What is a hereditary algebra?

(On Ext^2 and the vanishing of Ext^2)

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At the Münster workshop 2011, three short lectures were arranged in the style of the regular column in the Notices of the AMS: *What is ... ?*. The following text is a slightly expanded version of my report — I should stress that the title may be considered as being slightly odd, it better should read: *What is a hereditary ring?* since for being hereditary, it is only the ring structure of an algebra which plays a role. But questions about rings in general are considered now-a-days as being obsolete.

We are going to outline a number of equivalent conditions, the most comprehendible seems to us condition (9); but unfortunately, this condition hides the origin of the naming — thus we better start with condition (1). We also will draw attention to one class of examples: the path algebras of finite quivers. In the realm of commutative algebra, only few rings are hereditary, the most prominent ones are the Dedekind domains.

1. So, when a ring R is said to be hereditary? The property which is inherited is the projectivity which is passed down from a module to its submodules: A ring R is hereditary if and only if

- *any submodule of a projective module is projective,*

and there is also the dual property: injectivity is passed down to factor modules: R is hereditary if and only if

- *any factor module of an injective module is injective.*

Submodules of projective modules are sometimes called torsionless, factor modules of injective modules are called divisible, thus we can reformulate the two conditions by saying that *torsionless modules are projective,* or that *divisible modules are injective.* A further equivalent condition is the following:

- *for any R-module M, there is an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ with P_0, P_1 projective.*

Namely, on the one hand, given M, there is a free (thus projective) module P_0 which maps onto M, and if R is hereditary, then the kernel P_1 of this map has to be projective, as well. On the other hand, if U is a submodule of a projective module P, let $M = P/U$ and use Schanuel's lemma in order to conclude that U is projective. Of course, we also have the dual condition:

- *for any R-module M, there is an exact sequence $0 \to M \to I_0 \to I_1 \to 0$ with I_0, I_1 injective.*

It is usual in homological algebra, instead of looking at an R-module M, to look at a projective resolution $\cdots \to 0 \to P_1 \to P_0 \to 0 \to \dots$, this is a complex of projective modules with a unique homology group, and this homology group is precisely M. Now such projective resolutions are used in order to calculate derived functors, for example the functors Ext^i. One observes quite easily: R is a hereditary ring if and only if

- $\mathrm{Ext}^2_R = 0$.

The equivalence of the first two assertions mentioned above (that torisonless modules are projective and that divisible modules are injective) is usually shown by using as intermediate condition this vanishing of Ext^2 . On the other hand, the vanishing of Ext^2 is a convenient condition for defining heredity in more general settings, say for dealing with abelian categories in general which may not have sufficiently many projective (or injective) objects. To work with Ext^2 is sometimes considered as a burden (later we will try to show that one can handle the elements of Ext^2 quite easily), thus one often tries to avoid Ext^2. In terms of Ext^1, one gets the following reformulations:

- *If $g\colon Y \to Y'$ is a surjective module homomorphism, then the induced map* $\mathrm{Ext}^1(X, g)$ *is surjective, for any module X.*
- *If $f\colon X' \to X'$ is an injective module homomorphism, then the induced map* $\mathrm{Ext}^1(f, Z)$ *is surjective, for any module Z.*

Before we continue, a warning is necessary. Up to now, we have mentioned "modules" without specifying whether we mean left modules or right modules, and indeed in the formulations above, we always have to consider **both** left modules and right modules. If we only are interested in say left modules, we arrive at the notion of a **left** hereditary ring.

2. Left hereditary rings. Let us collect again the various properties which we have listed above, where now all the modules considered are **left** modules. We also insert some additional conditions.

