = \sigma\,\mathbf{dim}\,N$ (here, $S(1)$ and $S(2)$ are the simple representations of $K(3)$; they are defined by $\mathbf{dim}\,S(1) = (1,0), \mathbf{dim}\,S(2) = (0,1)$).

An indecomposable Λ-module M is regular if and only if all the modules $\sigma^n N$ and $\sigma^{-n} N$ with $n \in \mathbb{N}$ are nonzero. The restriction of σ to the full subcategory of all regular modules is a self-equivalence with inverse σ^- and a regular module M is elementary if and only if σM is elementary. We say that an indecomposable representation M of $K(3)$ is of *σ-type* (x, y) provided $\mathbf{dim}\,M$ belongs to the σ-orbit of (x, y).

In terms of σ, the main result can be formulated as follows:

Theorem. *The elementary $kK(3)$-modules are of σ-type $(1,1)$ and $(2,2)$. All the indecomposable representations of σ-type $(1,1)$ are elementary and tree modules. An indecomposable representation of σ-type $(2,2)$ is either elementary or else a tree module.*

The tree modules with dimension vector $(2,2)$ are precisely the representations of the form

$$\begin{tikzcd} \bullet \arrow[d, "\alpha"'] & \bullet \arrow[dl, "\beta"'] \arrow[d, "\gamma"] \\ \bullet & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[d, "\alpha"'] & \bullet \arrow[dl, "\beta"'] \arrow[d, "\alpha"] \\ \bullet & \bullet \end{tikzcd}$$

for some basis α, β, γ of A.

2. Reduction to the dimension vectors (x, y) with $\frac{2}{3}x \le y \le x$.

Let us denote by $\mathbf{R}$ the set of regular dimension vectors. As we have mentioned, σ maps $\mathbf{R}$ onto $\mathbf{R}$. There is the additional transformation δ on $K_0(\Lambda)$ defined by $\delta(x, y) = (y, x)$. Of course, it also sends $\mathbf{R}$ onto $\mathbf{R}$. If M is a representation of $Q(3)$, then $\delta(\mathbf{dim}\,M) = \mathbf{dim}\,M^*$, where M^* is the dual representation of M (defined in the obvious way: $(M^*)_1$ is the k-dual of M_2, $(M^*)_2$ is the k-dual of M_1, the map $(M^*)_\alpha$ is the k-dual of M_α, and so on).

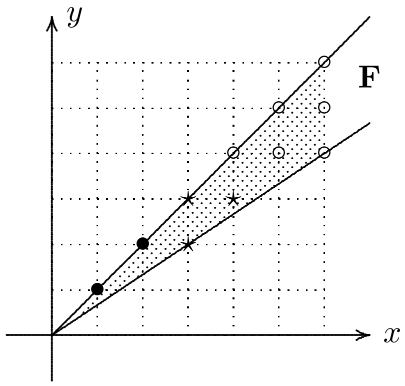
Lemma. *The subset*

$$\mathbf{F} = \{(x,y) \mid \tfrac{2}{3}x \le y \le x\}$$

is a fundamental domain for the action of σ *and* δ *on* $\mathbf{R}$.

The proof is easy. Let us just mention that $\sigma(3,2) = (2,3)$ and that for $(x,y) \in \mathbf{R}$ with $\sigma(x,y) = (x',y')$, we have $\frac{y}{x} > \frac{y'}{x'}$ (this condition explains why we call σ a shift). $\square$.

It follows that for looking at an elementary module, we may use the shift σ and duality in order to obtain an elementary module M with $\mathbf{dim}\, M \in \mathbf{F}$. Here is the set $\mathbf{F}$:



In the next section 3, we first will consider the pairs $(x,y) \in \mathbf{F}$ with $y \ge 4$, they are marked by a circle $\circ$. Then we deal with the three special pairs $(3,2), (3,3)$ and $(4,3)$ marked by a star $\star$ (actually, instead of $(4,3)$ and $(3,2)$, we will look at $(3,4)$ and $(2,3)$, respectively). As we will see in section 3, all these pairs cannot occur as dimension vectors of elementary modules.

