Invariant Subspaces
of Nilpotent Linear Operators

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I. The problem: Invariant subspaces of a nilpotent operator T.

Let k be a field.

Let T be a nilpotent linear operator on some vector space V.
Example 1. Take as T the $(n \times n)$ Jordan block (with eigenvalue 0)

$$J(n) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix},$$

it operates on $V = k^n$.
What are the invariant subspaces U of V ?
(invariant $=$ T-invariant, i.e. $T(U) \subseteq U$.)

The only possibilities for U are the subspaces

$$0, \qquad k \times 0^{n-1}, \qquad k^2 \times 0^{n-2}, \quad \dots \quad , \quad k^n,$$

they form a chain.

In general, T will not be a single Jordan block, but a diagonal sum of Jordan blocks.

Example 2. The first interesting case is $T = J(3) \oplus J(1)$, this is the matrix

$$J(3) \oplus J(1) = \left[\begin{array}{ccc:c} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ \hdashline & & & 0 \end{array}\right]$$

It operates on $V = k^4$. Let $e_1, \dots, e_4$ be its canonical basis.

Let $U = \langle e_2 + e_4,\ e_3 \rangle$.
Obviously, $T(U) = \langle e_3 \rangle \subset U$, thus U is an invariant subspace.

Claim: The triple (V, T, U) is “indecomposable”: If $V = V' \oplus V''$ with

$$T(V') \subseteq V', \quad T(V'') \subseteq V'', \quad \text{and} \quad U = (U \cap V') \oplus (U \cap V''),$$

then $V' = 0$ or $V'' = 0$.

Nilpotent operators

A *nilpotent operator* is a pair (V, T), where V is a vector space and $T\colon V \to V$ is a linear map with $T^n =$ for some n.

Typical example: The pair $M(n) = (k^n, J(n))$, where $J(n)$ is the Jordan block $J(n)$.

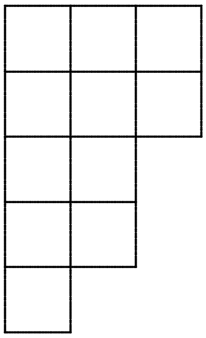
If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$ is a decreasing sequence of natural numbers (a "partition"), let

$$M(\lambda) = M(\lambda_1) \oplus \cdots \oplus M(\lambda_t).$$

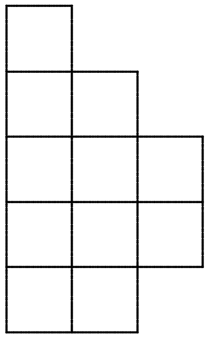
Any nilpotent operator is isomorphic to $M(\lambda)$ for a **unique** *partition λ.*

Visualization: Dealing with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$, we draw the corresponding *Young diagram.*
Our convention: the parts correspond to the **columns**,
the i-the column consists of λ_i boxes.

Example: $\lambda = (5, 4, 2)$

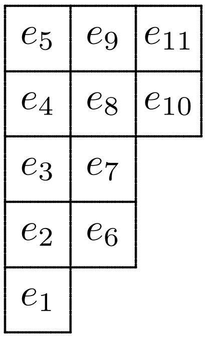


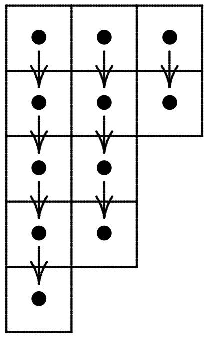
or, say



Consider the boxes as base vectors for k^n with T the shift downwards.

This yields
$J(5) \oplus J(4) \oplus J(2)$





Let $\mathcal{S}(n)$ be the class of *triples* (V, T, U), where

V is a finite-dimensional k-space V,
T a linear operator $T\colon V \to V$ with $T^n = 0$,
U a subspace of V with $T(U) \subseteq U$, write $W = V/U$ (if needed)

An *isomorphism* between (V, T, U) and (V', T', U') is an invertible linear map $f\colon V \to V'$ with $f(U) = U'$ and $fT = T'f$.

Aim. To classify the isomorphism classes of such triples.

The *direct sum* of $X = (V, T, U)$ and $X' = (V', T', U')$ is the triple

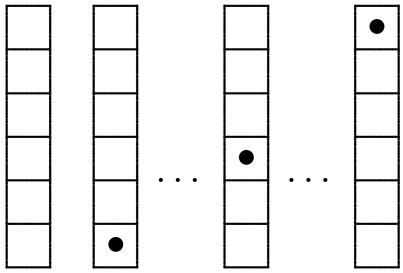
$$X \oplus X' = (V \oplus V', T \oplus T', U \oplus U').$$

(V, T, U) is said to be *indecomposable* provided it is not zero and not isomorphic to a direct sum $X \oplus X'$ with non-zero triples X, X' (the *zero triple* is $(0, 0, 0)$).

Any triple is a direct sum of indecomposable triples and these direct summands are unique up to isomorphism (Krull-Remak-Schmidt property).

Aim. To classify the isomorphism classes of indecomposable triples.

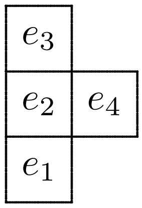
Again example 1 (“pickets”):
The triples (V,T,U) with $(V,T) = (k^n, J(n))$.
As we have mentioned,
the only invariant subspaces U
are the subspaces
$0, \qquad k \times 0^{n-1}, \qquad k^2 \times 0^{n-2}, \quad \dots \quad , \quad k^n.$



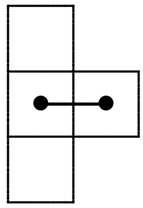
A bullet indicates a generator of (U,T).

Easy: If (V,T,U) is indecomposable and $\dim U = 1$, then (V,T) is a picket.

Again
Example 2:



with (U,T) generated
by $e_2 + e_4$



Note: Both U, W are of dimension 2 and indecomposable (with respect to T)
It follows: (V,T,U) is an indecomposable triple.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing n.

In $S(n)$ there are two special triples: $(k^n, J(n), 0)$ and $(k^n, J(n), k^n)$.

n	number of indecomposables	
1	2	$= 2 + 0$
2	5	$= 2 + 3$
3	10	$= 2 + 8$
4	20	$= 2 + 18$
5	50	$= 2 + 48$
6	?	

Richman-Walker (1999): If (V, T, U) in $\mathcal{S}(5)$ is indecomposable,
then $\dim V \leq 12$, $\dim U \leq 6$, and $\dim \operatorname{Ker} T \leq 3$,
and these bounds are optimal.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing n.

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n	number of indecomposables	
1	2	$= 2 + 0$
2	5	$= 2 + 3$
3	10	$= 2 + 8$
4	20	$= 2 + 18$
5	50	$= 2 + 48$
6	?	

The **Online Encyclopedia of Integer Sequences** provides for 2, 5, 10, 20, 50 two entries:

A051109 Hyperinflation sequence of banknotes, next numbers: 100, 200

A124146 USA currency denominations in dollars, next numbers: 100, 500

For 0, 3, 8, 18, 48, . . . , there is no entry.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing n.