A ring R is said to be *left hereditary* provided the following equivalent properties are satisfied:

(1) *any submodule of a projective module is projective,*
(1′) *any left ideal is projective,*
(2) *any factor module of an injective module is injective.*
(3) *for any R-module M, there exists an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ with P_0, P_1 projective.*
(4) *for any R-module M, there exists an exact sequence $0 \to M \to I_0 \to I_1 \to 0$ with I_0, I_1 injective.*
(5) *Any element of the derived category* $D^b(\mathrm{mod}\, R)$ *(or even* $D(\mathrm{Mod}\, R)$*) is isomorphic to its homology.*
(6) $\mathrm{Ext}^2_R = 0$.
(7) *If $g\colon Z \to Z'$ is a surjective module homomorphism, then the induced map* $\mathrm{Ext}^1(X, g)$ *is surjective, for any module X.*
(8) *If $f\colon X' \to X'$ is an injective module homomorphism, then the induced map* $\mathrm{Ext}^1(f, Z)$ *is surjective, for any module Z.*

Condition (1′) (a theorem of M. Auslander) asserts that in order to check whether R is hereditary, it is sufficient to consider in (1) just one single projective module, namely ${}_RR$, the free module of rank 1.

It was asked in Cartan-Eilenberg (1956) whether left hereditary rings are also right hereditary. The first counter example was given by Kaplansky in 1958, easier examples were given later (1961) by Small, for example the matrix ring

$$R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix},$$

with condition ($1'$) one easily shows that R is left hereditary. However, the opposite ring

$$R^{\mathrm{op}} = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix},$$

is not left hereditary, since the radical of R^{op} is (flat, but) not projective.

But one should know that *a left and right noetherian ring is left hereditary if and only if it is right hereditary.*

3. Path algebras of finite quivers. Let k be a field. Claim: *The path algebra kQ of a finite quiver Q is hereditary.* There are various ways to prove this. Let us outline a proof using the condition (7).

Thus, let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, here Q_0, Q_1 are finite sets and $s, t\colon Q_1 \to Q_0$ are set-maps. Recall that a representation of Q is of the form $M = (M_x, M_\alpha$, with vector spaces M_x for all $x \in Q_0$ and linear maps $M_\alpha\colon M_{s(\alpha)} \to M_{t(\alpha)}$. If M, M' are representations of Q, a homomorphism $f = (f_x)\colon M \to M'$ is given by a family of linear maps f_x with $x \in Q_0$ satisfying certain commutativity relations. To be precise, the homomorphism space $\mathrm{Hom}(M, M')$ is the kernel of the following map

$$\begin{aligned} \bigoplus_{x\in Q_0} \mathrm{Hom}_k(M_x, M'_x) &\to \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(M_{s(\alpha)}, M'_{t(\alpha)}), \\ (f_x)_x \quad &\mapsto (M'_\alpha f_{s(\alpha)} - f_{t(\alpha)} M_\alpha)_\alpha \end{aligned}$$

and it turns out (and is not difficult to see) that the cokernel is precisely $\mathrm{Ext}^1(M, M')$. Thus, we deal with the following exact sequence:

$$0 \to \mathrm{Hom}(M, M') \to \bigoplus_{x\in Q_0} \mathrm{Hom}_k(M_x, M'_x) \to \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(M_{s(\alpha)}, M'_{t(\alpha)}) \to \mathrm{Ext}^1(M, M') \to 0.$$

Now assume that there is given a further representation M'' of Q and a surjective homomorphism $g = (g_x)_x\colon M' \to M''$ (and the surjectivity means that all the vector space maps g_x are surjective), then we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(M_{s(\alpha)}, M'_{t(\alpha)}) \arrow[r] \arrow[d, "\bigoplus \mathrm{Hom}(M_{s(\alpha)}, g_{t(\alpha)})"] & \mathrm{Ext}^1(M, M') \arrow[r] \arrow[d, "\mathrm{Ext}^1(M,g)"] & 0 \\ \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(M_{s(\alpha)}, M''_{t(\alpha)}) \arrow[r] & \mathrm{Ext}^1(M, M'') \arrow[r] & 0 \end{tikzcd}$$

The vertical map $\bigoplus \mathrm{Hom}(M_{s(\alpha)}, g_{t(\alpha)})$ on the left is surjective, thus also $\mathrm{Ext}^1(M, g)$ is surjective.

