Hall algebras revisited

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Abstract. For any finite-dimensional algebra over a finite field, the corresponding Hall algebra has been introduced in order to handle the possible filtrations of modules with fixed factors. For the path algebra of a Dynkin diagram with a fixed orientation, it has been shown that the Hall algebra satisfies relations which are similar to the Drinfeld-Jimbo relations defining quantum groups, but they depend on the chosen orientation. The purpose of this note is to adjust the multiplication of a Hall algebra in order to obtain the Drinfeld-Jimbo relations themselves. The additional factor introduced in our change of multiplication involves the Euler characteristic, in this way we eliminate the dependence on the orientation.

Given a finite-dimensional connected hereditary algebra A of finite representation type, say with Dynkin diagram Δ, the indecomposable A-modules correspond bijectively to the positive roots of the simple complex Lie-algebra $\mathbf{g}$ of type Δ. Thus, the Grothendieck group $G(A, \mathbb{C})$ of the category of finitely generated A-modules relative to split exact sequences and with coefficients in $\mathbb{C}$ may be identified with $\mathbf{n}_+$, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of $\mathbf{g}$. Hall algebras have been introduced in order to recover the Lie multiplication on $G(A, \mathbb{C})$ using the representation theory of A. The Hall algebra $\mathcal{H}(A, \mathbb{C}[q])$ is rather similar to the Drinfeld-Jimbo quantization $U_q(\mathbf{n}_+)$ of the universal enveloping algebra $U(\mathbf{n}_+)$, however it depends on the orientation on Δ given by A. Our aim is to change the multiplication slightly in order to remove this dependence. We will explain the change of multiplication dealing with the integral Hall algebra $\mathcal{H}(A)$, where A is any finite-dimensional hereditary k-algebra with center k, with k a finite field.

1. Change of multiplication for graded rings

Let $R = \bigoplus_{g\in G} R_g$ be a graded ring, say with multiplication $\cdot$, where G is an abelian group (written additively). Let c be an invertible central element of R of degree 0, and let $\alpha\colon G\times G \to \mathbb{Z}$ be a bilinear form. On the underlying graded group of R, we define a new multiplication $* = \underset{\alpha}{*}$ as follows: Given non-zero elements $r \in R_g, s \in R_h$, let

$$r * s = c^{\alpha(g,h)} r \cdot s,$$

the ring obtained in this way will be denoted by $R_{[\alpha,c]}$. It is easy to check that $R_{[\alpha,c]}$ is again an associative ring, with the same unit element as R. Using induction, one shows:

Lemma 1. *Let $r_i \in R_{g_i}$, for $1 \le i \le n$. Then*

$$r_1 * r_2 * \cdots * r_m = c^a r_1 \cdot r_2 \cdot \ldots \cdot r_m,$$

where $a = \sum_{i<j} \alpha(g_i, g_j)$.

We denote by $r^{*t} = r * \cdots * r$ the t^{th} power of an element r in $R_{[\alpha,c]}$.

2. Hall algebras

Let A be a finite-dimensional hereditary k-algebra, with center k, and let $E_1, \ldots, E_n$ be the simple A-modules.

Let $K_0(A)$ be the Grothendieck group of all finite-dimensional A-modules relative to all exact sequences. For any A-module M, the corresponding element in $K_0(A)$ will be denoted by $\mathbf{dim}\, M$, thus $K_0(A)$ may be identified with the free abelian group with basis $\mathbf{dim}\, E_1, \ldots, \mathbf{dim}\, E_n$. Let ε be the Euler characteristic on $K_0(A)$, thus given A-modules M_1, M_2,

$$\begin{aligned}\varepsilon(\mathbf{dim}\, M_1, \mathbf{dim}\, M_2) &= \sum_{t \ge 0} (-1)^t \dim_k \operatorname{Ext}^t(M_1, M_2) \\ &= \dim_k \operatorname{Hom}(M_1, M_2) - \dim_k \operatorname{Ext}^1(M_1, M_2).\end{aligned}$$

Let us assume that k is a finite field, let $v = \sqrt{|k|}$. Let $\mathcal{H} = \mathcal{H}(A) \otimes \mathbb{Z}[v, v^{-1}]$; it is a $K_0(A)$-graded ring. We consider the ring $\mathcal{H}^* = \mathcal{H}_{[\varepsilon,v]}$. We will exhibit a direct description of $\mathcal{H}^*$ below. Given an A-module M, we denote its isomorphism class by $[M]$ and the corresponding element in $\mathcal{H}(A)$ and in $\mathcal{H}$ by $u_{[M]}$. Let $u_i = u_{[E_i]}$.

