Some minimal representation-infinite algebras

or

Minimal representation-infinite semigroups

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St. Petersburg Conference, September 24–28, 2007

The representation type of a finite-dimensional (associative) k-algebra

k field

Λ finite dimensional k-algebra (associative, with 1.

mod Λ the category of all (finite-dimensional) A-modules

Krull-Remak-Schmidt Theorem. *Any Λ-module can be written as a direct sum of indecomposable modules, and such a decomposition is unique up to isomorphism.*

A is called *representation-infinite* provided there are infinitely many isomorphism classes of indecomposable Λ-modules, otherwise *representation-finite.*

Λ is *minimal representation-infinite* provided Λ is representation-infinite, but any proper factor algebra is representation-finite.

The known representation types

Finite type: There is no 1-parameter-family of indecomposable modules

Tame: There are 1-par.-families of indec. modules, but no 2-par.-families.

> **Domestic:** There are only finitely many primitive 1-par.-family of indec. modules.
>
> **Non-domestic, but polynomial growth.**
>
> **Non-polynomial growth:** The number of 1-par.-families of indec. modules of dimension d is not bounded by a polynomial in d.

Wild: There are n-parameter-families of indec. modules, for any n.

Domestic algebras. An algebra is said to be *n-domestic* in case there are precisely n primitive 1-parameter family of indecomposable modules (and additional "isolated" indecomposables).

In the present lecture, we mainly will consider 1-domestic algebras.

Surprisingly, not much is known about domestic algebras Λ, not even about 1-domestic algebras! For example:

First question: Is the number of non-regular AR-components of Λ uniformly bounded? (An AR-component is regular if it contains no projective and no injective module.)

For all the previously known examples there are at most 4 such components.

Second question: Let M be a primitive homogeneous and absolutely indecomposable Λ-module. Has $\mathrm{End}(M)/\mathrm{ss}$ bounded dimension?
(Here, $\mathrm{End}(M)/\mathrm{ss}$ is the factor ring of $\mathrm{End}(M)$ modulo the ideal of endomorphisms with semisimple image.)

Semigroups (with zero)

Semigroups $S = (S, z)$ to be considered are semigroups with zero element z (e.g. $zs = z = sz$ for all $s \in S$). We call the number $|S|$ of non-zero elements the *order* of S. Consider the (reduced) semigroup algebra of S,

$$k[S] = kS/\langle z\rangle$$

here, kS is the (usual) semigroup algebra: the k-algebra with basis S and with multiplication being given by the multiplication in S.
The element z generates a one-dimensional ideal $\langle z\rangle$ which is factored out in order to obtain $k[S]$. Since z is a central idempotent of kS, there is an algebra decomposition

$$kS = kS(1-z) \times kSz.$$

Thus

$$\operatorname{mod} kS = \operatorname{mod} k[S] \times \operatorname{mod} k$$

$\operatorname{mod} k[S] =$ the category of all k-linear representations M of S with $zM = 0$.

Basic semigroups

We call a finite semigroup S *basic*, provided

(1) S has enough idempotents: for any $s \in S$ there are idempotents e, e' such that $se = s = e's$.
(2) Idempotents are pairwise orthogonal: if $e \neq e'$ are idempotents, then $ee' = z$.
(3) Any element of S is either idempotent or nilpotent.

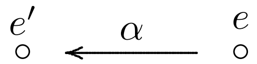
If S is a basic semigroup, $k[S]$ has the following properties:

(a) The non-zero nilpotent elements of S yield a basis of $\operatorname{rad} k[S]$.
(b) The non-zero idempotent elements of S yield a basis of $k[S]/\operatorname{rad}$.

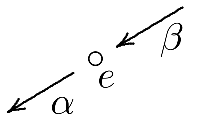
The quiver $Q(S)$ of a basic semigroup S

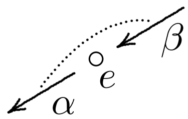
Call $\alpha \in S$ *irreducible* provided α is not idempotent, and given any factorization $\alpha = s_1 s_2$, at least one of the elements s_1, s_2 is idempotent.
The non-zero idempotents and the irreducible elements together generate S.

The **quiver** $Q(S)$ has as vertices the non-zero idempotents of S. The arrows α from e to e' are the irreducible elements $\alpha \in S$ with $\alpha e = s = e'\alpha$.
We call the vertex e the starting point of the arrow α and e' its endpoint, we visualize these arrows as pointing from right to left:



An arrow α starting at e and an arrow β ending at e yield a path of length two:





Dotted lines may be used to depict relations – for example the relation $\alpha\beta = z$.

The multiplicative basis theorem

Bautista, Gabriel, Roiter, Salmeron (1984)

Theorem. Let k be algebraically closed.

1. *Let* Λ *be a representation-finite* k*-algebra.*
Then there is a basic semigroup S *such that* $\operatorname{mod}\Lambda$ *is equivalent to* $\operatorname{mod} k[S]$.

2. *Let* Λ *be a minimal representation-infinite* k*-algebra which is chainless.*
Then there is a basic semigroup S *such that* $\operatorname{mod}\Lambda$ *is equivalent to* $\operatorname{mod} k[S]$.

Remark: In particular, this applies to the semigroup algebra of an arbitrary finite semigroup H: If the semigroup algebra kH is representation-finite, then kH is Morita equivalent to $k[S]$ for some basic semigroup S (but note that S depends not only on H, but also on k).