As a consequence, the only possible dimension vectors in $\mathbf{F}$ which can occur as dimension vectors of elementary modules are $(1,1)$ and $(2,2)$; they are marked by a bullet $\bullet$ and will be studied in section 4.

3. Dimension vectors without elementary modules.

Lemma 3.1. *Assume that* M *is a regular module with a proper non-zero submodule* U *such that both dimension vectors* $\mathbf{dim}\, U$ *and* $\mathbf{dim}\, M/U$ *are regular. Then* M *is not elementary.*

Proof. This is a direct consequence of the fact that M is elementary if and only if for any submodule U the submodule U is preprojective or the factor module M/U is preinjective, see the Appendix 1.

Lemma 3.2. *A* 3*-Kronecker module* M *with* $\mathbf{dim}\, M = (x,y)$ *such that* $2 \le y \le x+1$ *has a submodule* U *with dimension vector* $(1,2)$*.*

Proof. Let us show that there are non-zero elements $m \in M_1$ and $\alpha \in A$ such that $\alpha m = 0$. The multiplication map $A \otimes_k M_1 \to M_2$ is a linear map, let W be its kernel. Since $\dim A = 3$, we see that $\dim A \otimes_k M_1 = 3x$. Since $\dim M_2 = y$, it follows that $\dim W \ge 3x - y$. The projective space $\mathbb{P}(A \otimes M_1)$ has dimension $3x - 1$, the decomposable tensors in $A \otimes M_1$ form a closed subvariety $\mathcal{V}$ of $\mathbb{P}(A \otimes M_1)$ of dimension $(3-1)+(x-1) = x+1$. Since $\mathcal{W} = \mathbb{P}(V)$ is a closed subspace of $\mathbb{P}(A \otimes M_1)$ of dimension $3x-y-1$, it follows that

$$\dim(\mathcal{V} \cap \mathcal{W}) \geq (x+1) + (3x-y-1) - (3x-1) = x-y+1.$$

By assumption, $x-y+1 \geq 0$, thus $\mathcal{V} \cap \mathcal{W}$ is non-empty. As a consequence, we get non-zero elements $m \in V, \alpha \in A$ such that $\alpha m = 0$, as required.

Given non-zero elements $m \in M_1$ and $\alpha \in A$ such that $\alpha m = 0$, the element m generates a submodule U' which is annihilated by α, thus $\mathbf{dim}\, U' = (1, u)$ with $0 \leq u \leq 2$. Since $y \geq 2$, there is a semi-simple submodule U'' of M with dimension vector $(0, 2-u)$ such that $U' \cap U'' = 0$. Let $U = U' \oplus U''$. This is a submodule of M with dimension vector $\mathbf{dim}\, U = \mathbf{dim}\, U' \oplus U'' = (1, 2)$.

Remark. Under the stronger assumption $2 \leq y < x$, we can argue as follows: We have $\langle (1,2), (x,y) \rangle = x + 2y - 3y = x - y > 0$, where $\langle -, - \rangle$ is the canonical bilinear form on $K_0(\Lambda)$ (see [R1]), thus $\mathrm{Hom}(N.M) \neq 0$ for any module N with $\mathbf{dim}\, N = (1,2)$ The image of any non-zero map $f \colon N \to M$ has dimension vector $(1, u)$ with $0 \leq u \leq 2$.

Lemma 3.3. *If* $(x,y) \in \mathbf{F}$ *and* $y \geq 4$*, then* $(x-1, y-2)$ *is a regular dimension vector.*

Proof. Since $y \leq x$, we have $y - 2 \leq x - 1$. On the other hand, the inequalities $y \geq 4$ and $y \geq \frac{2}{3}x$ imply the inequality $y - 2 \geq \frac{2}{5}(x-1)$. Thus $\frac{2}{5}(x-1) \leq y-2 \leq x-1$. As a consequence, $(x-1, y-2)$ is a regular dimension vector.