The Hall algebra $\mathcal{H}^*$ may be defined directly as follows: Given A-modules N_1, N_2, M, let $F^M_{N_1 N_2}$ be the number of submodules M' of M such that M/M' is isomorphic to N_1, whereas M' is isomorphic to N_2. Let $\mathcal{H}^*$ be the free $\mathbb{Z}[v, v^{-1}]$-module with basis $(u_{[M]})_{[M]}$, indexed by the set of isomorphism classes of finite A–modules. We define on $\mathcal{H}^*$ a multiplication $*$ by the following rule

$$u_{[N_1]} * u_{[N_2]} = v^{\varepsilon(\mathbf{dim}\, N_1, \mathbf{dim}\, N_2)} \sum_{[M]} F^M_{N_1 N_2} u_{[M]}.$$

For any i, let $f_i = \dim_k \operatorname{End}(E_i)$. Fix some pair $i \ne j$, with $\operatorname{Ext}^1(E_i, E_j) = 0$. Let

$$\begin{aligned} a_{ij} &= -\dim \operatorname{Ext}^1(E_j, E_i)_{\operatorname{End}(E_i)} \\ a_{ji} &= -\dim_{\operatorname{End}(E_j)} \operatorname{Ext}^1(E_j, E_i), \end{aligned}$$

thus $f_i a_{ij} = f_j a_{ji}$. Let $q_i = v^{2 f_i}$.

Recall the Drinfeld-Jimbo relations

$$\rho_n(q, X, Y) = \sum_{t=0}^{n} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_q q^{-\frac{t(n-t)}{2}} X^t Y X^{n-t},$$

they have been introduced in order to define the quantizations of the universal enveloping algebras of the semisimple complex Lie algebras, and, more general, of the Kac-Moody Lie algebras.

Proposition. *In $\mathcal{H}^*$, we have*

$$\rho_{1-a_{ij}}(q_i,u_i,u_j)=0, \quad \textit{and} \quad \rho_{1-a_{ji}}(q_j,u_j,u_i)=0.$$

Proof. We consider a pair $i \neq j$, with $\operatorname{Ext}^1(E_i,E_j)=0$. Let

$$\varepsilon_{ij}=\varepsilon(\mathbf{dim}\,E_i,\mathbf{dim}\,E_j),$$

thus

$$\varepsilon_{ii}=f_i,\ \varepsilon_{ij}=0,\ \varepsilon_{ji}=f_ia_{ij}=f_ja_{ji}.$$

Let $m=1-a_{ij}$. We have to consider products of the form $u_i^{*t}*u_j*u_i^{*(m-t)}$, and Lemma 1 shows that

$$u_i^{*t}*u_j*u_i^{*(m-t)}=v^a u_i^t u_j u_i^{m-t},$$

where

$$\begin{aligned} a &= \binom{m}{2}\varepsilon_{ii}+t\varepsilon_{ij}+(m-t)\varepsilon_{ji} \\ &= \binom{m}{2}f_i+(m-t)f_ia_{ij} \\ &= f_i(\frac{m(m-1)}{2}+(m-t)(1-m)) \\ &= f_i(-\binom{m}{2}+tm-t). \end{aligned}$$

It follows that

$$-f_it(m-t)+a=f_i(-tm+t^2-\binom{m}{2}+tm-t)=f_i(-\binom{m}{2}+2\binom{t}{2}).$$

