Cluster-Concealed Algebras

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Nanjing, November 2010

Cluster-tilted algebras.

Let k be an algebraically closed field.

Let B be a tilted algebra
(the endomorphism ring of a tilting A-module T,
where A is a finite-dimensional hereditary algebra)

$C = B^c = B \ltimes I$, the trivial extension with the B-B-bimodule $I = \mathrm{Ext}^2_B(DB, B)$,
with $D = \mathrm{Hom}(-, k)$ the k-duality.

B^c is called a *cluster tilted* algebras.
(Buan-Marsh-Reiten, Zhu, Assem-Brüstle-Schiffler.)

Note: B is a subalgebra as well as a factor algebra of B^c,

The C-modules N with $IN = 0$ are just the B-modules.

Dimension vectors.

R finite-dimensional k-algebra

$K_0(R)$ its Grothendieck group
of finitely generated R-modules modulo exact sequences:

$K_0(R)$ is a free abelian group with basis the simple R-modules,
using this basis, identify $K_0(R)$ with $\mathbb{Z}^n$,
n the number of (isomorphism classes of) simple R-modules.

For an R-module N, let $\mathbf{dim}\, N$ be its element in $K_0(R)$,
the coefficients of $\mathbf{dim}\, N$ are the Jordan-Hölder multiplicities of N.

The simple R-modules which are composition factors of N
form the *support* of N.

Question.

Given a finite-dimensional algebra R, one may ask:

Are the indecomposable R-modules determined by their dimension vectors?

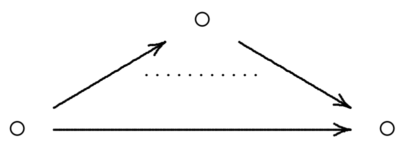
This means: If N, N' are indecomposable R-modules with $\mathbf{dim}\, N = \mathbf{dim}\, N'$, does it follow that N and N' are isomorphic?

This can be true only for algebras with finitely many isomorphism classes of indecomposable modules.

True: for R hereditary and representation-finite.

Not true: for the connected serial self-injective algebra
with 2 indecomposable projective modules, both of length 2,
both have dimension vector $(1, 1)$.

Not true for the algebra given by the following quiver with one zero relation:



Theorem 1. (Geng-Peng, R.)
Let C be a representation-finite cluster-tilted algebra.
If N, N' are indecomposable C-modules with $\mathbf{dim}\, N = \mathbf{dim}\, N'$*,*
then N and N' are isomorphic.

See Geng-Peng also for the link to cluster algebras:
Theorem 1 settles a conjecture of Fomin and Zelevinsky concerning cluster variables.

Note: If $C = B^c$ is representation-finite, and $B = \mathrm{End}({}_AT)$,
then B is a concealed algebra

Concealed means: $B = \mathrm{End}({}_AT)$,
with A finite-dimensional hereditary algebra
and T a preprojective tilting A-module.

If B is concealed, B^c is said to be **cluster-concealed.**

Representation-finite cluster-tilted algebras are cluster-concealed algebras,
but there are also many cluster-concealed algebras which are tame or wild.

Kac Theorem: *The dimension vectors of the indecomposable A-modules are the positive roots of the (Kac-Moody) root system $\Phi_A \subset K_0(A)$.*

Note that $q_A(x) \le 1$ for any $x \in \Phi_A$, with q_A Euler form on $K_0(A)$

Since $B^c = B \ltimes I$, identify $K_0(B^c)$ and $K_0(B)$,
and consider q_B the Euler form of B on $K_0(B) = K_0(B^c)$.

T multiplicity-free tilting A-module with $B = \operatorname{End}(T)$,
let $G = \operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ be the tilting functor.

Let $T_1, \dots, T_n$ be indecomposable direct summands.
Then $\mathbf{dim}\, T_1, \dots, \mathbf{dim}\, T_n$ is a basis of $K_0(A)$,
whereas $\mathbf{dim}\, G(T_1), \dots, \mathbf{dim}\, G(T_n)$ is a basis of $K_0(B)$,

$g\colon K_0(A) \to K_0(B)$ the linear bijection with $g(\mathbf{dim}\, T_i) = \mathbf{dim}\, G(T_i)$.
Let $\Phi_B = g(\Phi_A)$.

If $x \in \Phi_A$, then x or $-x$ belongs to $\mathbb{N}$,
but Φ_B may have elements with some coefficients positive, some negative.

For any element $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, let $\operatorname{abs} x = (|x_1|, \dots, |x_n|)$.