We are now able to provide a proof for the first assertion of the Theorem: *The elementary* $kK(3)$*-modules are of* σ*-type* $(1,1)$ *and* $(2,2)$.

Proof. Let M be elementary with dimension vector $\mathbf{dim}\, M = (x,y) \in \mathbf{F}$. First, assume that $y \geq 4$. According to Lemma 3.2, there is a submodule U with the regular dimension vector $\mathbf{dim}\, U = (1,2)$. The factor module M/U has dimension vector $(x-1, y-2)$ and $(x-1, y-1)$. According to Lemma 3.3, also $(x-1, y-2)$ is a regular dimension vector. Using Lemma 3.1, we obtain a contradiction.

It remains to show that the dimension vectors $(3,2), (3,3), (4,3)$ cannot occur. Using duality, we may instead deal with the dimension vectors $(2,3), (3,3), (3,4)$. Thus, assume there is given an elementary module N with dimension vector $(2,3)$, $(3,3)$ or $(3,4)$. According to Lemma 3.2, it has a submodule U with dimension vector $(1,2)$. The corresponding factor module M/U has dimension vector $(1,1), (2,1), (2,2)$, respectively. But all these dimension vectors are regular. Again Lemma 3.1 provides a contradiction. $\square$.

4. The indecomposable modules with dimension vector $(1,1)$ and $(2,2)$.

Dimension vector $(1,1)$**.** *Any indecomposable* Λ*-module* M *with dimension vector* $(1,1)$ *is of the form*

$$\begin{tikzcd} \bullet \arrow[d, "\alpha"'] \\ \bullet \end{tikzcd}$$

for some basis α, β, γ of A, thus a tree module. Namely, $M = P(1)/U$, where $P(1)$ is the indecomposable projective module corresponding to the vertex 1 and U is a two-dimensional submodule of $P(1)$. Actually, we may consider U as a two-dimensional subspace of $P(1)$. Let α, β, γ be a basis of A such that $U = \langle \beta, \gamma \rangle$.

Of course, *any indecomposable Λ-module with dimension vector $(1, 1)$ is elementary.*

The indecomposable Λ-modules with dimension vector $(2, 2)$.

Lemma 4.1. *An indecomposable module with dimension vector $(2, 2)$ is elementary if and only if it is of the form $X(\alpha, \beta, \gamma)$.*

Proof. First we show: *The modules $M = X(\alpha, \beta, \gamma)$ are elementary.* We have to verify that any non-zero element of M_1 generates a 3-dimensional submodule. We see this directly for the elements $(1, 0)$ and $(0, 1)$ of $M_1 = k^2$. If (a, b) with $a \neq 0, b \neq 0$, then $\beta(a, b) = (b, 0)$ and $\gamma(a, b) = (0, a)$ are linearly independent elements of $M_2 = k^2$. This completes the proof.

Conversely, let M be an elementary module with dimension vector $(2, 2)$. Let us show that *the restriction of M to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring.* Let α, β, γ be a basis of the arrow space and consider the restriction M' of M to the subquiver $K(2)$ with basis β, γ. If M' has a simple injective direct summand, then either M' is annihilated by α, then M' is a simple injective submodule of M, therefore M is not indecomposable, impossible. If M' is not annihilated by α, then $M' + \alpha(M')$ is an indecomposable submodule of dimension 2, thus M is not elementary. Dually, M' has no simple projective direct summand. It remains to exclude the case that $M' = R \oplus R$ for some simple regular representation R of $K(2)$. Without loss of generality, we can assume that M' is annihilated by γ. Since M is annihilated by γ, it is just a regular representation of the 2-Kronecker quiver with arrow basis α and β. But any 4-dimensional regular representation of a 2-Kronecker quiver has a 2-dimensional regular submodule. This shows that M is not elementary. Altogether we have shown that the restriction of M to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring.