4. What is Ext^2 ? It is well-known and often used that for any abelian category the first derived functor Ext^1 of the Hom-functor can be defined using equivalence classes of short exact sequences: the Baer-definition of Ext^1 (note that the Baer definition assumes that we obtain really a set, not a class — but otherwise we go to another universe ...).

Similarly (at least for small categories), there is a Baer definition of Ext^n, for $n \geq 2$ using long exact sequences. For example, for Ext^2 (and this is what here we are interested in), we use long exact sequences of the form

$$0 \to Z \to M_1 \to M_2 \to X \to 0,$$

again, we need equivalence classes given by maps between such sequences which are the identity on X and on Z. But note: such maps are no longer invertible, thus we have to take maps in both possible directions — this is the origin of the construction of the derived category (calculus of fractions), and to look at Ext^2 is a convenient playing ground for getting familiar with quasi-isomorphisms. Let me stress that dealing with such exact sequences is nothing else than looking just at morphisms $f\colon M_1 \to M_2$ in the category (the clumsy notation of writing down a long exact sequence gives names to the kernel and the cokernel), or, to formulate it more fancy, we consider complexes of the form with one map f and otherwise using zero modules and $(\cdots 0 \to M_1 \to M_2 \to 0 \cdots)$ with fixed homology).

Thus, let us describe the vanishing of Ext^2 in terms of maps $f\colon M_1 \to M_2$. Here it is: A ring R is left hereditary if and only if

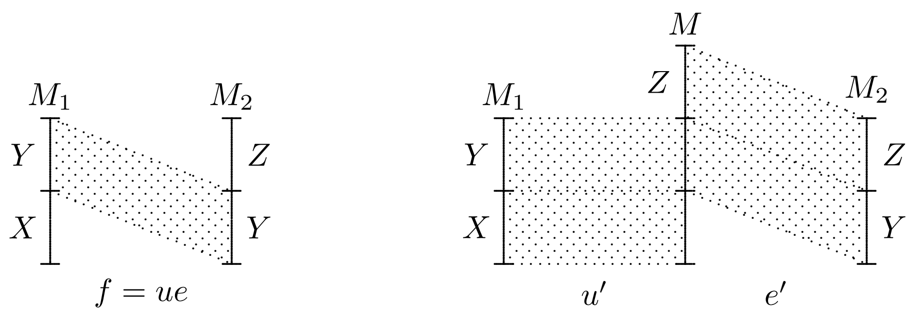
(9) *for any homomorphism $f\colon M_1 \to M_2$ with epi-mono-factorization $M_1 \xrightarrow{e} Y \xrightarrow{u} M_2$, there is a pushout-pullback diagram of the form*

$$\begin{array}{ccc} M_1 & \xrightarrow{\;e\;} & Y \\ {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ M & \xrightarrow{\;e'\;} & M_2. \end{array}$$

Before we look at the proof, let us indicate some way of visualizing of what is going on: Given the map $f\colon M_1 \to M_2$, denote by Z its kernel, by Y the image and by X the cokernel. Thus, we deal with two short exact sequences

$$0 \to X \to M_1 \xrightarrow{e} Y \to 0, \quad \text{and} \quad 0 \to Y \xrightarrow{u} M_2 \to Z \to 0.$$

and $f = ue$. We may depict this as shown on the left. And we are looking for a module M and maps u', e' as shown on the right:



Now, we show the equivalence of (8) and (9). First, let us start with a homomorphism $f\colon M_1 \to M_2$ with epi-mono-factorization $M_1 \xrightarrow{e} Y \xrightarrow{u} M_2$ and denote by Z the kernel of f, thus of e. If the condition (8) holds, then the element $[0 \to X \to M_1 \to Y \to 0]$ of $\mathrm{Ext}^1(Y,Z)$ is in the image of $\mathrm{Ext}^1(Z,u)$, thus there is a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & M_1 & \xrightarrow{e} & Y & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & Z & \longrightarrow & M & \xrightarrow{e'} & M_2 & \longrightarrow & 0,
\end{array}$$

this provides the required pushout-pullback diagram. Conversely, assume condition (9) is satisfied, let $u\colon M' \to M''$ be an injective homomorphism. We want to show that $\mathrm{Ext}^1(u,Z)$ is surjective. Thus take an element $[0 \to Z \to M_1 \xrightarrow{e} M' \to 0]$ in $\mathrm{Ext}^1(M',Z)$ and apply condition (9) to the map $ue\colon M_1 \to M''$ with its given epi-mono-factorization: we obtain a pushout-pullback-diagram

$$\begin{array}{ccc}
M_1 & \xrightarrow{e} & M' \\
{\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\
M & \xrightarrow{e'} & M''.
\end{array}$$