Theorem 2. *Let B be a concealed algebra and $C = B^c$ the corresponding cluster-concealed algebra,*

(a) *The dimension vectors of the indecomposable C-modules are precisely the vectors* $\operatorname{abs} x$ *with $x \in \Phi_B$.*

(b) *If Z is an indecomposable C-module, then $q_B(\mathbf{dim}\, N) \leq 1$ if and only if Z is a B-module; if N is not a B-module, then $q_B(\mathbf{dim}\, N)$ is an odd integer (greater than 2).*

(c) *If N is an indecomposable C-module which is not a B-module, then* $\operatorname{End}(N) = k$.

Remarks.

1. The quadratic form q_B depends on the choice of B (and not only on B^c).

2. Invariants such as quadratic forms or root systems have often been used for classifying indecomposable modules.

Usually, one guesses all these objects,
— then one shows that they are pairwise non-isomorphic
— and that all the indecomposable modules have been obtained;
— finally, one tries to describe the structure of the module category.

In our case the procedure is completely reversed:
the module category is known from the beginning, but one is lacking sufficient information concerning the modules themselves.

The missing A-modules. Recall the principles of tilting theory:

$$\begin{aligned}
\mathcal{F} = \mathcal{F}(T) &= \{M \in \operatorname{ind} A \mid \operatorname{Hom}(T,M) = 0\},\\
\mathcal{G} = \mathcal{G}(T) &= \{M \in \operatorname{ind} A \mid \operatorname{Ext}^1(T,M) = 0\},\\
\mathcal{X} = \mathcal{X}(T) &= \{M \in \operatorname{ind} B \mid T \otimes_B M = 0\},\\
\mathcal{Y} = \mathcal{Y}(T) &= \{M \in \operatorname{ind} B \mid \operatorname{Tor}_1(T,M) = 0\}.
\end{aligned}$$

The pair $(\mathcal{F},\mathcal{G})$ is a torsion pair in $\operatorname{mod} A$.
The pair $(\mathcal{Y},\mathcal{X})$ is a torsion pair in $\operatorname{mod} B$ which is even split.

The functor $G = \operatorname{Hom}(T,-)$ gives an equivalence $\mathcal{G}(T) \to \mathcal{Y}(T)$
the functor $F = \operatorname{Ext}^1(T,-)$ gives an equivalence $\mathcal{F}(T) \to \mathcal{X}(T)$.

We are concerned with

$$\mathcal{M} = \mathcal{M}(T) = \{M \in \operatorname{ind} A \mid \operatorname{Hom}(T,M) \neq 0,\ \operatorname{Ext}^1(T,M) \neq 0\},$$

as well as with

$$\mathcal{N} = \mathcal{N}(B) = \{N \in \operatorname{ind} B^c \mid IN \neq 0\},$$

$\mathcal{M}$ are the A-modules which are not send to B-modules,
$\mathcal{N}$ are the B^c-modules which are not B-modules.

The mixed modules of a torion pair.

In both cases, we deal with the mixed modules of torsion pairs:

$\mathcal{M}$ are the mixed modules for the torion pair $(\mathcal{F}, \mathcal{G})$ in $\operatorname{mod} A$,
$\mathcal{N}$ are the mixed modules for the torion pair $(\mathcal{Y}, \mathcal{X})$ in $\operatorname{mod} C$.

There is a general procedure for recovering the mixed modules of a torsion pair.

Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in the abelian category $\mathcal{A}$.
For $A \in \mathcal{A}$, let tA be its torsion subobject.

Proposition (Kiev school, $\sim$ 1972). *There is a functor*

$$\eta \colon \mathcal{A} \to \operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$$

which is full, dense and with kernel the ideal generated by all maps $\mathcal{F} \to \mathcal{G}$,

namely for $A \in \mathcal{A}$, *let* $\eta(A) = (A/tA, tA, \epsilon_A)$,
with ϵ_A *the equivalence class of the exact sequence* $0 \to tA \to A \to A/tA \to 0$.

What is $\operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$?

The matrix category of a bimodule.

$\mathcal{A}$, $\mathcal{B}$ additive categories
$E = {}_{\mathcal{A}}E_{\mathcal{B}}$ an $\mathcal{A}$-$\mathcal{B}$-bimodule (= a bilinear functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \operatorname{mod} k$)

$\operatorname{Mat} E$ is the category of E-*matrices* (introduced by Drozd 1972):

Objects are the triples (A, B, m),
with A object of $\mathcal{A}$, B object of $\mathcal{B}$ and $m \in E(A, B)$.

Morphisms $(A, B, m) \to (A', B', m')$ are pairs (α, β) of a morphism $\alpha\colon A \to A'$ in $\mathcal{A}$ and a morphism $\beta\colon B \to B'$ in $\mathcal{B}$, such that $m\beta = \alpha m'$.