If u is a non-zero element of M_1, then Λu contains M_2 and $\dim \Lambda u = 3$. Namely, if Λu is of dimension 1, then Λu simple injective, thus M cannot be indecomposable. If Λu is of dimension 2, then Λu is a proper non-zero regular submodule and then M is not elementary. It follows that $\dim \Lambda u = 3$ and that $M_2 \subset \Lambda u$. Given any non-zero element $u \in M_1$, there is a non-zero element which annihilates u, say $0 \neq \beta \in A$. No element in $A \setminus \langle \beta \rangle$ annihilates u, since otherwise the dimension of Λu is at most 2. Let u, v be a basis of M_1. Let β, γ be non-zero elements of A with $\beta(u) = 0, \gamma(v) = 0$. Then the elements β, γ are linearly independent, since otherwise we would have $\gamma(u) = 0$, thus the submodule Λu would be of dimension at most 2. The elements $\beta(v), \gamma(u)$ must be linearly independent, since otherwise the restriction of M to β, γ would be the direct sum of a simple projective and an indecomposable injective. We take $\beta(v), \gamma(u)$ as an ordered basis of $M_2 = k^2$, so that $\beta(v) = (1, 0)$ and $\gamma(u) = (0, 1)$. Choose an element $\alpha \in A \setminus \langle \beta, \gamma \rangle$, thus α, β, γ is a basis of A. Let $\alpha(u) = (\kappa, \lambda)$ and $\alpha(v) = (\mu, \nu)$ with $\kappa, \lambda, \mu, \nu$ in k. Since $\alpha(u)$ cannot be a multiple of $\gamma(u) = (0, 1)$, we see that $\kappa \neq 0$. Since $\alpha(v)$ cannot be a multiple of

$\beta(v) = (1, 0)$, we see that $\nu \neq 0$. Let $\alpha' = \alpha - \mu\beta - \lambda\gamma$. Then

$$\begin{aligned}
\alpha'(u) &= \alpha(u) - \mu\beta(u) - \lambda\gamma(u) = (\kappa, \lambda) - (0, 0) - \lambda(0, 1) = (\kappa, 0), \\
\alpha'(v) &= \alpha(v) - \mu\beta(v) - \lambda\gamma(v) = (\mu, \nu) - \mu(1, 0) - (0, 0) = (0, \nu).
\end{aligned}$$

Let $\beta' = \kappa\beta$ and $\gamma' = \nu\gamma$. Then we have

$$\begin{aligned}
\beta'(u) &= (0, 0), \quad \beta'(v) = (\kappa, 0), \\
\gamma'(u) &= (0, \nu), \quad \gamma'(v) = (0, 0),
\end{aligned}$$

Altogether, we see that

$$\begin{tikzcd}
u \arrow[d, "\alpha'"'] \arrow[dr, "\gamma'"] & v \arrow[d, "\alpha'"] \arrow[dl, "\beta'"'] \\
(\kappa, 0) & (0, \nu)
\end{tikzcd}$$

Since both elements κ and ν are non-zero, the elements $(\kappa, 0)$ and $(0, \nu)$ form a basis of k^2, and α', β', γ' form a basis of A. Thus M is isomorphic to $X(\alpha', \beta', \gamma')$. This completes the proof.

Lemma 4.2. *A tree module with dimension vector* $(2, 2)$ *cannot be elementary.*

Proof. If M is a tree module with dimension vector $(2, 2)$, then the coefficient quiver has to be of the form

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[d] \arrow[dl] \\
\bullet & \bullet
\end{tikzcd}$$

But then M has a submodule U such that both U and M/U have dimension vector $(1, 1)$.

Lemma 4.3. *If M is indecomposable with dimension vector* $(2, 2)$ *and not elementary, then M is of one of the following forms*

$$\begin{tikzcd}
\bullet \arrow[d, "\alpha"'] & \bullet \arrow[d, "\gamma"] \arrow[dl, "\beta"'] \\
\bullet & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[d, "\alpha"'] & \bullet \arrow[d, "\alpha"] \arrow[dl, "\beta"'] \\
\bullet & \bullet
\end{tikzcd}$$

for some basis α, β, γ of A.

