Quiver Grassmannians for Wild Acyclic Quivers

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Abstract. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as a quiver Grassmannian for a suitable representation of some wild acyclic quiver. We show that this happens for *any* wild acyclic quiver.

Let k be an algebraically closed field, and Q a finite acyclic quiver. The modules which we consider are the (finite-dimensional) kQ-modules, where kQ is the path algebra of Q, thus the (finite-dimensional) representations of Q (with coefficients in k). We denote by $\operatorname{mod} kQ$ the corresponding module category.

Let M be a representation of Q and $\mathbf{d}$ a dimension vector for Q. The *quiver Grassmannian* $\mathbb{G}_{\mathbf{d}}(M)$ is the set of submodules of M with dimension vector $\mathbf{dim}\, M = \mathbf{d}$; this is a projective variety. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as the quiver Grassmannian for a representation of **some** wild acyclic quiver Q, see for example [R2]. We are going to show:

Theorem. *Let Q be* **any** *wild acyclic quiver. Any projective variety occurs as a quiver Grassmannian $\mathbb{G}_{\mathbf{d}}(M)$ for some representation M of Q and some dimension vector* $\mathbf{d}$.

Typical wild acyclic quivers are the Kronecker quivers $Q = K(n)$ with $n \geq 3$ (the Kronecker quiver $K(n)$ has two vertices 1 and 2 and n arrows pointing from 2 to 1). A representation of $K(n)$ will be said to be *reduced* provided N has no simple injective direct summand. In [R3] we have shown that for any projective variety $\mathcal{V}$ there is a natural number n (depending on $\mathcal{V}$) such that $\mathcal{V}$ can be realized as the quiver Grassmannian $\mathbb{G}_{(1,1)}(N)$ of a reduced representation N of $K(n)$ (see also [H]). Our present investigation relies on this special case.

Note that the elements of $\mathbb{G}_{(1,1)}(N)$ are certain submodules of N of length 2, and all the indecomposable submodules of length 2 belong to $\mathbb{G}_{(1,1)}(N)$. We call indecomposable modules of length 2 *bristles*. For any representation N of $K(n)$, the set $\beta(N)$ of bristle submodules of N is an open subset of $\mathbb{G}_{(1,1)}(N)$ which we call the *bristle variety* of N. In general, $\beta(N)$ is a proper subset of $\mathbb{G}_{(1,1)}(N)$, but for a reduced representation N, we have $\beta(N) = \mathbb{G}_{(1,1)}(N)$.

The procedure of the present paper is as follows: Given any wild acyclic quiver Q, and a natural number m, we will construct for some $n \geq m$ an orthogonal pair X, Y of bricks with $\dim \operatorname{Ext}^1(Y, X) = n$ (a *brick* is a module with endomorphism ring k and X, Y are said to be *orthogonal* provided $\operatorname{Hom}(X, Y) = 0 = \operatorname{Hom}(Y, X)$). Always, $\mathbf{x}$ and $\mathbf{y}$ will denote the dimension vectors of X and Y, respectively. Let $\mathcal{E} = \mathcal{E}(Y, X)$ be the full subcategory of all kQ-modules M with an exact sequence of the form

$$0 \to X^a \to M \to Y^b \to 0,$$

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where a, b are natural numbers. Note that $\mathcal{E}$ is equivalent to $\operatorname{mod} kK(n)$ with an equivalence being given by an exact fully faithful functor

$$\eta\colon \operatorname{mod} kK(n) \to \operatorname{mod} kQ$$

with image $\mathcal{E}$. We say that a module M in $\mathcal{E}$ is *$\mathcal{E}$-reduced* provided it has no direct summand isomorphic to Y, thus provided it is the image of a reduced $kK(n)$-module under η.

An indecomposable kQ-module U will be called an *$\mathcal{E}$-bristle* provided there is an exact sequence of the form $0 \to X \to U \to Y \to 0$, thus provided U is the image of a bristle in $\operatorname{mod} kK(n)$ under η. For any $kK(n)$-module N with $M = \eta N$, the functor η identifies the bristle variety $\beta(N)$ of N with the set $\beta_{\mathcal{E}}(M)$ of submodules of M which are $\mathcal{E}$-bristles. Since $\mathcal{E}$-bristles have dimension vector $\mathbf{x}+\mathbf{y}$, we have $\beta_{\mathcal{E}}(M) \subseteq \mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$. It remains to find conditions such that any submodule U of M with dimension vector $\mathbf{x}+\mathbf{y}$ is indeed an $\mathcal{E}$-bristle.

