Rhombal Algebras (Survey)

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Symmetric algebras. Recall that a finite-dimensional k-algebra A is said to be symmetric provided there exists a linear form $f\colon A \to k$ with the following two properties:
(1) $f(xy) = f(yx)$ for any pair $x, y \in A$.
(2) The only ideal of A which is contained in the kernel of f is the zero ideal.
Note that it follows from (1) and (2) that the kernel of f contains no non-zero left ideal and no non-zero right ideal. (Namely, if L is a left ideal of A which is contained in the kernel of f, then also the two-sided ideal LA generated by L is contained in the kernel of f, since for $x \in L$ and $a \in A$ we have $f(xa) = f(ax) = 0$.)

Let Q be a finite quiver without loops or double arrows. Let $C(Q)$ be the set of cyclic paths in Q (of non-zero length). Call a function $f\colon C(Q) \to k$ *admissable,* provided the following conditions are satisfied:
(0) There exists a number b such that $f(w) \neq 0$ implies that the length $|w|$ of w satisfies $1 \leq |w| \leq b$, and that w is a cyclic path.
(1) If w, w' are obtained from each other by rotation, then $f(w) = f(w')$.
(2) For every arrow α, there exists a cyclic path w starting with α such that $f(w) \neq 0$.

We extend the function f linearly to kQ (with f being zero on non-cyclic paths).

Let $I(f)$ be the union of all ideals I of kQ such that $f(I) = 0$. Then:
(i) $I(f)$ is an ideal and satisfies $f(I(f)) = 0$.
(ii) f induces a function on $A/I(f)$ which makes the algebra symmetric.
(iii) The quiver of $A/I(f)$ is just Q.

Proof: should be well-known.

The zero edges of a rhombal tiling.

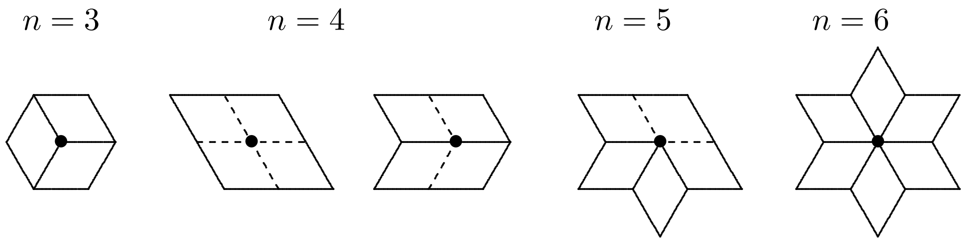
Given a graph Q, we consider it also as a quiver, by replacing any edge by a pair of arrows in opposite directions. If α is an arrow in such a quiver, we denote by α^* the arrow in opposite direction.

We start with a rhombal tiling Q of the plane (there are many such tilings, both periodic ones as well as non-periodic ones; they can be constructed by starting with the unique tiling of the plane which uses equilateral triangles of a fixed size and choosing pairs of neighboring triangles as rhombuses); we consider such a tiling as a graph (or as a quiver).

We call an edge of Q a *zero-edge* provided the two adjacent rhombuses are obtained from each other by a planar rotation of 180°. Note that for an edge with vertices x, y, there are the following two possibilities:

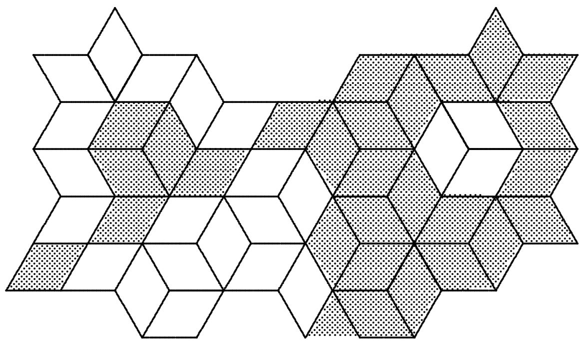
on the left hand, one sees the zero edge $x — y$, on the right hand the edge $x — y$ is not a zero edge.