For a bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$, introduce a quadratic form r_E on the direct sum of $K(\mathcal{A}, \oplus)$ and $K(\mathcal{B}, \oplus)$,
X object in $\mathcal{A}$, and Y object in $\mathcal{B}$, let

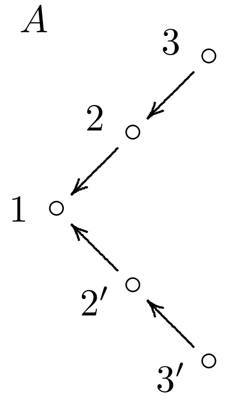
$$r_E((X, Y)) = \dim \operatorname{End}_{\mathcal{A}}(X) + \dim \operatorname{End}_{\mathcal{B}}(Y) - \dim E(X, Y),$$

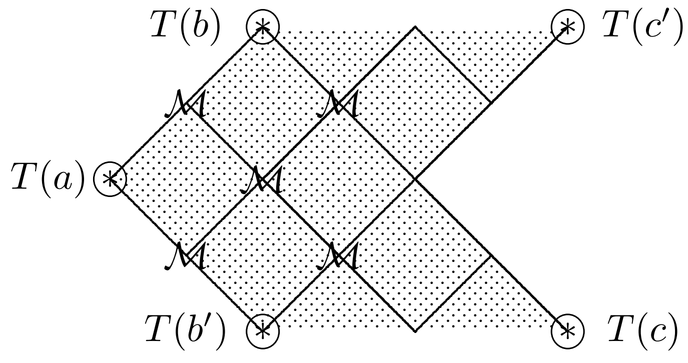
and extend this to a quadratic form on $K(\mathcal{A}, \oplus) \oplus K(\mathcal{B}, \oplus)$.

Recall the usual graphical presentation of such quadratic forms,
using two kinds of edges, solid ones and dotted ones.

Example. A path algebra of a quiver Q of type $\mathbb{A}_5$,
with its Auslander-Reiten quiver, and a tilting module with marks $*$

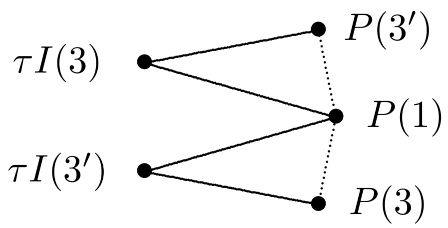
$\mathcal{G}$ = the modules $T(a), T(b), T(b')$ and the indecomposable injective modules,
$\mathcal{F}$ = the two modules $\tau I(3)$ and $\tau I(3')$.





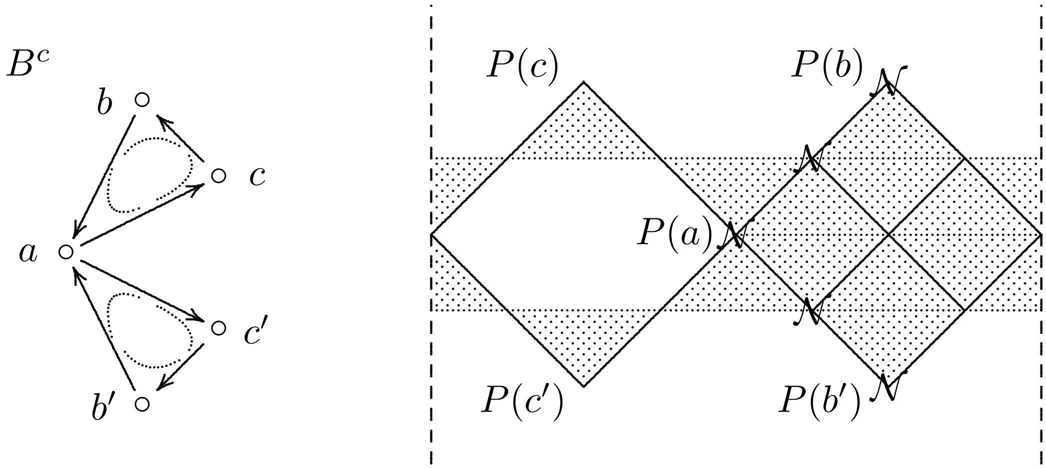
$\mathcal{M}$ are the mixed modules.

Recall that $\operatorname{mod} A$ is described by $\operatorname{Mat} E$ with $E = \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$.
Here is the essential part of r_E:



(we have deleted the isolated vertices).

Here is the algebra B^c, with its Auslander-Reiten quiver and the positions of the five mixed modules in $\mathcal{N}$.



Here we deal with the category $\operatorname{Mat} E'$ with $E' = \operatorname{Ext}^1(\mathcal{Y}, \mathcal{X})$.

The bijection.

Proposition. *Let T be a preprojective tilting module. There is a bijection*

$$\iota\colon\ \operatorname{ind} A \to \operatorname{ind} B^c,$$

such that for $M \in \operatorname{ind} A$, the restriction of $\iota(M)$ to B is $G(M) \oplus F(M)$.