If we add on the left Z as the kernel of both e and e', we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & M_1 & \xrightarrow{e} & M' & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & Z & \longrightarrow & M & \xrightarrow{e'} & M'' & \longrightarrow & 0,
\end{array}$$

which shows that the upper sequence is the image of the lower sequence under $\mathrm{Ext}^1(u,Z)$.

More precisely, one sees: $\mathrm{Ext}^2(X,Z) = 0$ *if and only if* (9) *is satisfied for all maps* f *with kernel* Z *and cokernel* X.

It may be instructive to point out in which way a pushout-pullback diagram as given in condition (9) provides different representatives of an element of $\mathrm{Ext}^2(X,Z)$. Thus, consider the following pushout-pullback diagram

$$\begin{array}{ccc}
M_1 & \xrightarrow{e} & Y \\
{\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\
M & \xrightarrow{e'} & M_2,
\end{array}
\quad \text{and therefore} \quad
\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & M_1 & \xrightarrow{e} & Y & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & Z & \longrightarrow & M & \xrightarrow{e'} & M_2 & \longrightarrow & 0,
\end{array}$$

with exact rows. In addition, there is also the exact sequence $(0 \to Y \xrightarrow{u} M_2 \to Z \to 0)$. If we form the exact sequence induced from it by u, we obtain an exact sequence which is split exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{u} & M_2 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow & & \| & & \\
0 & \longrightarrow & M_2 & \longrightarrow & N & \xrightarrow{p} & X & \longrightarrow & 0
\end{array}$$

(here, p is a split epimorphism). Concatenation of the exact sequences yields

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] & M \arrow[r] & N \arrow[r, "p"] & X \arrow[r] & 0,
\end{tikzcd}$$

in this way we see that the two rows yield the same element in $\operatorname{Ext}^2(X, Z)$, and since p is split epi, this is the zero element of $\operatorname{Ext}^2(X, Z)$.

5. Modules of finite length. Since the functor Ext^2 is half exact in both variables, the vanishing $\operatorname{Ext}^2(T, S) = 0$ for all simple modules S, T implies that one has $\operatorname{Ext}^2(X, Z)$ for all finite length modules X, Z.

So what does it mean that $\operatorname{Ext}^2(T, S) = 0$ for simple modules S, T ? Let $I(S)$ be the injective envelope of S and let us assume from now on that there exists a projective cover $P(T)$ (this holds true in case we deal with an artinian ring, or more generally, a semi-perfect ring).

Lemma. *Any element in* $\operatorname{Ext}^2(T, S)$ *has a representative*

$$0 \to S \to M_1 \to M_2 \to T \to 0,$$

where M_1 *is a (non-zero) submodule of* $I(S)$, *and* M_2 *is a (non-zero) factor module of* $P(T)$.

To say that M_1 is a non-zero submodule of $I(S)$ means that M_1 is a module with simple essential socle S; dually, to say that M_2 is a non-zero factor module of $P(T)$ means that M_2 is a module with a small maximal submodule with factor module T.