The zero edges (and the vertices lying on zero edges) form a subgraph $Z(Q)$ of Q with vertices having degree 2 or 4. Namely, here are the five different possibilities for the local structure around a vertex x: There are n neighbors, with $3 \leq n \leq 6$. If $n = 3$ or $n = 6$, the vertex x does not belong to $Z(Q)$. If $n = 4$, the vertex x has degree 2 or 4 as a vertex of $Z(Q)$. If $n = 5$, it has degree 2:



The graph $Z(Q)$ provides a partition of the plane which can be coloured with (at most) two colours, say + and − (and there are precisely two such colourings: colour one rhombus arbitrarily...). We fix such a colouring and denote it by c. Thus c is a function on the set of rhombuses with values in $\{1, -1\}$.

Here is such a colouring:

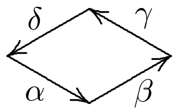


Remark. Given the triangle tiling, then the graph $Z(Q)$ and the choice of one rhombus determines uniquely the rhombal tiling: for any rhombus R and any edge $x — y$ of R, the position of the second rhombus with edge $x — y$ is uniquely determined by knowing $Z(Q)$.

Remark. The parts P of the partition determined by $Z(Q)$ are *regular* in the following sense: Given a vertex x, then all the rhombuses in P which involve x have the same angle at x. In particular, any vertex in the interior of P has valency 3 or 6. Also, the parts P are maximal with respect to this regularity condition.

Rhombal algebras (as introduced by Peach).

Let Q be a rhombal tiling, considered as a quiver. We consider cyclic paths of length 4 which involve only arrows from a single rhombus R



There are the following kinds:

Type	Examples:
Going around:	$\delta\gamma\beta\alpha, \quad \alpha^*\beta^*\gamma^*\delta^*$
Closed scissor	$\alpha^*\alpha\alpha^*\alpha$
Open scissor (acute)	$\delta\delta^*\alpha^*\alpha$
Open scissor (obtuse)	$\alpha\alpha^*\beta^*\beta$

In order to define the rhombal algebra $A(Q,c) = kQ/I(f)$, we define a function $f = f_c$ on the set $C(Q)$ of cyclic paths with values in $\{0,1,-1\}$ as follows:

(1) $f(w) = 0$ if w is not a cyclic path of length 4.
(2) $f(w) = 0$ if there is no rhombus R such that all the arrows involved in w are on the boundary of R.
(3) $f(w) = -c(R)$ if w is a path around the rhombus R.
(4) $f(w) = c(R)$ for any scissor path at the rhombus R.
(5) $f(\alpha\alpha^*\alpha\alpha^*) = 0$ if α is a zero edge, otherwise
$f(\alpha\alpha^*\alpha\alpha^*) = c(R)$ for any rhombus R with α on its boundary.

Claim 1. *The ideal $I(f)$ contains all the relations introduced by Peach,* in the form modified by Turner:

(a) Paths of length 2 can be non-zero only if there exists at least one rhombus which contains the two arrows involved in the path on its boundary.
(b) Two paths of length 2 going from a vertex of a rhombus to the opposite vertex of this rhombus add up to zero.
(c) Given a vertex x, there may be at most six paths of length 2 which go to a neighbor and return. Add such a path and the path in the opposite direction (if they exist, otherwise use zero). This gives three elements a_x, b_x, c_x of the path algebra. Take as relations: $a_x - b_x$ and $b_x - c_x$.

Proof: For example, consider a rhombus R as displayed above. It follows from

$$f(\delta\gamma(\beta\alpha + \gamma^*\delta^*)) = f(\delta\gamma\beta\alpha) + f(\delta\gamma\gamma^*\delta^*)) = -c(R) + c(R) = 0$$

and similar calculations that $\beta\alpha + \gamma^*\delta^*$ belongs to $I(f)$.

And so on.

Claim 2. The algebra $A(Q,c)$ is the rhombal algebra of Peach.

Remark. Conway-Thurston. Two rhombal tilings of a finite simply connected region can be transformed into each other by a succession of hexagon changes: the convex hexagon formed by six triangles allows precisely two divisions into three rhombuses; such a local change will be called a hexagon change.

This is easily seen after rotating our pictures by 90°: then a rhombus tiling can be interpreted as a visualization of a configuration of cubes. In this interpretation, the hexagon change correspond to adding or removing one cube.

Remark. Physics. Consider the triangle tiling, with the vertices being considered as particles with a charge $+$ or $-$. Assume the following condition is satisfied: the three vertices of any triangle do not have the same charge (thus, for any triangle, there are either two $+$ and one $-$, or else one $+$ and two $-$). Deleting the edges $x — y$ where x, y have the same charge, we obtain a rhombal tiling.