In $S(n)$ there are two special triples: $(k^n, J(n), 0)$ and $(k^n, J(n), k^n)$.

n	number of indecomposables		
1	2	$= 2 + 0$	$= 2 + 0$
2	5	$= 2 + 3$	$= 2 + \frac{3}{2} \cdot 2$
3	10	$= 2 + 8$	$= 2 + 2 \cdot 4$
4	20	$= 2 + 18$	$= 2 + 3 \cdot 6$
5	50	$= 2 + 48$	$= 2 + 6 \cdot 8$
6	∞		$= 2 + \frac{6}{0} \cdot 10$

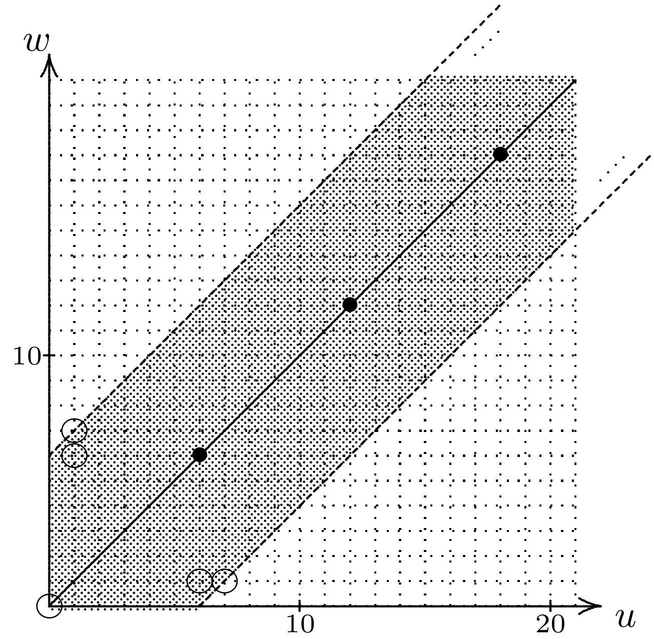
$$2 + \frac{6}{6-n} 2(n-1)$$

We are going to study the case $n = 6$.

The *dimension pair* of a triple (V, T, U) is the pair $(\dim U, \dim W)$ (where $W = V/U$).

The case $n = 6$.

Theorem 1. *A pair (u, w) of natural numbers is the dimension pair of an indecomposable triple in $\mathcal{S}(6)$ if and only if (u, w) satisfies $|u - w| \leq 6$ and is different from* $(0, 0)$, $(1, 6)$, $(1, 7)$, $(6, 1)$, $(7, 1)$.



We may reformulate part of Theorem 1 as follows:

Assume (V,T,U) *is in* $\mathcal{S}(6)$ *and* $|\dim U - \frac{1}{2}\dim V| > 3$. *Then there are non-zero subspaces* V', V'' *with* $V = V' \oplus V''$ *such that*

$$T(V') \subset V', \quad T(V'') \subset V'' \quad \text{and} \quad U = (U \cap V') \oplus U \cap V'').$$

For (V,T,U) indecomposable in $\mathcal{S}(6)$, the dimension of U is roughly half of the dimension of V.

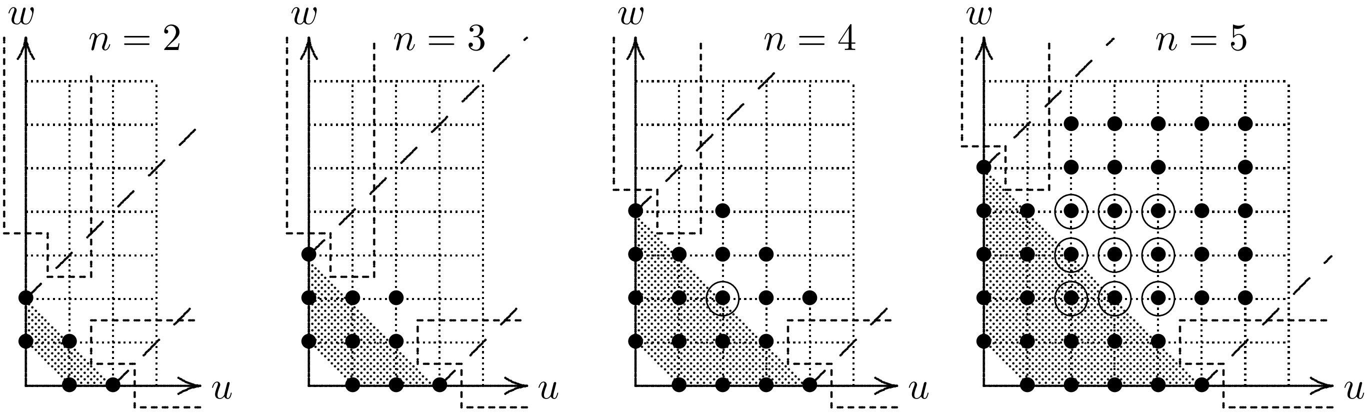
Why do we have to exclude the pairs $(1,6)$, $(1,7)$, $(6,1)$, $(7,1)$?

We have already noted that for (V,T,U) indecomposable and $\dim U = 1$, then (V,T) is indecomposable. This shows that $(1,6)$ and $(1,7)$ do not occur.

The pairs $(6,1)$ and $(7,1)$ are excluded, using duality: If (V,T,U) is in $\mathcal{S}(n)$, then $(V^*, T^*, (V/U)^*)$ is also a triple in $\mathcal{S}(n)$.

For $n \leq 6$ and $(V, T, U) \in \mathcal{S}(n)$, we even have $|u - w| \leq n$,
and this bound is always optimal.

Here are the cases $n = 2, 3, 4, 5$ (always, the picket region is shaded).



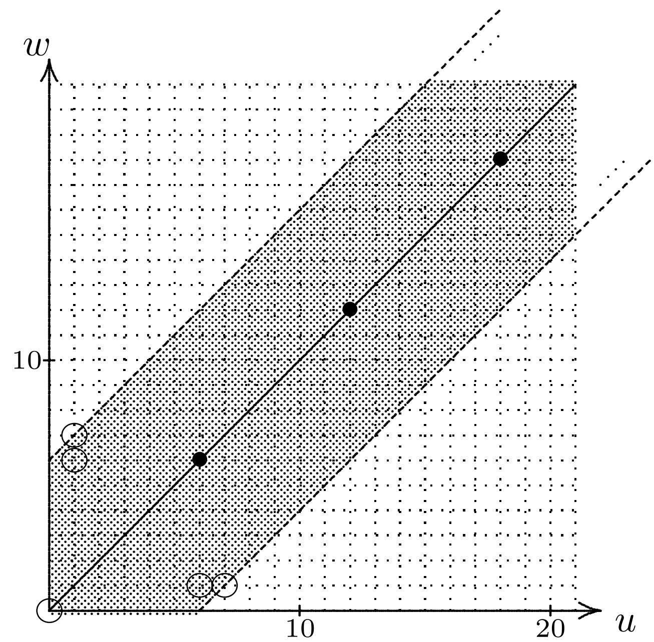
Encircled bullets: there are (precisely) two indecomposable triples.

Note: For $n \leq 5$, the pair (n, n) does not occur as dimension pair!