(Algebras Λ, Λ' with equivalent module categories are called *Morita equivalent.*)

Biserial semigroups

S is *biserial* provided

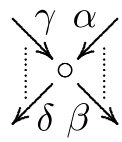
(1) Any vertex of $Q(S)$ is endpoint of at most two arrows

(1′) Any vertex of $Q(S)$ is starting point of at most two arrows.

(2) If two different arrows γ and δ start in the endpoint of the arrow α, then at least one of the paths $\gamma\alpha, \delta\alpha$ is a relation.

(2′) If two different arrows α and β end in the starting point of the arrow γ, then at least one of the paths $\gamma\alpha, \gamma\beta$ is a relation.

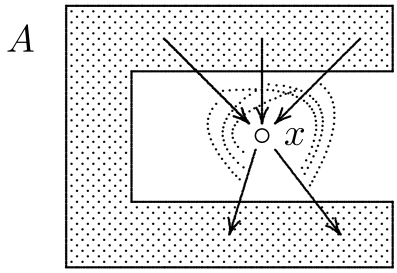
Thus, a biserial algebra has locally at most the following shape (maybe there are less arrows or more relations).

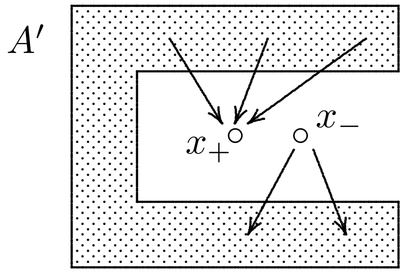


A vertex s is called a *node* in case $\beta\alpha = z$ for arrows α ending in s and β starting in s.

Main result: Preparation

It is sufficient to deal with algebras without a node, since an algebra A is minimal representation-infinite if and only if the algebra A' obtained from A by "separating" the node is minimal representation-infinite.





(There is a canonical functor $\operatorname{mod} A \to \operatorname{mod} A'$ which yiels a bijection between the indecomposable A-modules and the indecomposable A'-modules different from the simple A'-module $S(x_-)$.)

Main result

Theorem. *The biserial semigroups which are minimal representation-infinite and have no nodes are*

- *the cycles without relations,*
- *the barbells with non-serial bars and*
- *the wind wheels.*

We are going to exhibit these semigroups.

But first we recall what is known about the representations of biserial semigroups.

The representations of biserial semigroups

Gelfand-Ponomarev (1968):

Biserial semigroups are representation-finite or tame.

There are two kinds of indecomposable representations:
- *string modules and*
- *band modules.*

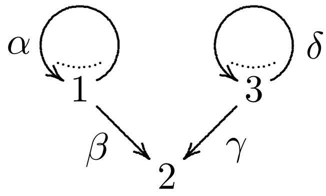
In order to construct all indecomposable representations, one starts with a word using as letters the arrows of the quiver and formal inverses of the arrows: in this way walking around in the quiver.

Any such word gives rise to a string module (and all are obtained in this way).

In order to obtain a band module, one needs a primitive cyclic word as well as a finite-dimensional k-space endowed with an indecomposable vector-space automorphism.

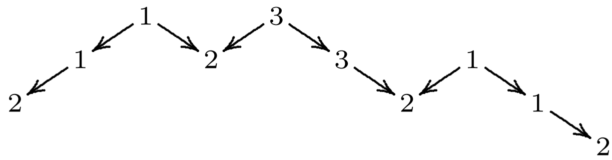
Example

Consider the following biserial semigroup



Here is a typical word

$$w = \beta \quad \alpha \quad \beta^{-1} \; \gamma \quad \delta^{-1} \gamma^{-1} \; \beta \quad \alpha^{-1} \beta^{-1}$$

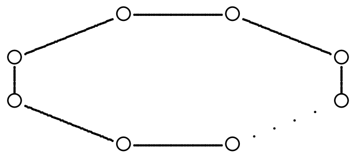


the corresponding string module $M(w)$ is of length 10.

I.

Cycles

The underlying graph of a cycle has the following form (the underlying graph of a quiver is obtained by deleting the orientation of the arrows):