To be precise, we are looking for kQ-modules X, Y so that the following closure condition (C) is satisfied:

(C) *If M is an $\mathcal{E}$-reduced module in $\mathcal{E}(Y,X)$ and U is a submodule of M with* $\mathbf{dim}\, U = \mathbf{x}+\mathbf{y}$*, then U is an $\mathcal{E}$-bristle.*

If the condition (C) is satisfied, then for any reduced representation N of $K(n)$, there is a canonical bijection between $\mathbb{G}_{(1,1)}(N)$ and $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$, where $M = \eta N$. Namely, if B is a submodule of the $kK(n)$-module N with $\mathbf{dim}\, B = (1,1)$, then ηB is a submodule of M with dimension vector $\mathbf{x}+\mathbf{y}$. Conversely, if U is a submodule of M with $\mathbf{dim}\, U = \mathbf{x}+\mathbf{y}$, then, by condition (C), U belongs to $\mathcal{E}(Y,X)$, say $U = \eta B$ for some $K(n)$-submodule B and the dimension vector of B is $(1,1)$.

The minimal wild acyclic quivers.

As we have mentioned, our aim is to exhibit for any wild acyclic quiver Q and any natural number m an orthogonal pair X, Y of kQ-modules which are bricks such that $\dim_k \operatorname{Ext}^1(Y,X) = n \geq m$ and such that the condition (C) is satisfied. Of course, it is sufficient to deal with minimal wild acyclic quivers. (We recall that a quiver Q is *wild* provided it is not the disjoint union of Dynkin and Euclidean quivers, and Q is said to be *minimal wild* provided it is wild, and no quiver obtained from Q by deleting a vertex or an arrow is wild.)

The following well-known proposition suggests to deal with two different cases.

Proposition. *A minimal wild acyclic quiver Q different from $K(3)$ is obtained from a Euclidean quiver Q' by adding a vertex ω and a single arrow which connects ω with some vertex of Q' (in particular, ω is a sink or a source).*

Sketch of proof. If Q has cycles, then there is a subquiver Q' of type $\widetilde{\mathbb{A}}_n$ for some n such that Q' is obtained from Q by deleting one vertex and one arrow.

Now assume that Q is a tree. If there is a vertex with at least four neighbors, then Q' is obtained from a quiver of type $\widetilde{\mathbb{D}}_4$ by deleting one vertex and one arrow. If Q has two vertices which have three neighbors each, then Q' is obtained from a quiver of type $\widetilde{\mathbb{D}}_n$ with $n \geq 5$ by deleting one vertex and one arrow. If Q has is a star with three arms, then Q' is obtained from a quiver of type $\widetilde{\mathbb{E}}_m$ with $m = 6, 7, 8$ by deleting one vertex and one arrow. $\square$

Case 1. One-point extensions of representation-infinite quivers.

We assume now that Q is a connected quiver with a vertex ω which is a sink or a source such that the quiver Q' obtained from Q by deleting ω and the arrows which start or end in ω is connected and representation-infinite. Up to duality, we can assume that ω is a source, thus there is an arrow $\omega \to p$ with $p \in Q'_0$.

Let $Y = S(\omega)$, the simple kQ-module corresponding to the vertex ω. Since Q' is connected and representation-infinite, there is an exceptional kQ'-module X with $\dim_k X_p \geq m$. The arrow $\omega \to p$ shows that $\dim_k \operatorname{Ext}^1(Y, X) \geq \dim_k X_p$. This pair X, Y is the orthogonal pair of bricks which we use in order to look at $\mathcal{E}(Y, X)$.

Lemma 1. *Let a be a natural number. Any submodule W of X^a with $\mathbf{dim}\, W = \mathbf{x}$ is isomorphic to X.*

Proof. We denote by $\langle -, - \rangle$ the bilinear form on the Grothendieck group $K_0(kQ)$ with $\langle \mathbf{dim}\, M, \mathbf{dim}\, M' \rangle = \dim_k \operatorname{Hom}(M, M') - \dim_k \operatorname{Ext}^1(M, M')$. Since X is exceptional, we have $\langle X, W \rangle = \langle X, X \rangle > 0$, Therefore, there is a non-zero homomorphism $f \colon X \to W$. Let $\iota \colon W \to X^a$ be the inclusion map. The composition $\iota f \colon X \to X^a$ is nonzero. Since X is a brick, we see that $f \colon X \to W$ is a split monomorphism, in particular injective. Now $\mathbf{dim}\, X = \mathbf{dim}\, W$ implies that f is an isomorphism. $\square$