For $n \geq 7$, the numbers $|u - w|$ are not bounded,
but the possible dimension pairs are not yet known.

We return to $n = 6$.

Theorem 2. *If (u, w) is not in $\mathbb{N}(6,6)$, then the number of isomorphism classes of indecomposable objects in $\mathcal{S}(6)$ with dimension pair (u, w) is finite (and independent of k).*

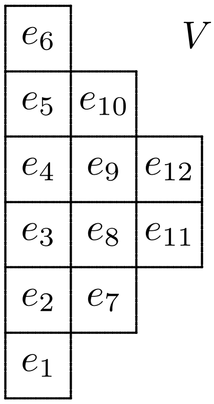


By contrast, the triples with dimension pair in $\mathbb{N}(6,6)$ depend on the field k.

Let $(u, w) = t(6,6)$ with $t \in \mathbb{N}_1$.

There are t disjoint one-parameter families of indecomposable triples with dimension pair (u, w).

$t = 1$



$V = M(6,4,2)$

generators of U: $e_8 + e_{11}$

$e_4 + (1-c)e_9 - ce_{12}$

for any $c \in k$

$t = 2$

$V = M(6,6,4,4,2,2)$

or

$V = M(6,6,5,3,3,1)$

Weakly homogeneous triples. An indecomposable triple (V, T, U) in $\mathcal{S}(n)$ is called *weakly homogeneous* provided $(V/U, T)$ is isomorphic to (U, T) and (V, T) is isomorphic to $(U, T) \oplus M(n)^t$ for some t. Then

$$\dim U + tn = \dim V = \dim U + \dim V/U = 2\dim U,$$

thus $\dim(V, T, U) = t(n, n)$.

For $n \leq 5$, there are no weakly homogeneous triples. Return to $n = 6$.

Theorem 3. *For any $t \in \mathbb{N}_1$, there are t pairwise disjoint one-parameter families of weakly homogeneous triples in $\mathcal{S}(6)$ with dimension pair $t(6,6)$, each being indexed by $k \setminus \{0, 1\}$.*
If k is algebraically closed, then there are only finitely many additional isomorphism classes indecomposable triples in $\mathcal{S}(6)$ with dimension pair $t(6,6)$ (and these triples are defined independently of k).

If (V, T, U) is weakly homogeneous, then $U = M(4,2)^r \oplus M(5,3,3,1)^s$ for some r, s (and $t = r + 2s$).

For $r > 0, s > 0$, there are **two** one-parameter families of weakly homogeneous triples with $U = M(4,2)^r \oplus M(5,3,3,1)^s$. Later we will see how to distinguish these two families.

Recall: (V,T,U) indecomposable in $\mathcal{S}(6)$. Then $|\dim U - \frac{1}{2}\dim V| \le 3$.
This means: The dimension of U is **roughly** half of the dimension of V.

If (V,T,U) is weakly homogeneous, then we even have: $\dim U = \frac{1}{2}\dim V$.

The structure theorem for weakly homogeneous triples asserts:
$V = M(4,2)^r \oplus M(5,3,3,1)^s \oplus M(6)^{r+2s}$. This implies:

$$\begin{aligned} \dim \operatorname{Ker} T &= \tfrac{1}{4}\dim V \\ \dim \operatorname{Ker} T^3 &= \tfrac{2}{3}\dim V \\ \dim \operatorname{Ker} T^5 &= \tfrac{11}{12}\dim V \end{aligned}$$

$$\begin{aligned} \tfrac{11}{24}\dim V \le \dim \operatorname{Ker} T^2 \le \tfrac{1}{2}\dim V \\ \tfrac{19}{24}\dim V \le \dim \operatorname{Ker} T^4 \le \tfrac{5}{6}\dim V \end{aligned}$$

$$\dim \operatorname{Ker} T^4 / \operatorname{Ker} T^2 = \tfrac{1}{3}\dim V$$

Again, for indecomposable triples which are not weakly homogeneous, these (in)equalities are **roughly** true: they hold up to small differences ...

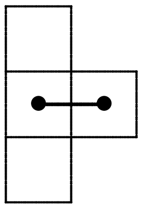
Graded triples. A grading of the triple (V, T, U) is a direct decomposition $V = \bigoplus_{i \in \mathbb{Z}} V_i$ such that $T(V_i) \subseteq V_{i-1}$ and $U = \bigoplus (U \cap V_i)$.

Theorem 4. *For $n = 6$, any triple in $\mathcal{S}(6)$ can be graded. For indecomposable triples, the grading is unique up to shift.*

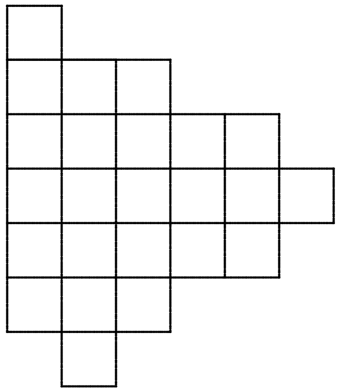
Interpretation. Write $(V, T) = M(\lambda)$ where λ is a partition.
Visualize $M(\lambda)$ using the Young diagram of λ, present the parts as columns.
A grading of (V, T, U) means **to adjust the columns** conveniently.

Examples:

The indecomposable triple with dim pair $(2, 2)$



The columns of a weakly homogeneous triple with $U = M(5, 3, 3, 1)$ are adjusted as follows:



The grading theorem is the essential result!
It provides a lot of new invariants: We can refine

$$u = \sum u_i, \quad w = \sum w_i,$$

where

$$u_i = \dim U \cap V_i, \quad w_i = \dim V_i/(U \cap V_i).$$

A graded triple is a system of vector spaces and linear maps as follows:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{T} & U_0 & \xleftarrow{T} & U_1 & \xleftarrow{T} & U_2 & \xleftarrow{T} & \cdots \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
\cdots & \xleftarrow[T]{} & V_0 & \xleftarrow[T]{} & V_1 & \xleftarrow[T]{} & V_2 & \xleftarrow[T]{} & \cdots
\end{array}$$

The squares are commutative (and $T^6 = 0$).

A remark concerning the proof of all the results presented here:

- One first deals with graded triples: Classification of the indecomposables, determination of the global structure of the category.
- The structure of the category of graded triples implies that any triple can be graded: Theorem 4.
- This then leads to the remaining assertions.

The classification of indecomposable triples follows a well-known procedure in the representation theory of finite dimensional algebras:

- "Covering".
- "Knitting".
- "Tilting".

However all these techniques had been available only for module categories itself — here we deal with a proper subcategory of a module category, thus the techniques had to be modified. (Note that any module category is abelian, the category $\mathcal{S}(n)$ is not abelian!)

Up to now, $\mathcal{S}(n)$ was considered only as a class of objects, not as a category. The appropriate notion of *maps* $(V,T,U) \to (V',T',U')$ is the following: take the linear maps $f\colon V \to V'$ with $fT = T'f$ and $f(U) \subseteq U'$.

The Auslander-Reiten quiver of categories such as $\mathcal{S}(n)$ or $\mathcal{S}(\widetilde{n})$.