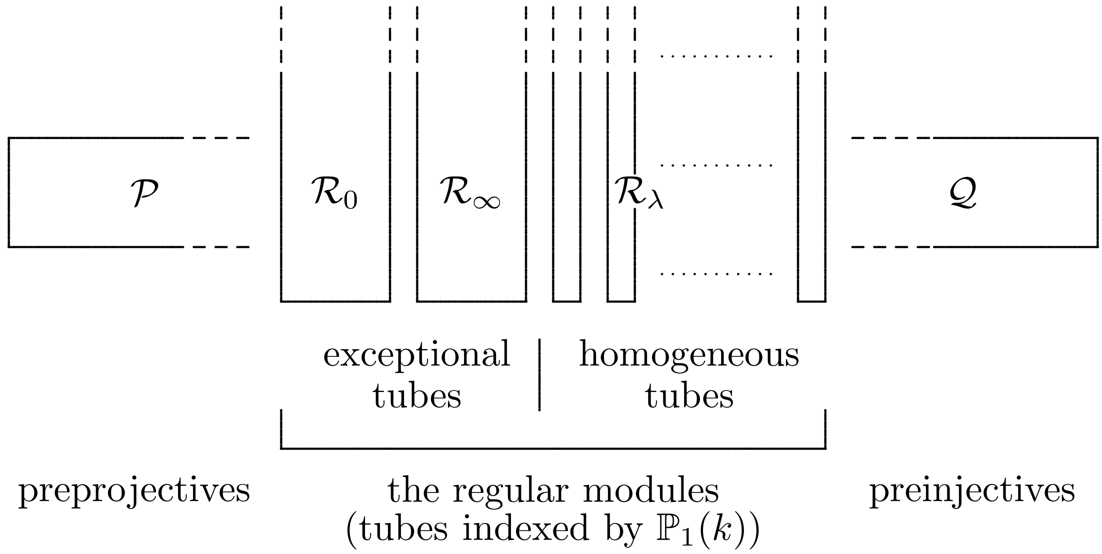


If $Q(S)$ is a cycle and no relations are given, then there has to be at least one sink and at least one source in $Q(S)$ so that S is finite.

These algebras $k[S]$ are hereditary algebras (i.e. global dimension ≤ 1), they are usually called the finite dimensional algebras of type $\widetilde{A}$.

The barbells and the wind wheels will be obtained from cycles by barification (and adding further relations).

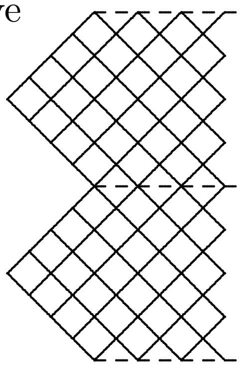
The Auslander-Reiten quiver of a cycle

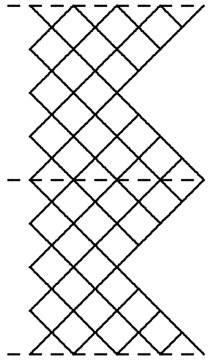


All components are surfaces with boundary, all are homeomorphic to $[0, \infty[\times S^1$.

The four non-homogeneous components: These are the components which contain the string modules.

The preprojective component $\mathcal{P}$

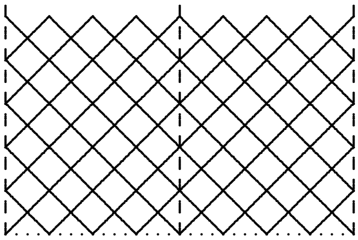


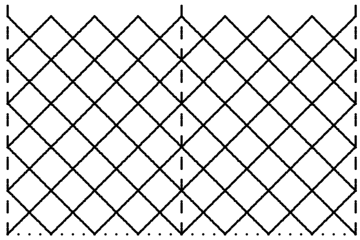


The preinjective component $\mathcal{Q}$

The two exceptional tubes:

$\mathcal{R}_0$





$\mathcal{R}_\infty$

The Auslander-Reiten quiver for any fin-dim k-algebra.

Let Λ be a finite-dimensional k-algebra (k algebraically closed).
The Auslander-Reiten quiver $\Gamma(\Lambda)$ describes
the factor category of $\operatorname{mod}\Lambda$ modulo its infinite radical, as follows:

- The vertices are the isomorphism classes $[X]$ of the indec Λ-modules X.
- The number of arrows $[X] \to [X']$ is $\dim \operatorname{Hom}(X, X')/\operatorname{rad}(X, X')$.
- In addition, the Auslander-Reiten translation defines a bijection between the non-projective modules and the non-injective modules.

The translation is used in order to define relations (the "mesh relations") on $\Gamma(\Lambda)$. The mesh relations are used in order to consider $\Gamma(\Lambda)$ as a (two-dimensional) **simplicial complex**.

In case S is a biserial semigroup, the mesh relations are zero-relations or commutativity relations, thus we deal again with a semigroup with zero (but not necessarily finite).

The serial modules for a cycle

A module is called *serial* provided it has a unique composition series.

We consider the serial $k[S]$-modules, where S is a cycle.

Lemma. *Any serial $k[S]$-module M is projective or regular or injective.*

Let M be a serial $k[S]$-module of length at least two.
- *If M is projective, then $M/\operatorname{soc} M$ belongs to $\mathcal{R}_0$ or $\mathcal{R}_\infty$.*
- *If M is injective, then $\operatorname{rad} M$ belongs to $\mathcal{R}_0$ or $\mathcal{R}_\infty$.*

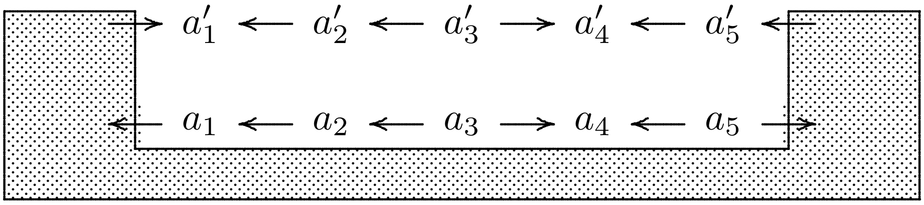
In the barification process (to be defined later), we often will "barify"
a projective serial module M of length $b > 1$ with $M/\operatorname{soc} M$ in $\mathcal{R}_0$
and an injective serial module M of the same length b with $\operatorname{rad} M$ in $\mathcal{R}_\infty$

II.