Proof. Let M be indecomposable with dimension vector $(2, 2)$. If M is not faithful, say annihilated by $0 \neq \gamma \in A$, then M is a $K(2)$-module and therefore as shown on the right.

Now assume that M is faithful, and not elementary. Since M is not elementary, there is an element $0 \neq u \in M_1$ such that Λu has dimension vector $(1, 1)$. The annihilator B of u is a 2-dimensional subspace of A. Let $v \in M_1 \setminus \langle u \rangle$. Since M is indecomposable, we see that Λv has to be 3-dimensional and there is a non-zero element $\alpha \in A$ with $\alpha v = 0$. Since M is faithful, $\alpha(u) \neq 0$. Also, since M is faithful, we have $Bv = M_2$. Thus, there is $\beta \in B$ with $\beta(v) = \alpha(u)$. Let $\gamma \in B \setminus \langle \beta \rangle$. Then $\alpha(u), \beta(\gamma)$ is a basis of M_2. With respect to the basis α, β, γ of A, the basis u, v of M_1 and the basis $\alpha(u), \beta(\gamma)$ of M_2, the module M has the form as depicted on the left. $\square$.

5. The structure of the modules $\sigma^t X(\alpha, \beta, \gamma)$.

The 3-Kroncker modules $I_i = \sigma^i S(2)$ are the preinjective modules, see [FR1].

Proposition. *For $t \geq 1$, there is an exact sequence*

$$0 \to X(\alpha, \beta, \gamma) \to \sigma^t X(\alpha, \beta, \gamma) \to \bigoplus_{0 \leq i < t} I_i{}^2 \to 0.$$

Proof. First we consider the case $t = 1$. There is an obvious embedding of $X(\alpha, \beta, \gamma)$ into $Y(\alpha, \beta, \gamma) = \sigma X(\alpha, \beta, \gamma)$, thus there is an exact sequence of the form

$$0 \to X(\alpha, \beta, \gamma) \to Y(\alpha, \beta, \gamma) \to S(2)^2 \to 0.$$

Now we use induction. We start with the sequence

$$0 \to X(\alpha, \beta, \gamma) \to \sigma^t X(\alpha, \beta, \gamma) \to \bigoplus_{0 \leq i < t} I_i{}^2 \to 0,$$

for some $t \geq 1$ and apply σ. In this way, we obtain the sequence

$$0 \to \sigma X(\alpha, \beta, \gamma) \to \sigma^{t+1} X(\alpha, \beta, \gamma) \to \bigoplus_{1 \leq i \leq t} I_i{}^2 \to 0.$$

This shows that $M = \sigma^{t+1} X(\alpha, \beta, \gamma)$ has a submodule U isomorphic to $\sigma X(\alpha, \beta, \gamma)$, with M/U isomorphic to $\bigoplus_{1 \leq i \leq t} I_i{}^2$. But the case $t = 1$ shows that U has a submodule U' isomorphic to $X(\alpha, \beta, \gamma)$ with U/U' isomorphic to $S(2)^2 = I_0{}^2$. The embedding of U/U' into M/U' has to split, since I_0 is injective. This completes the proof.

Appendix 1. Elementary modules.

According to [K], Proposition 4.4, a regular representation M is elementary if and only if for any nonzero regular submodule U of M, the factor module M/U is preinjective. Let us include the proof of a slight improvement of this criterion.

We deal with the general setting where Λ is a hereditary artin algebra.

Proposition. *Let M be non-zero regular module M. Then M is elementary if and only if given any submodule U of M, the submodule U is preprojective or the factor module M/U is preinjective.*

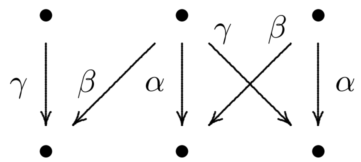
Proof. Let M be non-zero and regular. First, assume that for any submodule U of M, the submodule U is preprojective or the factor module M/U is preinjective. Then M cannot be a proper extension of regular modules, thus M is elementary.