Let $\mathcal{C}$ a category such as $\mathcal{S}(n)$ or $\mathcal{S}(\widetilde{n})$.
The Auslander-Reiten quiver $\Gamma(\mathcal{C})$ describes
the factor category of $\mathcal{C}$ modulo its infinite radical, as follows:

- The **vertices** are the isomorphism classes $[X]$ of the indecomposable objects in $\mathcal{C}$.
- The number of **arrows** $[X] \to [X']$ is $\dim \operatorname{Hom}(X, X')/\operatorname{rad}(X, X')$ Such arrows represent "irreducible maps".
- In addition, there is a canonical bijection (the Auslander-Reiten translation) between the non-projective objects and the non-injective objects.

The translation is used to define relations (the "mesh relations") on $\Gamma(\mathcal{C})$.

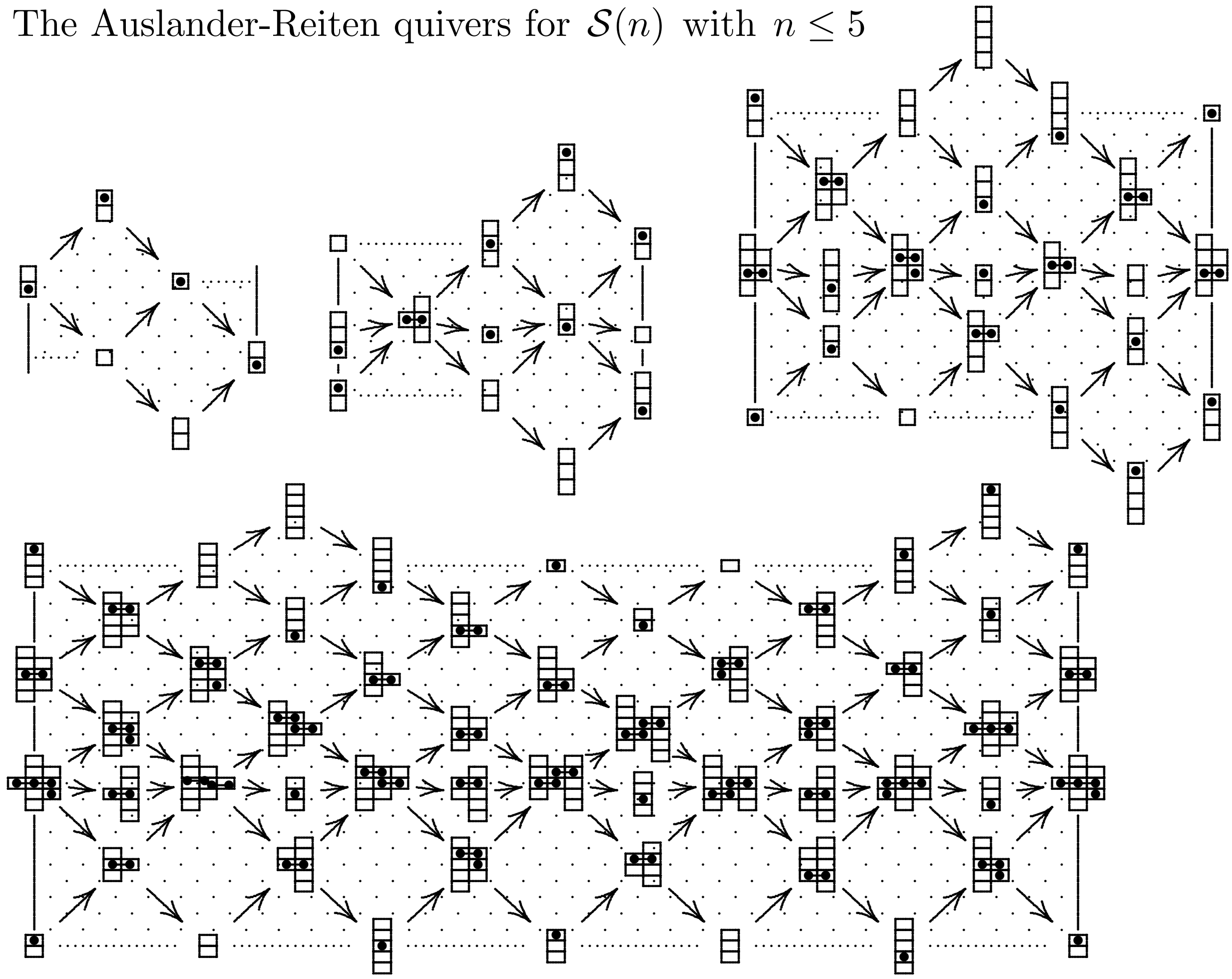
In general, the Auslander-Reiten translation is not defined for all objects, some objects may be "projective" or "injective".

In the categories $\mathcal{S}(n)$, there are jut two triples of this kind, the triples

$$(k^n, J(n), 0), \quad \text{and} \quad (k^n, J(n), k^n),$$

both are "projective" and "injective" as well.

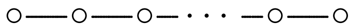
The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$



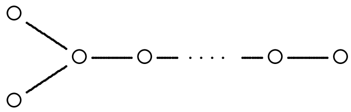
n	number of indecomposables			hidden Lie type
1	2	$= 2 + 0$	$= 2 + 0$	$\emptyset$
2	5	$= 2 + 3$	$= 2 + \frac{3}{2} \cdot 2$	A_2
3	10	$= 2 + 8$	$= 2 + \frac{6}{3} \cdot 4$	D_4
4	20	$= 2 + 18$	$= 2 + \frac{6}{2} \cdot 6$	E_6
5	50	$= 2 + 48$	$= 2 + 6 \cdot 8$	E_8
6	∞			

↑ a tree (pointing to the last factor in the fourth column)

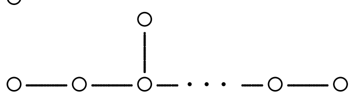
Here is the list of the simply laced Dynkin diagrams considered in Lie theory:



A_n

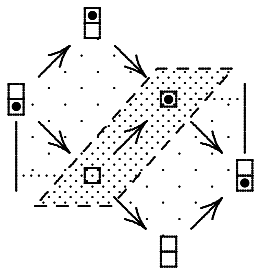


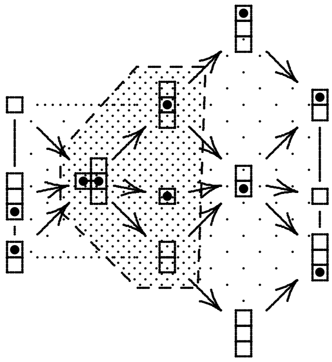
D_n

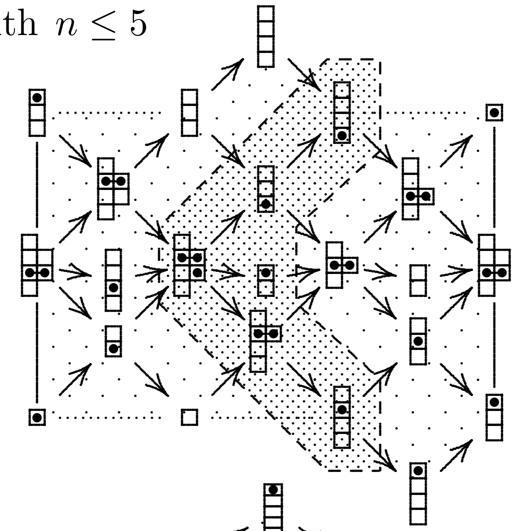


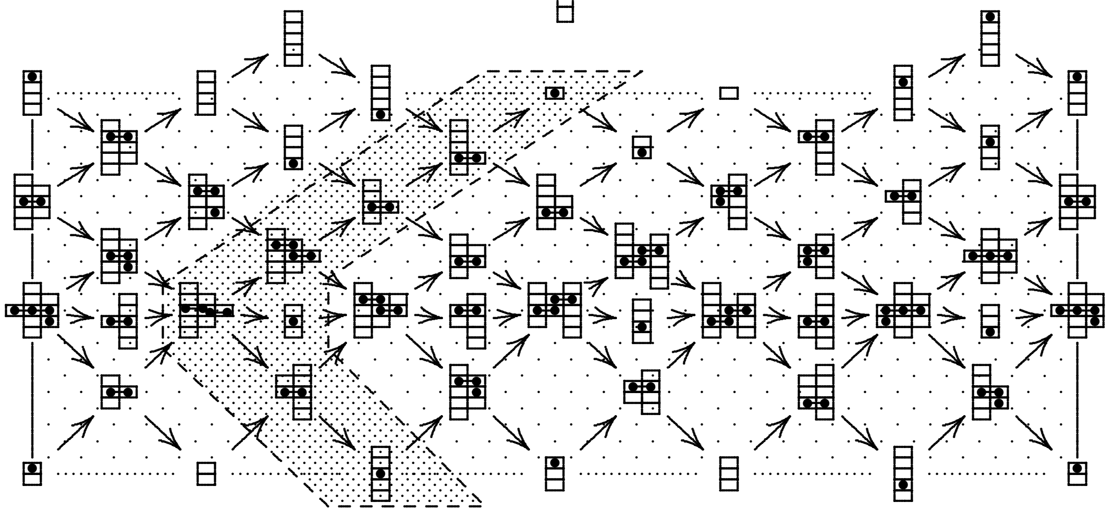
$E_m \quad (m = 6, 7, 8)$

The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$









A more general setting. For Λ a ring, let $\mathcal{S}(\Lambda)$ denote the class of pairs (M, U), with M a finitely generated Λ-module and U a submodule of M.

We consider the case where Λ is a uniserial commutative ring of length n,
such as $\Lambda = k[T]/T^n$ (here, $k[T]$ is the polynomial ring in one variable T)
or $\Lambda = \mathbb{Z}/p^n$ (with p a prime number).

Note that $\mathcal{S}(n) = \mathcal{S}(k[T]/T^n)$.

The problem to study $\mathcal{S}(\mathbb{Z}/p^n)$ was raised by Garrett Birkhoff in 1934:
To determine *relativ invariants of subgroups;*
i.e. invariants under automorphisms of the given group.

For any uniserial commutative ring Λ of length n, the isomorphism classes of the Λ-modules correspond bijectively to the partitions with parts of size at most n, thus we can use the same box diagrams as in the special case of nilpotent operators.

Theorem. *For $n \leq 5$, the Auslander-Reiten quiver of $\mathcal{S}(\Lambda)$ depends only on n, not on Λ.*

n	number of indecomposables		
1	2	$= 2 + 0$	
2	5	$= 2 + 3$	
3	10	$= 2 + 8$	1984 Hunter-Richman-Walker
4	20	$= 2 + 18$	1999 Richman-Walker
5	50	$= 2 + 48$	1999 Richman-Walker
6	∞		1934 (Birkhoff)

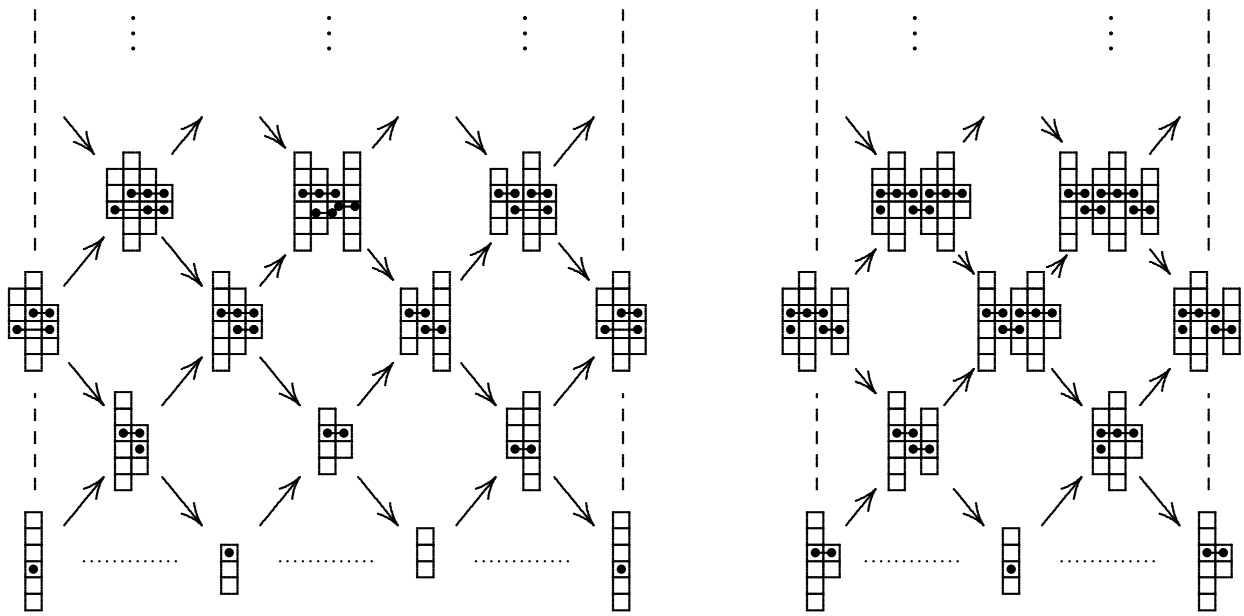
Birkhoff has shown that the indecomposable objects (M, U) in $\mathcal{S}(\mathbb{Z}/p^6)$ with M of partition type $(6, 4, 2)$ depend on the prime p (and the number of isomorphism classes tends to infinity, if $p \to \infty$)

But a complete classification of the indecomposable objects in $\mathcal{S}(\mathbb{Z}/p^6)$ is still unknown (in contrast to $\mathcal{S}(k[T]/T^6) = \mathcal{S}(6)$).

We return to consider $\mathcal{S}(6)$. Here we deal mostly with **stable tubes.**

What is a tube?