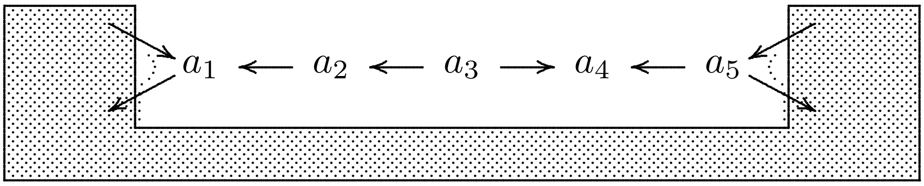
Barification

Barification

We start with two disjoint maximal isomorphic paths in $Q(S)$. For example

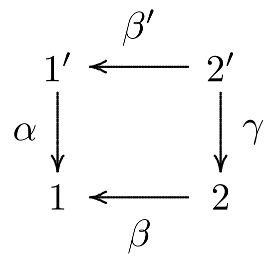


The paths are identified in order to form a *bar*. This barification yields a quiver of the following form (the dottes box is not changed):

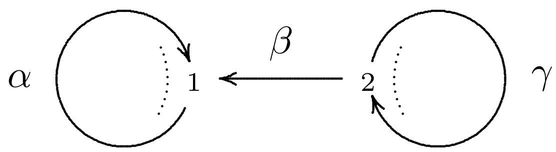


Of importance are the new zero relations on the left and on the right!

Example (a barbell with serial bar). Start with the following cycle:



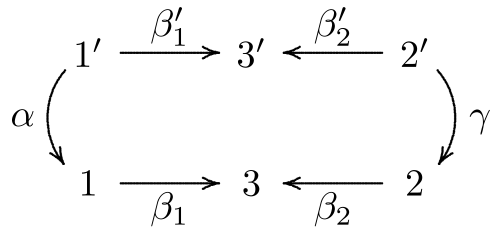
and identify the arrows β and β' to form the bar β:



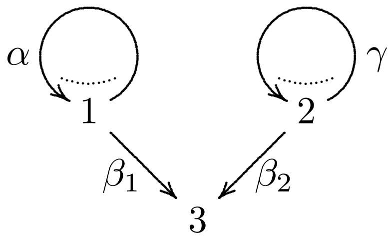
Barification adds the zero relations $\alpha^2 = z = \beta^2$.

This semigroup is not minimal representation-infinite, unless we add the relation $\alpha\beta\gamma$ (**"serial barification"**).

Barbells with non-serial bar. Example: start with



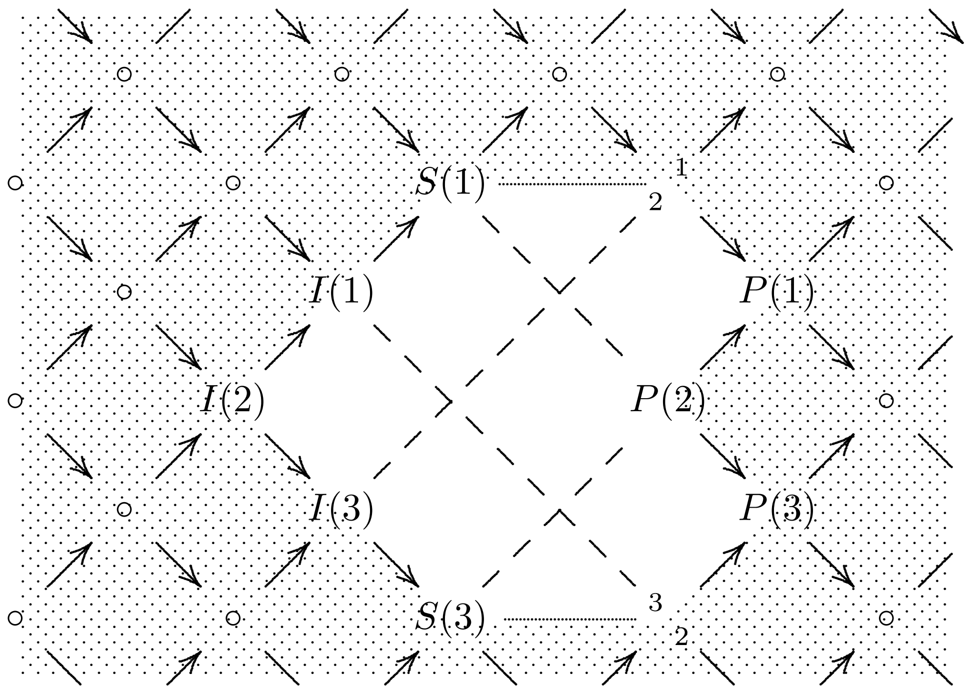
Barification yields:



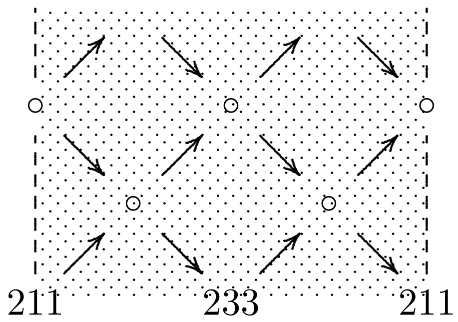
Here the bar (given by the arrows $1 \to 3 \leftarrow 2$) is not serial.