Conversely, let M be elementary. Let U be a submodule which is not preprojective. Since M has no non-zero preinjective submodules, we can write $U = U_1 \oplus U_2$ with U_1 preprojective and U_2 regular. Since U is not preprojective, we know that U_2 is non-zero. Since M has no non-zero preprojective factor modules, we decompose M/U_2 as a direct sum of a regular and a preinjective module: there are submodules V_1, V_2 of M with $V_1 \cap V_2 = U, V_1 + V_2 = M$ (thus $M/U = V_1/U_2 \oplus V_2/U_2$) such that V_1/U_2 is regular, and V_2/U_2 is preinjective.

Consider V_2. First of all, $V_2 \neq 0$, since U_2 is a non-zero submodule of V_2. Second, we claim that V_2 is regular. Namely, V_2 is an extension of the regular module U_2 by the preinjective module V_2/U_2, thus is has no non-zero preprojective factor module. Thus, we can decompose $V_2 = W_1 \oplus W_2$ with W_1 regular, W_2 preinjective. But W_2 is a preinjective submodule of M, therefore $W_2 = 0$. This shows that $V_2 = W_1$ is regular.

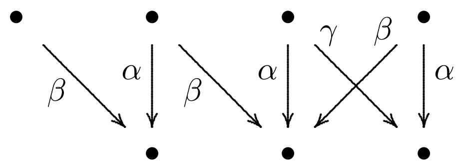
On the other hand, W/V_2 is isomorphic to V_1/U_2, thus regular. But since M is not a proper extension of regular modules, it follows that $W/V_2 = 0$, thus $V_2 = M$. Therefore $M/U_2 = V_2/U_2$ is preinjective. But $M/U = M/(U_1 + U_2)$ is a factor module of M/U_2, and a factor module of a preinjective module is preinjective. This shows that M/U is preinjective. $\square$

The definition of an elementary module implies that any regular module has a filtration by elementary modules. But such filtrations are not at all unique. This is well-known, but we would like to mention that the 3-Kronecker modules provide examples which are easy to remember. Here is the first such example M:



On the right we see that $X(\alpha, \beta, \gamma)$ is a factor module, and the corresponding kernel is $B(\gamma)$ (it is generated by the first base vector of M_1). On the other hand, on the left we see that $V(\beta, \gamma)$ is a factor module, and the corresponding kernel has dimension vector $(1, 2)$ (it is generated by the last base vector of M_1).

Here is the second example N:



On the right we see again that $X(\alpha, \beta, \gamma)$ is a submodule, and the corresponding factor module is $V(\alpha, \beta)$ (generated by the first two base vectors of N_1). On the other hand, going from left to right, we see that the module has a filtration whose lowest two factors are of the form $B(\beta)$, whereas the upper factor is $V(\alpha, \gamma)$ (generated by the last two base vectors of N_1).

Appendix 2: The representations of the 2-Kronecker quiver.

Proposition. *Any indecomposable $K(2)$-module is a tree module (with respect to some basis of the arrow space of $K(2)$), and its coefficient quiver is of type $\mathbb{A}$.*

Proof. The preprojective and the preinjective modules are exceptional modules, thus they are tree modules with respect to any basis. The remaining indecomposable representations of $K(2)$ are of the form $R[t]$ where R is simple regular, and $R[t]$ denotes the indecomposable regular module of dimension 2t with regular socle R. We may choose a basis of the arrow space such that R is isomorphic to $(k, k; 1, 0)$. Then $R[t]$ is a tree module such that the underlying graph of the coefficient quiver is of type $\mathbb{A}_{2t}$.

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Claus Michael Ringel

Fakultät für Mathematik, Universität Bielefeld
D-33501 Bielefeld, Germany

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China.

e-mail: `ringel@math.uni-bielefeld.de`