As a consequence

$$q_i^{-\frac{t(m-t)}{2}}u_i^{*t}*u_j*u_i^{*(m-t)}=q_i^{-\frac{1}{2}\binom{m}{2}}q_i^{\binom{t}{2}}u_i^tu_ju_i^{m-t},$$

thus

$$\begin{aligned} \rho_m(q_i,u_i,u_j) &= \sum_{t=0}^{m}(-1)^t\begin{bmatrix} m \\ t \end{bmatrix}_{q_i} q_i^{-\frac{t(m-t)}{2}}u_i^{*t}*u_j*u_i^{*(m-t)} \\ &= \sum_{t=0}^{m}(-1)^t\begin{bmatrix} m \\ t \end{bmatrix}_{q_i} q_i^{-\frac{1}{2}\binom{m}{2}}q_i^{\binom{t}{2}}u_i^tu_ju_i^{m-t} \\ &= q_i^{-\frac{1}{2}\binom{m}{2}}\sum_{t=0}^{m}(-1)^t\begin{bmatrix} m \\ t \end{bmatrix}_{q_i} q_i^{\binom{t}{2}}u_i^tu_ju_i^{m-t}, \end{aligned}$$

and according to [**R3**] we know that the latter sum vanishes.

Similarly, let $m' = 1 - a_{ji}$. Observe that

$$u_j^{*(m'-t)} * u_i * u_j^{*t} = v^b u_j^{m'-t} u_i u_j^t,$$

with

$$b = \binom{m'}{2}\varepsilon_{jj} + (m'-t)\varepsilon_{ji} + t\varepsilon_{ij} = f_j(-\binom{m'}{2} + tm' - t),$$

therefore

$$\begin{aligned}
\rho_{m'}(q_j, u_j, u_i) &= (-1)^{m'} \sum_{t=0}^{m'} (-1)^t \begin{bmatrix} m' \\ t \end{bmatrix}_{q_j} q_j^{-\frac{t(m'-t)}{2}} u_j^{*(m'-t)} * u_i * u_j^{*t} \\
&= (-1)^{m'} \sum_{t=0}^{m'} (-1)^t \begin{bmatrix} m' \\ t \end{bmatrix}_{q_j} q_j^{-\frac{1}{2}\binom{m'}{2}} q_j^{\binom{t}{2}} u_j^{m'-t} u_i u_j^t \\
&= (-1)^{m'} q_j^{-\frac{1}{2}\binom{m'}{2}} \sum_{t=0}^{m'} (-1)^t \begin{bmatrix} m' \\ t \end{bmatrix}_{q_j} q_j^{\binom{t}{2}} u_j^{m'-t} u_i u_j^t,
\end{aligned}$$

and again the latter sum vanishes according to [**R3**]**.** This completes the proof.

In a similar way, we may change the multiplication for the generic Hall algebras, and for the Loewy and composition algebras as defined in [**R2,3,4**]**.**

3. The Euler characteristic for a quiver

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, with Q_0 the set of vertices, Q_1 the set of arrows; these arrows are of the form $\alpha : s(\alpha) \to t(\alpha)$. If we allow the existence of cyclic paths, the path algebra kQ will not be finite-dimensional, however the Hall algebra $\mathcal{H}(kQ)$ still is defined provided we assume that there are only finitely many arrows between any pair of vertices, see [**R2**]. We consider $\mathcal{H}(kQ)$ as a graded G-ring, where $G = \mathbb{Z}^{Q_0}$. Note that **dim** furnishes a surjective map from the Grothendieck group $K_0(kQ)$ of all finite-dimensional kQ-modules modulo exact sequences onto G, but this map is bijective only in case there are no cyclic paths.