Theorem. *The barbells with non-serial bars are minimal representation-infinite. They are tame and of non-polynomial growth, but not domestic.*

The non-regular component of the Auslander-Reiten quiver looks as follows:



Here we have seen only part of the boundary, namely 10 of the 12 boundary modules. The remaining two boundary modules form the boundary of a stable tube of rank 2; the boundary meshes are those provided by the arrows $1 \to 2$ and $3 \to 2$ respectively:



Proposition. *The barbells with non-serial bar yield Gorenstein algebras of Gorenstein dimension 1.*

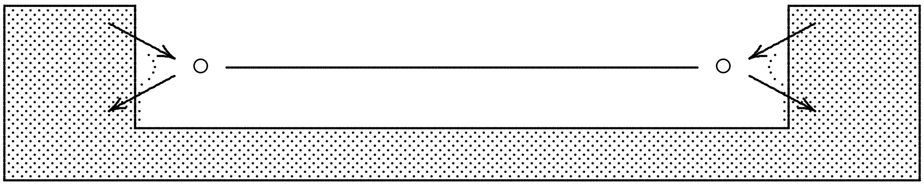
Sketch of the proof of the main theorem. Let S be a biserial semigroup which is minimal representation-infinite

Let x be a vertex. In case there are a arrows ending in x and b-arrows starting in x, then we say that x is an $(a+b)$-vertex.

(1) *Assume S is a biserial semigroup and minimal representation-infinite. If the vertex x is a 4-vertex, then x is a node.*

Since we may assume that there is no node, we easily see that there are only 2-vertices and 3-verices. In case all the vertices are 2-vertices, then we deal with a cycle.

Thus we can assume that there are 3-vertices, and therefore there will be a subquiver of the form (with only 2-vertices between the two given 3-vertices):



In this way one shows that S is obtained from a cycle by a sequence of barifications, adding, if necessary, further relations. It remains to analyse these barifications!

(2) Assume that S is obtained from a cycle by barification with bar B. In case B is not serial, any further barification yields a semigroup which no longer is minimal representation-infinite.

Thus we can assume that S is obtained from a cycle by a sequence of serial barifications, say identifying the serial paths w_i and w_i', for $1 \le i \le t$.

(3) Any module $M(w_i), M(w_i')$ is projective or injective.

(4) Assume that $M(w_i)$ is projective with radical in $\mathcal{R}_0$, then

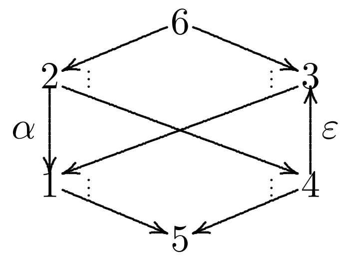
- all the other modules $M(w_j)$ are also projective with radical in $\mathcal{R}_0$,
- all the modules $M(w_j')$ are injective and $M(w_j')/\operatorname{soc}$ belongs to $\mathcal{R} > \infty$.

The semigroups obtained in this way are called "wind wheels".

III.

Wind wheels

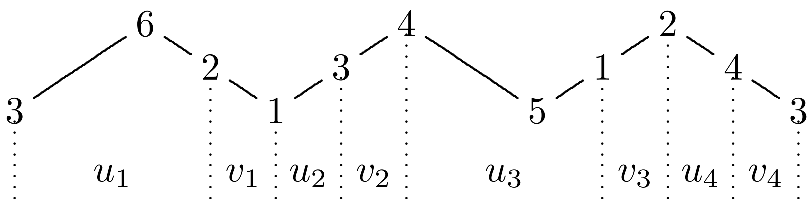
Recall that wind wheels are semigroups obtained from a cycle by a sequence of serial barifications. Here is an example:

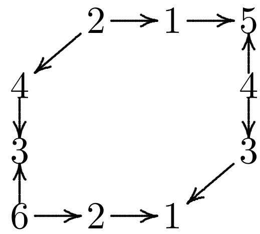


with the further relations $6 \to 2 \to 1 \to 5$ and $2 \to 4 \to 3 \to 1$.

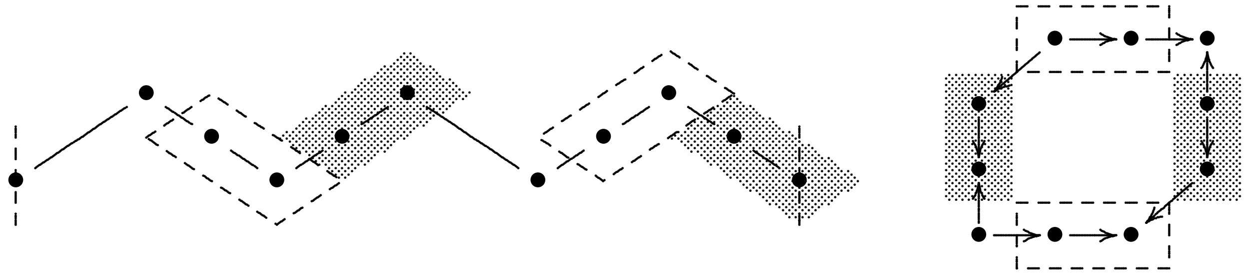
There are two bars, namely α and ϵ.