Proof of condition (C). Let M be an $\mathcal{E}$-reduced kQ-module in $\mathcal{E}(Y, X)$, say with an exact sequence

$$0 \to X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \to 0.$$

Let U be a submodule of M with dimension vector $\mathbf{x} + \mathbf{y}$ and inclusion map $\iota \colon U \to M$. The composition $\pi\iota$ is non-zero, since otherwise U would be a submodule of X^a, but $\dim_k U_\omega = 1$ whereas $X_\omega = 0$. If follows that the image of $\pi\iota$ is isomorphic to Y. If we denote the kernel of $\pi\iota$ by W, we obtain the following commutative diagram with exact rows and vertical monomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, "\iota"'] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^a \arrow[r, "\mu"] & M \arrow[r, "\pi"] & Y^b \arrow[r] & 0.
\end{tikzcd}$$

Of course, $\mathbf{dim}\, W = \mathbf{x}$, thus Lemma 1 shows that W is isomorphic to X. In particular, U belongs to $\mathcal{E}$.

It remains to show that U is indecomposable. Otherwise, U would be isomorphic to $W \oplus Y$. Thus M would have a submodule isomorphic to Y. But Y is relative injective inside $\mathcal{E}$, thus M would have a direct summand isomorphic to Y, in contrast to our assumption that M is $\mathcal{E}$-reduced. This shows that U is indecomposable, thus an $\mathcal{E}$-bristle. $\square$

Case 2. The Kronecker quiver $K(3)$.

Here we consider the Kronecker quiver $Q = K(3)$, with the three arrows $\alpha, \beta, \gamma : 2 \to 1$. Let $\lambda_1, \dots, \lambda_n$ be pairwise different non-zero elements of k with $n \geq 2$. Let $X = X(\lambda_1, \dots, \lambda_n) = (k^n, k^n; \alpha, \beta, \gamma)$ be defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(i)) = e(i+1),$$

for $1 \leq i \leq n$, where $e(1), \dots, e(n)$ is the canonical basis of k^n and $e(n+1) = e(1)$. Let $Y = (k, k; 1, 0, 0)$. We denote by Q' the subquiver of Q with arrows α, β, this is the 2-Kronecker quiver $K(2)$. For the structure of the category $\operatorname{mod} K(2)$, see for example [R1]. The restriction of X, Y to Q' shows that $\operatorname{Hom}(X, Y) = \operatorname{Hom}(Y, X) = 0$. The endomorphism ring of $X|Q'$ is $k \times \cdots \times k$; and the only endomorphisms of $X|Q'$ which commute with γ are the scalar multiplications. This shows that X is a brick. Also, it is easy to see that $\dim_k \operatorname{Ext}^1(Y, X) = n$.

Lemma 2. *Let a be a natural number. Any submodule W of X^a with* $\mathbf{dim}\, W$ *of the form (w, w) is isomorphic to X^s for some s.*

Proof: Let $M = X^a$ and decompose $M|Q' = \bigoplus_{i=1}^n M(i)$, where $\beta(x) = \lambda_i x$ for $x \in M(i)_1$. Here, we use α in order to identify M_1 and M_2. Now we consider the submodule W of M. Note that $W|Q'$ has to be regular, since it cannot have any non-zero preinjective direct summand. As a regular submodule of a semisimple regular Kronecker module it has to be a direct summand of $M|Q'$, thus we have a similar direct decomposition $W = \bigoplus W(i)$, where $W(i) = W \cap M(i)$.

The linear map γ restricted to $W(i)_1$ is a monomorphism $W(i)_1 \to W(i+1)_2 = W(i+1)_1$ for $1 \leq i \leq n$; we obtain in this way a monomorphism $W(1)_1 \to W(1)_2 = W(1)_1$. This shows that all the monomorphisms $W(i)_1 \to W(i+1)_2 = W(i+1)_1$ are actually bijections. Let $\dim_k W(1)_1 = s$. It follows that W is isomorphic to X^s. $\square$

Proof of condition (C). Let M be an $\mathcal{E}$-reduced kQ-module in $\mathcal{E}$ and let U be a submodule of M with dimension vector $\mathbf{x} + \mathbf{y} = (n+1, n+1)$ and with inclusion map $\iota \colon U \to M$.