(Recall: $G = \operatorname{Hom}(T, -)$, and $F = \operatorname{Ext}^1(T, -)$)

Remark. For any A-module M, we have

$$G(M) = G(tM) \quad \text{and} \quad F(M) = F(M/tM).$$

Thus we could write $\iota(M) = G(tM) \oplus F(M/tM)$.

This shows:
we deal with the middle terms of the exact sequences

$$0 \to tM \to M \to M/tM \to 0,$$
$$0 \to F(M/tM) \to \iota(M) \to G(tM) \to 0.$$

The relation between the bimodules E and E'.

Recall that $\mathcal{F} \simeq \mathcal{X}$ and $\mathcal{G} \simeq \mathcal{Y}$.

We try to compare the bimodule $E' = \operatorname{Ext}^1(\mathcal{Y}, \mathcal{X})$ (or better, its dual) with an $\mathcal{F}$-$\mathcal{G}$-bimodule.

Using tilting theory as well as the basic Auslander-Reiten formula

$$\operatorname{Ext}^1(F, G) \simeq D \operatorname{Hom}(G, \tau F)$$

(for modules over a hereditary algebra), it turns out:

E and E' are dual bimodules.

Theorem (de la Peña - Simson, 1992). *Let* E *be a bimodule.*
Assume that the category $\operatorname{Mat} E$ *is directed.*
Then r_E *is weakly positive,*
and the indecomposable objects in $\operatorname{Mat} E$ *correspond bijectively to the positive roots of* r_E (via coordinate vectors).

Application:

Recall that the bimodules E, E' are dual to each other,
thus r_E and $r_{E'}$ have the same root system.

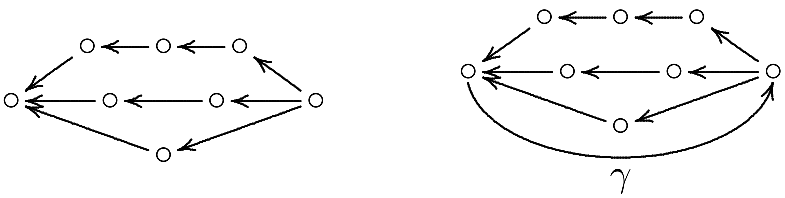
$\operatorname{Mat} E$ is (essentially) a factor category of the preprojective A-modules,
thus directed.

Similarly, $\operatorname{Mat} E'$ is (essentially) a factor category of a connecting component of B_∞-modules, thus also directed.
Recall: $B^c = B \ltimes I$, with $I = \operatorname{Ext}^2_B(DB, B)$,
and this has the Galois covering

$$B_\infty = \begin{bmatrix} \ddots & \ddots & & & \\ & B & I & & \\ & & B & I & \\ & & & B & \ddots \\ & & & & \ddots \end{bmatrix}$$

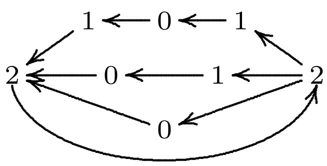
Separation Property.
If T is a preprojective tilting module, and M is indecomposable, then the supports of $F(M)$ and $G(M)$ are disjoint.

Example. Consider the canonical algebra B of type $\mathbb{E}_7$ and the corresponding cluster-tilted algebra B^c:

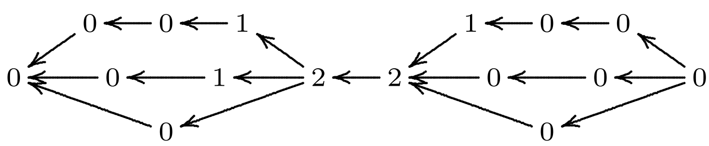


Here is a typical indecomposable B^c-module Z
as well as the corresponding B_∞-module $\widetilde{Z}$

$\mathbf{dim}\, Z$



$\mathbf{dim}\, \widetilde{Z}$



This explains why we have to use the absolute value $\operatorname{abs} x$.
The left part of $\widetilde{Z}$ belongs to $\mathcal{X}$, the right to $\mathcal{Y}$.

Recall the separation property:

Let T be a preprojective tilting module, and M an indecomposable A-module. Then the B-modules $G(M) = \operatorname{Hom}_A(T, M)$ and $F(M) = \operatorname{Ext}^1(T, M)$ have disjoint supports.

This is the reason for the appearance of absolute values in Theorem 2.

If T has an indecomposable regular direct summand,
the separation property no longer holds.

Thus, one cannot expect that a generalization of the main theorem
for arbitrary cluster-tilted algebras
will use the vectors $\operatorname{abs} x$ with $x \in \Phi_B$.