The cycle it is obtained from is seen by the only primitive cyclic word





Thus, we start with a quiver which can be drawn either as a zigzag (with arrows pointing downwards), where the left end and the right end have to be identified, or else as a proper cycle:



and we barify on the one hand the two subquivers which are enclosed in rectangular boxes, on the other hand also the two subquivers with shaded background. In both cases, the barification yields an identification of a projective serial module of length 2 with an injective serial module of length 2.

Theorem. *The wind wheels are minimal representation-infinite.*
They are 1-*domestic.* (There is a a unique primitive 1-parameter-family of indecomposable modules.)

If the wind wheel has t bars, then there are at most t non-regular components.

Theorem. *For any* $t \in \mathbb{N}$*, there are wind wheels with* t *non-regular AR-components, where* t *is the number of bars.*

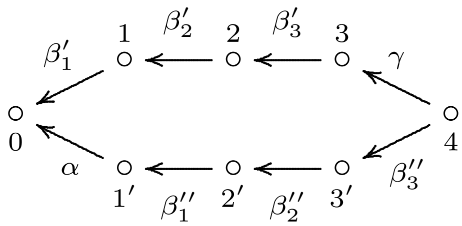
In particular, the number of non-regular components of 1-domestic algebras is not uniformly bounded.

Also: Let Λ we a wind wheel with t bars and let M be a primitive homogeneous and absolutely indecomposable Λ-module. Then the dimension of $\mathrm{End}(M)/\mathrm{ss}$ is equal to $t + 1$.

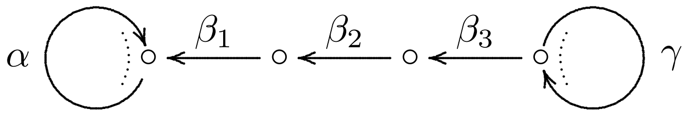
IV.

The Auslander-Reiten quiver of a wind wheel

An example. We consider the cycle H



and identify the paths $\beta_1'\beta_2'\beta_3'$ and $\beta_1''\beta_2''\beta_3''$ to obtain the bar $\beta_1\beta_2\beta_3$. The corresponding wind wheel W looks as follows:



with relations
$\alpha^2 = \gamma^2 = \alpha\beta_1\beta_2\beta_3\gamma = z$

Note that there is an embedding $k[W] \subset k[H]$, we consider the corresponding restriction functor

$$\eta\colon \operatorname{mod} k[H] \to \operatorname{mod} k[W].$$

Proposition. *The restriction functor*

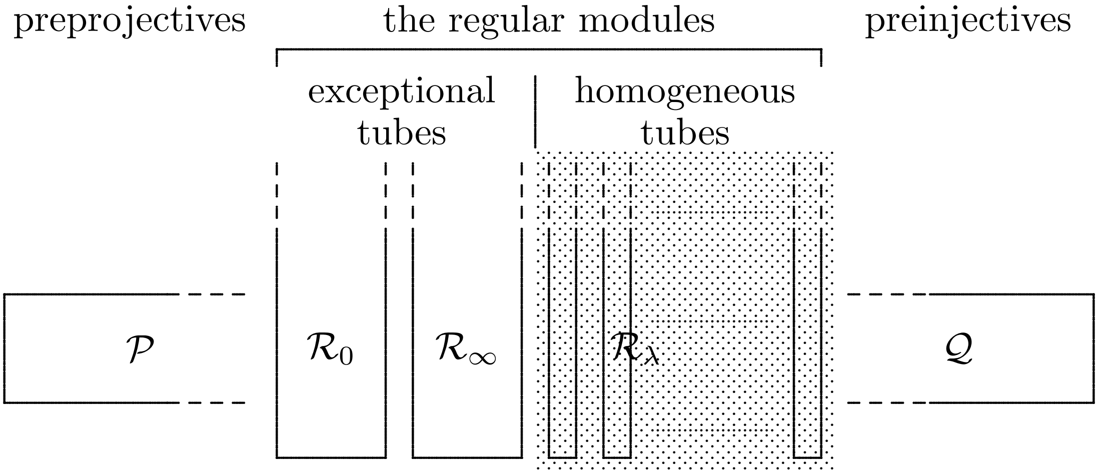
$$\eta\colon\ \operatorname{mod} k[H] \to \operatorname{mod} k[W].$$

has the following properties:

- *Indecomposable modules are sent to indecomposable modules.*
- *Corresponding modules on the two A_4-quivers which yield the bar become isomorphic, otherwise non-isomorphy is preserved.*
- *The indecomposable W-modules which are not in the image of the functor are the string modules for words which contain $\alpha^{-1}\beta_1\beta_2\beta_3\gamma^{-1}$ as a subword.*

We study the relationship between $\Gamma(k[H])$ and $\Gamma(k[W])$ in more detail.

Recall the shape of the category mod $k[H]$:

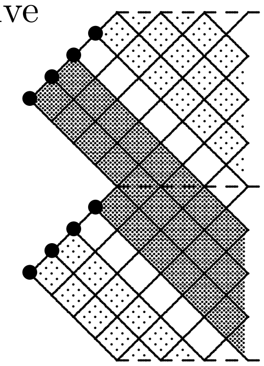


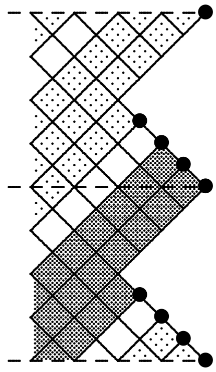
What happens under the restriction functor $\eta\colon\ \mathrm{mod}\, k[H] \to \mathrm{mod}\, k[W]$?