Starting with the exact sequence $0 \to X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \to 0$ and the inclusion map $\iota \colon U \to M$, let W be the kernel and $\overline{U}$ the image of $\pi\iota \colon U \to Y^b$. We obtain the following commutative diagram with exact rows and injective vertical maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & \overline{U} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow & & \\
0 & \longrightarrow & X^a & \xrightarrow{\mu} & M & \xrightarrow{\pi} & Y^b & \longrightarrow & 0;
\end{array}$$

Let us consider the restriction of these modules to Q'. Since $M|Q'$ is regular, it has no non-zero preinjective direct summand. Thus any submodule of $M|Q'$ with dimension vector $(n+1, n+1)$ has to be regular. This shows that $U|Q'$ is regular. Actually, $M|Q'$ is semisimple regular, thus also its regular submodule $U|Q'$ is semisimple regular (and a direct summand of $M|Q'$). Next, $\pi\iota$ is a map between regular kQ'-modules, it follows that the kernel $W|Q'$ and the image $\overline{U}|Q'$ are regular kQ'-modules. In particular, the dimension vector of W is of the form $\mathbf{dim}\, W = (w, w)$ for some $0 \leq w \leq n+1$.

Now $\overline{U}|Q'$ is a regular submodule of the semisimple regular kQ'-module $Y^b|Q'$, thus $\overline{U}|Q'$ is a direct sum of copies of $Y|Q'$. By construction, Y is annihilated by γ. Since $\overline{U}$ is a submodule of Y^b, it follows that $\overline{U}$ is annihilated by γ. Altogether, we see that $\overline{U}$ is the direct sum of copies of Y.

We claim that $W \neq 0$. Otherwise $U = \overline{U} = Y^{n+1}$, thus Y is a submodule of M. But Y is relative injective in $\mathcal{E}$, thus Y would be a direct summand of M. However, by assumption, M is $\mathcal{E}$-reduced. This contradiction shows that $W \neq 0$.

Now W is a submodule of X^a with dimension vector (w, w), thus, according to Lemma 2, W is a direct summand of say s copies of X and $s \geq 1$. The equality $(w, w) = (sn, sn)$ implies that that $s = 1$, since $w \leq n + 1$ and $n \geq 2$. In this way, we see that W is isomorphic to X. It follows that $\mathbf{dim}\,\overline{U} = (1, 1)$ and therefore $\overline{U} = Y$.

Finally, as in Case 1, we see that U is indecomposable, using again the assumption that M is $\mathcal{E}$-reduced. This shows that U is an $\mathcal{E}$-bristle. $\square$

Remark. We should stress that given orthogonal bricks X, Y in $\operatorname{mod} kQ$, the condition (C) is usually not satisfied. Here is a typical example for $Q = K(3)$. As above, let $Y = (k, k; 1, 0, 0)$, but for X we now take $X = X'(\lambda_1, \lambda_2) = (k^2, k^2; \alpha, \beta, \gamma)$, defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(1)) = e(2), \quad \gamma(e(2)) = 0$$

for $1 \leq i \leq 2$. Again, $e(1), e(2)$ is the canonical basis of k^2 and $\lambda_1 \neq \lambda_2$ are assumed to be non-zero elements of k. Since $\dim_k \operatorname{Ext}^1(Y, X) = 2$, there is an equivalence $\eta\colon \operatorname{mod} kK(2) \to \mathcal{E}(Y, X)$. Let N be an indecomposable $kK(2)$-module with dimension vector $(2, b)$ (note that b has to be equal to $1, 2$ or 3) and $M = \eta N$. Thus there is an exact sequence

$$0 \to X^2 \to M \to Y^b \to 0.$$

Since we assume that N is indecomposable, it is reduced, thus M is $\mathcal{E}$-reduced. Note that X has a (unique) kQ-submodule V with dimension vector $(1, 1)$: the vector spaces V_1 and V_2 both are generated by $e(2)$. The submodule $U = X \oplus V$ of X^2 is a submodule of M with dimension vector $(3, 3) = \mathbf{x} + \mathbf{y}$, and it is not an $\mathcal{E}$-bristle. Thus, condition (C) is not satisfied. Here, η defines a proper embedding of $\beta(N) = \mathbb{G}_{(1,1)}(N)$ into $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$.

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