Proof: We start with the exact sequence $0 \to S \xrightarrow{m} M_1 \xrightarrow{f} M_2 \xrightarrow{q} T \to 0$, and the inclusion $v\colon S \to I(S)$. Since m is a monomorphism and $I(S)$ is injective, there is $v'\colon M_1 \to I(S)$ with $v = v'm$. Let $v' = v''h$, where $h\colon M_1 \to M_1'$ is surjective and $v''\colon M_1' \to I(S)$ is injective. Since $v''hm = v'm = v$ is a monomorphism, also hm is a monomorphism. Thus, there is the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "m"] \arrow[d, equal] & M_1 \arrow[r, "e"] \arrow[d, "h"'] & Y \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r, "hm"] & M_1' & &
\end{tikzcd}$$

which we can complete by inserting a cokernel Y' of hm to obtain:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "m"] \arrow[d, equal] & M_1 \arrow[r, "e"] \arrow[d, "h"'] & Y \arrow[r] \arrow[d, "h'"] & 0 \\
0 \arrow[r] & S \arrow[r, "hm"] & M_1' \arrow[r] & Y' \arrow[r] & 0.
\end{tikzcd}$$

Since by assumption h is surjective, also h' is surjective. Next, we start with the exact sequence $0 \to Y \xrightarrow{u} M_2 \to T \to 0$ and form the sequence induced by h':

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, "h'"'] & M_2 \arrow[r, "e"] \arrow[d, "h''"'] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & M_2' \arrow[r] & T \arrow[r] & 0.
\end{tikzcd}$$

By concatenation, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "m"] \arrow[d, equal] & M_1 \arrow[r, "f"] \arrow[d, "h"'] & M_2 \arrow[r, "e"] \arrow[d, "h''"'] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S \arrow[r, "hm"] & M_1' \arrow[r] & M_2'' \arrow[r] & T \arrow[r] & 0,
\end{tikzcd}$$

thus, the two sequences yield the same element of $\mathrm{Ext}^2(T,S)$. This shows that we can replace the upper sequence by the lower sequence: here, M_1' is a submodule of $I(S)$ and in addition, M_2' is a factor module of M_2. By duality, we similarly see that we can replace the second series by a third one

$$0 \to S \to M_1'' \to M_2'' \to T \to 0,$$

where M_2'' is a factor module of $P(T)$ and where M_1'' is a submodule of M_1', thus of $I(S)$. This completes the proof.

It remains to analyze the vanishing of $\mathrm{Ext}^2(T,S)$. As we just have seen, we have to look at maps $f\colon M_1 \to M_2$ with kernel S and cokernel T such that S is an essential submodule of M_1 and the image of f is a small submodule of M_2. If the condition (9) holds, we obtain a module M with M_1 as a submodule, such that M/S is isomorphic to M_2.

Let us assume in addition that f is non-zero: then M is indecomposable. It is even what we call a *diamond*: a module M with submodules $M' \subseteq M''$ such that both M' and M/M'' are simple and such that any proper non-zero submodule of M contains M' and is contained in M''. (Proof: Take $M'' = M_1$ and $M' = \operatorname{soc} M_1$, then clearly both M' and M/M'' are simple. Let N be a proper non-zero submodule of M. Consider $M'' \cap N$. If $M'' \cap N = 0$, then N must be simple and $M = M'' \oplus N$, and therefore $M_2 = M/M' = M''/M' \oplus N$. But by assumption, M_2 is a module with a small maximal submodule, therefore $M''/M' = 0$, but then $f = 0$, in contrast to our assumption. This shows that $M'' \cap N \neq 0$, and therefore N contains the socle M' of M''. The dual argument shows that N is contained in M''.)

We can rephrase the previous considerations as follows: *The vanishing of* $\mathrm{Ext}^2(T,S)$ *means the existence of all possible diamonds with socle S and top T.* Conversely, we stress that if $\mathrm{Ext}^2(T,S) \neq 0$, this may kill not only one, but usually a lot of diamonds ...

Finally, one should be aware that when dealing with quivers with relations, the relations in question just correspond to non-zero elements of Ext^2 (so that in this case the vanishing of Ext^2 may be described by saying that there are no relations). Recall that a relation ρ for the quiver Q is by definition a non-zero linear combination ρ of paths of length at least 2 starting in some vertex x and ending in some vertex y (such a relation is an element of the path algebra kQ). For example, if ρ is a single path, then this is called a *monomial relation* or also a *zero relation;* if ρ is the difference of two paths, one says that ρ is a *commutativity relation.*