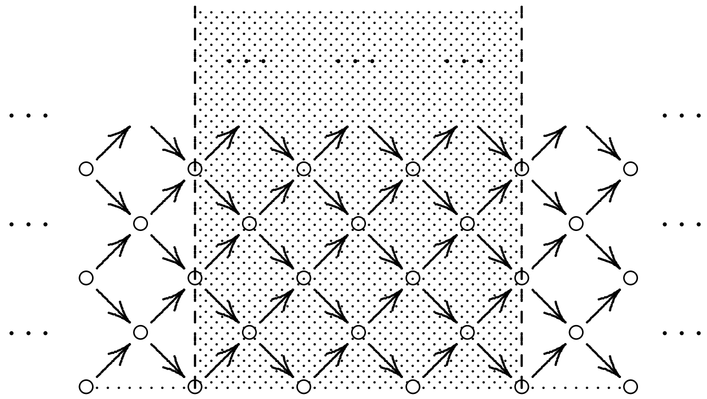
Two typical tubes in $\mathcal{S}(6)$:



These are the exceptional tubes of rank 3 and 2 in $\mathcal{T}_0$.

Stable tubes of rank r.

here: $r = 3$



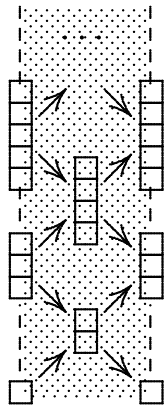
Vertices: the pairs $(x, y) \in \mathbb{Z}^2$ with $y \geq 0$ and $x \equiv y \bmod 2$

Arrows: $(x, y) \to (x+1, y+1)$ and $(x, y) \to (x+1, y-1)$

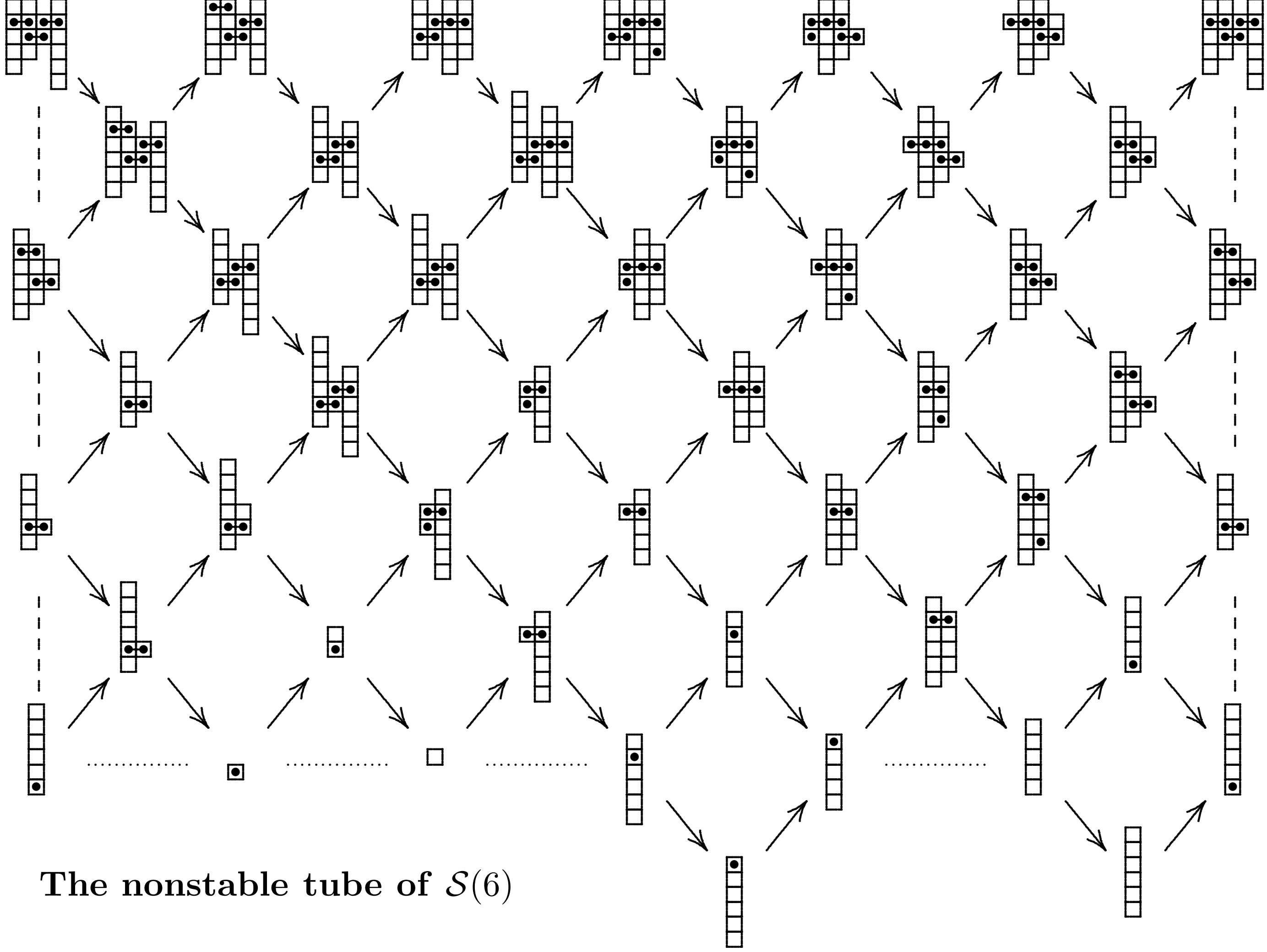
Translation $\tau(x, y) = (x-2, y)$

Identify (x, y) with $(x+2r, y)$

A *homogeneous tube* is a tube of rank 1



The Auslander-Reiten quiver of the category of nilpotent operators is a typical homogeneous tube!

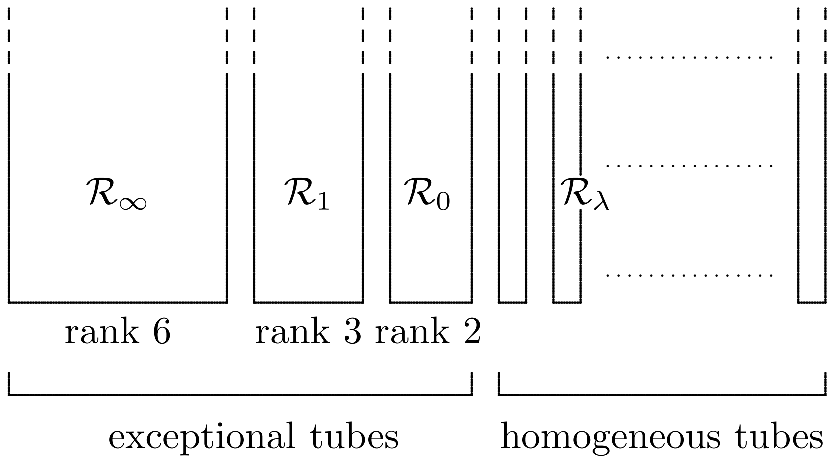


The nonstable tube of $\mathcal{S}(6)$

The structure of $\mathcal{S}(n)$ for $n = 6$:

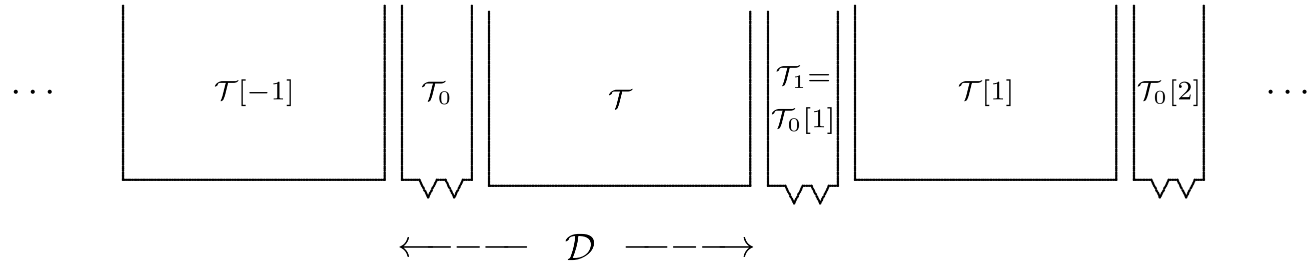
There are countable many stable tubular families in $\mathcal{T}$, all of type $(6, 3, 2)$.