We consider the bilinear form

$$\varepsilon(x, y) = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)},$$

for $x, y \in G$, it satisfies

$$\varepsilon(\mathbf{dim}\, M_1, \mathbf{dim}\, M_2) = \dim_k \operatorname{Hom}(M_1, M_2) - \dim_k \operatorname{Ext}^1(M_1, M_2),$$

for finite-dimensional kQ-modules M_1, M_2, see [**R1**]. The quadratic form obtained from the Euler characteristic ε may be described also in terms of algebraic geometry: Given $m_1, m_2 \in \mathbb{N}$, let $\mathrm{M}(m_1, m_2)$ be the set of $(m_1 \times m_2)$-matrices with entries in k, and $\mathrm{Gl}(m_1)$ the group of invertible $(m_1{\times}m_1)$-matrices. For $d \in \mathbb{N}^{Q_0}$, the affine space $\mathcal{M}(d) = \bigoplus_{\alpha\in Q_1} \mathrm{M}(d_{s(\alpha)}, d_{t(\alpha)})$ may be considered as the set of representations M with $\mathbf{dim}\, M = d$ using fixed vector spaces; the group $\mathcal{G}(d) = \prod_{i\in Q_0} \mathrm{Gl}(d_i)$ operates on $\mathcal{M}(d)$ so that the orbits are just the isomorphism classes. Then

$$\varepsilon(d,d) = \dim_k \mathcal{G}(d) - \dim_k \mathcal{M}(d).$$

One particular quiver should be mentioned explicitly, the quiver with one vertex and one arrow: its path algebra is the polynomial ring $k[T]$ in one variable, thus its Hall algebra $\mathcal{H}(k[T])$ is the tensor product of classical Hall algebras (one for each maximal ideal of $k[T]$) as studied by Steinitz and Ph. Hall. In this case, the bilinear form ε is the zero form, thus in forming $\mathcal{H}^*$, the multiplication is not changed at all.

4. Other bilinear forms

The deviation of the multiplication in the Hall algebra $\mathcal{H}$ as compared to $U_q(\mathbf{n}_+)$ was considered by Lusztig in **[L1]** when he worked with Hall algebras in order to exhibit canonical bases for $U_q(\mathbf{n}_+)$. The process of changing multiplication is implizitly used by Lusztig in **[L2]**, see in particular 10.2. He stresses the cocycle condition, but does not indicate the nature of the bilinear form. In fact, the bilinear form he works with differs from the Euler characteristic ε by diagonal entries. However, bilinear forms α, β which differ only by diagonal entries lead to isomorphic rings $R_{[\alpha,c]}, R_{[\beta,c]}$, as we are going to show.

Lemma 2. *Let G be a free abelian group with basis $e_1, \dots, e_n$. Let α, β be bilinear forms on G with values in $\mathbb{Z}$ such that $\alpha(e_i,e_j) = \beta(e_i,e_j)$ for all $i \neq j$. Let R be a G-graded ring, and let $c \in R_0$ be an invertible central element. Then the map $\eta\colon R_{[\alpha,c]} \to R_{[\beta,c]}$ defined for $r \in R_g,\ g = \sum_i g_i e_i$, by*

$$\eta(r) = c^{\delta(g)} r, \quad \textit{with} \quad \delta(g) = \sum_i \binom{g_i}{2} (\beta(e_i,e_i) - \alpha(e_i,e_i)),$$

is an isomorphism of rings.

Let us stress that the restriction of η to R_{e_i} is the identity, for all i.

Proof. Clearly, η is additive, thus, let us consider $r \in R_g$, $s \in R_h$ where $g = \sum_i g_i e_i$, $h = \sum_i h_i e_i$. We have

$$\begin{aligned} \eta(r \underset{\alpha}{*} s) &= c^{\delta(g+h)} r \underset{\alpha}{*} s = c^{\delta(g+h)+\alpha(g,h)} r \cdot s, \\ \eta(r) \underset{\beta}{*} \eta(s) &= c^{\delta(g)+\delta(h)} r \underset{\beta}{*} s = c^{\delta(g)+\delta(h)+\beta(g,h)} r \cdot s. \end{aligned}$$

The equality

$$\binom{g_i+h_i}{2} - \binom{g_i}{2} - \binom{h_i}{2} = g_i h_i$$

implies that

$$\begin{aligned}\delta(g+h)-\delta(g)-\delta(h) &= \sum_i g_i h_i(\alpha(e_i,e_i)-\beta(e_i,e_i)) \\ &= \alpha(g,h)-\beta(g,h)\end{aligned}$$

since we assume that $\alpha(e_i,e_j)-\beta(e_i,e_j)=0$ for $i\neq j$.

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