- The homogeneous tubes remain untouched.
- The remainung four components are cut between two rays or corays and embedded (with some overlap) into a planar component **("cut and paste").**

The four non-homogeneous components:

The preprojective component $\mathcal{P}$

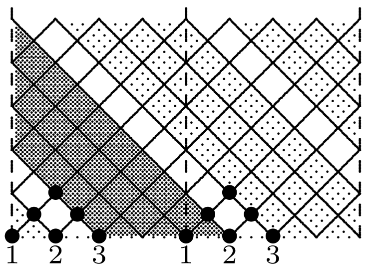


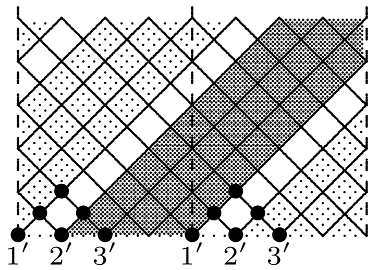


The preinjective component $\mathcal{Q}$

The two exceptional tubes:

$\mathcal{R}_0$

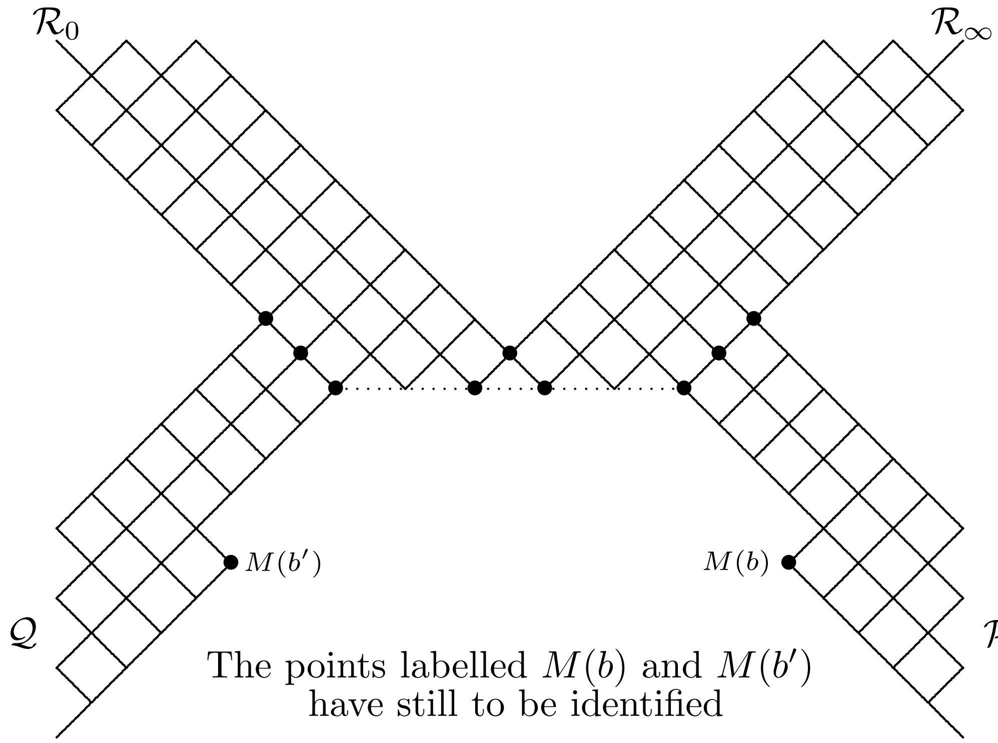


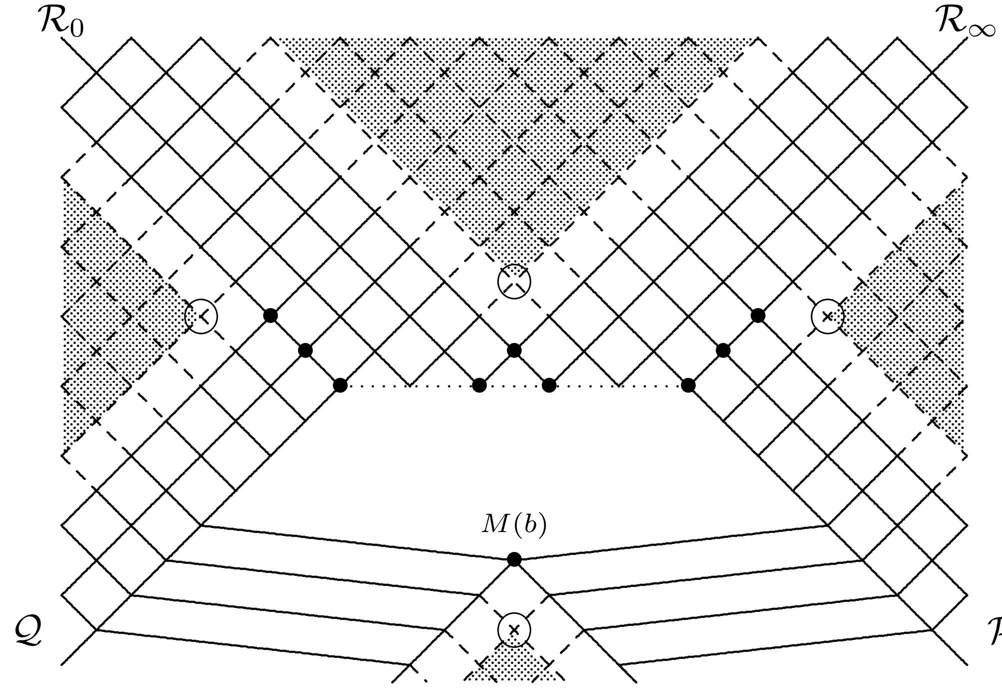


$\mathcal{R}_\infty$

(the boundaries of $\mathcal{R}_0$ and $\mathcal{R}_\infty$ contain three simple modules; they are labelled.)

By cutting and pasting we obtain this partial translation quiver.

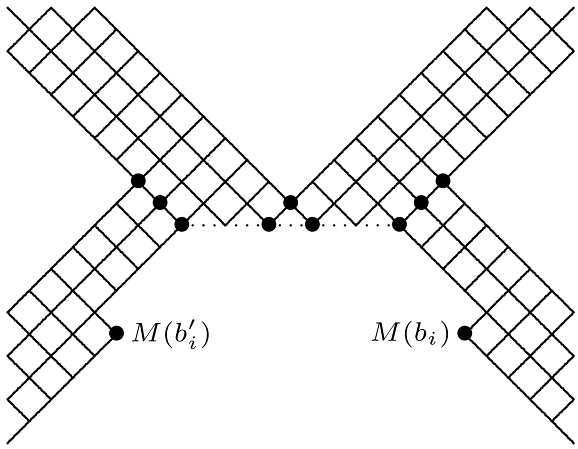




The added modules are those indecomposable W-modules which do not belong to the image of η. They form four separate quarter-planes.

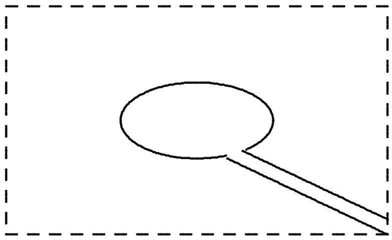
Wind wheels with t bars. In this case,

- any of the components $\mathcal{P}, \mathcal{Q}, \mathcal{R}_0, \mathcal{R}_\infty$ is cut into t pieces,
- using one piece of each kind, we obtain t planar partial translation quivers of the form

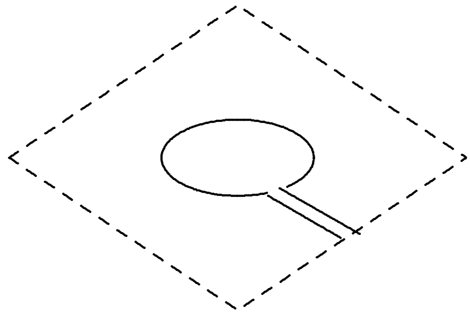


- There is a permutation π such that $M(b_i) = M(b'_{\pi(i)})$.
- The indecomposable W-modules which do not belong to the image of η form $4 \cdot t$ quarter-planes which can be inserted as in the case $t = 1$.

In this way, we obtain t partial translation quivers of the form



or better



They are sewn together in the same way as one constructs the Riemann surfaces of the n-the root functions in complex analysis (of course taking into account the permutation π).