A tubular family of type $(6, 3, 2)$ has the following form:



The index set for the tubular families we are interested in, will always be $\mathbb{P}_1(k)$.

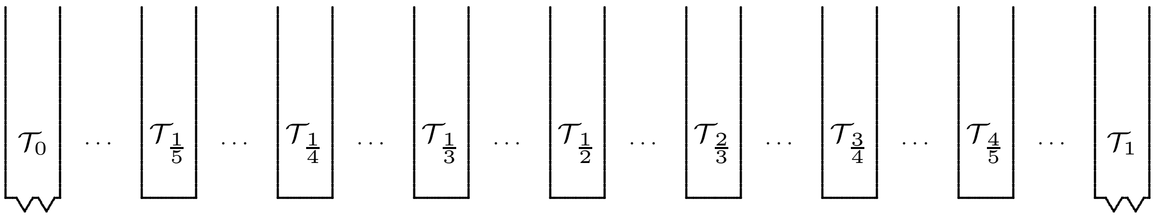
Here is the category $\mathcal{S}(\widetilde{6})$, and $\mathcal{D}$ is a fundamental domain for the shift σ.



Any object in $\mathcal{S}(\widetilde{6})$ has a "slope" $\gamma \in \mathbb{Q}$, the shift σ increases the slope by 1.

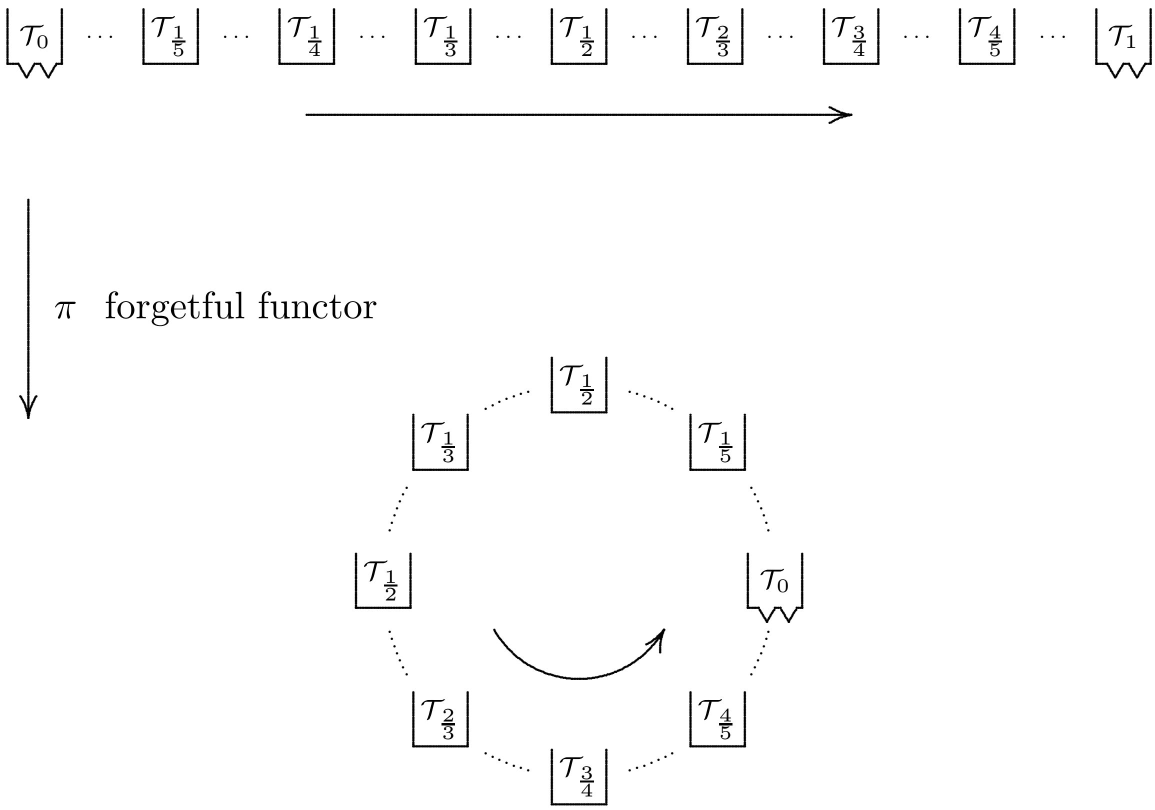
The objects with slope γ form the subcategory $\mathcal{T}_\gamma$.

The part containing the objects with slope in $\mathbb{Q}^+ \cap [0,1]$ looks as follows:



The graded triples with a fixed slope form a tubular family of type $(6,3,2)$.

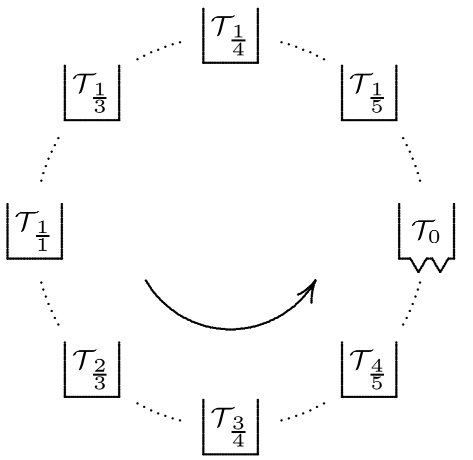
Forgetting the grading



To repeat: The classification of the indecomposable triples in $\mathcal{S}(6)$ (in case k is algebraically closed):

There are two projective-injective triples, with dimension pair $(0,6)$ and $(6,0)$.

For the remaining triples, one needs three invariants .



First invariant: The slope,
a rational number $0 \le \gamma < 1$

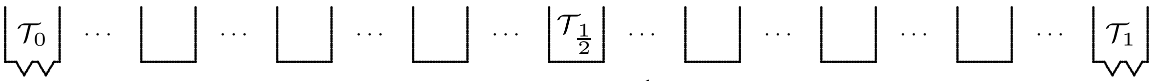
Second invariant: The spectral parameter c,
an element of $\mathbb{P}_1(k) = k \cup \{\infty\}$

Third invariant: A vertex x in a tube.
If $c \notin \{0, 1, \infty\}$, then $x \in \mathbb{N}$
If $c \in \{0, 1, \infty\}$, then $x = (i, m)$, $m \in \mathbb{N}$

$1 \le i \le 2$ for $c = 0$
$1 \le i \le 3$ for $c = 1$
$1 \le i \le 6$ for $c = \infty$

Recall: Almost all indecomposable triples (V, T, U) with fixed dimension pair are **weakly homogeneous** (i.e. $U \simeq V/U,\ V \simeq U \oplus M(6)^t$), and then $U = M(4,2)^r \oplus M(5,3,3,1)^s$ for some pair r, s.