The visualisation on the right hand side takes into account the embedding of these AR-components into the corresponding "AR-quilt" Ω_A (by adding as new vertices the possible infinite word — thus the isomorphism classes of indecomposable algebraically compact modules).

The non-regular AR-components of a wind wheel

The number c of AR-components obtained by the cut-and-paste procedure depends on the permutation π, it is just the number of orbits of π.

Recall that the remaining components of $\Gamma(k[W])$ are homogeneous tubes. It follows that *c is the number of non-regular AR-components.*

- For any $t \in \mathbb{N}$, one can construct wind wheel algebras with π being the identity. In this case way, we obtain algebras with $c = t$,

  thus c can be arbitrarily large.
- For t odd, one can construct algebras with $c = 1$.

The Auslander-Reiten quilt

Auslander-Reiten quilts have been considered until now only for suitable special biserial algebras Λ. A general definition can be given in case Λ is 1-domestic:

- The vertices are (finite or infinite) words in some alphabet,
- and there are arrows,
- meshes,
- but also a convergence relation.

The geom. realization of the AR-quilt is a compact surface Ω_Λ with boundary.

Infinite words are either $\mathbb{N}$-words or (non-periodic) $\mathbb{Z}$-words.

Lemma. *If Λ is a wind wheel algebra with t bars, then there are precisely t non-periodic (but bi-periodic) $\mathbb{Z}$-words.* Anyone gives rise to four quarters — these are the quarters which we have inserted into the cut-and-paste components.

Theorem. *Let Λ be a wind wheel with t bars. The Auslander-Reiten quilt Ω_Λ is a* **connected** *surface with boundary, its Euler characteristic is $\chi(\Omega_\Lambda) = -t$.*