Let $X = (V, T, U)$ be weakly homogeneous with slope γ.



For $\gamma = 0$
$U = M(4,2)^r$

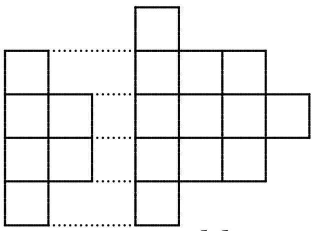
For $\gamma = \frac{1}{2}$
$U = M(5,3,3,1)^s$

For $\gamma = 1$
$U = M(4,2)^r$

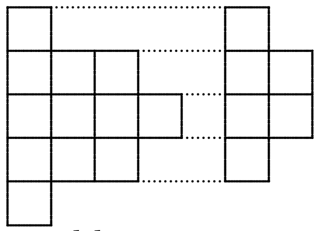
For $0 < \gamma < \frac{1}{2}$
there is an exact sequence
$0 \to X' \to X \to X'' \to 0$
slope 0   slope $\frac{1}{2}$

For $\frac{1}{2} < \gamma < 1$
there is an exact sequence
$0 \to X' \to X \to X'' \to 0$
slope $\frac{1}{2}$ slope 1

Adjustment
of the columns



even   odd



odd   even

References:

Ringel-Schmidmeier: Invariant subspaces of nilpotent linear operators. I. J. Reine Ang. Math. 614 (2008). 1-52.

Ringel-Schmidmeier: Invariant subspaces of nilpotent linear operators. II. (in preparation).

Categorical properties of $\mathcal{S}(6)$. The category $\mathcal{S}(n)$ is additive, thus the endomorphisms of any triple form a ring.

Let $n = 6$. The endomorphism rings $\operatorname{End}(X)$ of an indecomposable triple is usually rather large, however the bulk of endomorphisms will be nilpotent with nilpotency index at most 8:

Theorem. *Let X be an indecomposable triple in $\mathcal{S}(6)$. There is an ideal I in $\operatorname{End}(X)$ with $I^8 = 0$ such that $\operatorname{End}(X)/I$ is a local uniserial ring.*

The ideal I can be described as follows:
Given a Krull-Remak-Schmidt category such that the indecomposable objects have local endomorphism rings (such as $\mathcal{S}(n)$), its *radical* $\mathcal{R}$ is generated by all non-invertible maps between indecomposable objects.
The *infinite radical* is $\mathcal{I} = \bigcap_{i\in\mathbb{N}} \mathcal{R}^i$.

In the category $\mathcal{S}(6)$, the infinite radical $\mathcal{I}$ is an idempotent ideal, that is, $\mathcal{I}^2 = \mathcal{I}$ holds. If X is an indecomposable triple in $\mathcal{S}(6)$, then the ideal $\mathcal{I}(X,X)$ of $\operatorname{End}(X)$ has nilpotency index at most 8.

The ideal I to be used in Theorem is $I = \mathcal{I}(X,X)$.

The endomorphism theorem.

Denote by $\pi\colon \mathcal{S}(\widetilde{6}) \to \mathcal{S}(6)$ the forgetful functor (= forgetting the grading).

Recall that σ denotes the shift of the grading by 1.

The main formula is the following: Let X, Y be graded triples. Then

$$\mathrm{Hom}_{\mathcal{S}(6)}(\pi(X), \pi(Y)) = \bigoplus\nolimits_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(Y)).$$

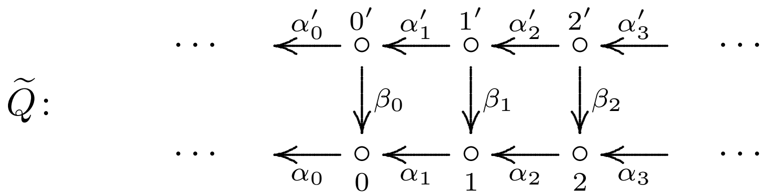
Now, in $\mathcal{S}(\widetilde{6})$ the following holds true: If X has slope γ and X' has slope γ', then $\mathrm{Hom}(X, X') = 0$ unless $\gamma \le \gamma'$

Therefore,

$$\begin{aligned}\mathrm{End}_{\mathcal{S}(6)}(\pi(X)) &= \bigoplus\nolimits_{i\ge 0} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X)) \\ &= \mathrm{End}_{\mathcal{S}(\widetilde{6})}(X) \oplus \bigoplus\nolimits_{i\ge 1} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X))\end{aligned}$$

If X is an indecomposable graded triple, then $\mathrm{End}_{\mathcal{S}(\widetilde{6})}(X)$ is a local uniserial ring, and $\mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X)) = 0$ for $i \ge 8$.

We see: A graded triple is a representation of the following quiver:



Usually, we will refrain from mentioning the indices of the arrows $\alpha_i, \alpha'_i, \beta_i$ and write α, α', β.

Let $\mathcal{S}(\widetilde{n})$ be the category of all representations of $\widetilde{Q}$ which satisfy

- the commutativity relations $\beta\alpha' = \alpha\beta$ (for all squares),
- the nilpotency relations $(\alpha')^n = \alpha^n = 0$ (for all compositions of n arrows α' or α)
- and for which all the maps β are realized by monomorphisms.

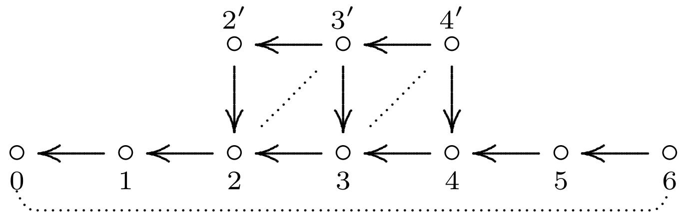
Note that this is just the category of triples in $\mathcal{S}(n)$ endowed with a grading.

Denote by σ the shift of the grading (by 1). Then $\mathcal{S}(\widetilde{6})/\sigma$ are the triples in $\mathcal{S}(6)$ which are gradable.

Aim. Classify the iso classes of indecomp representations of $\widetilde{Q}$.

It is sufficient to look at a suitable finite subquiver of $\widetilde{Q}$. It turns out that the following one provides sufficient information (and can be handled):

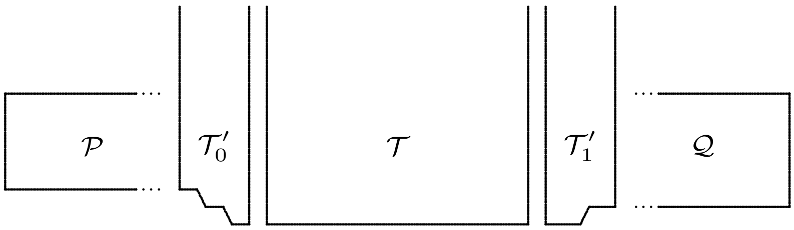
Q_Θ:



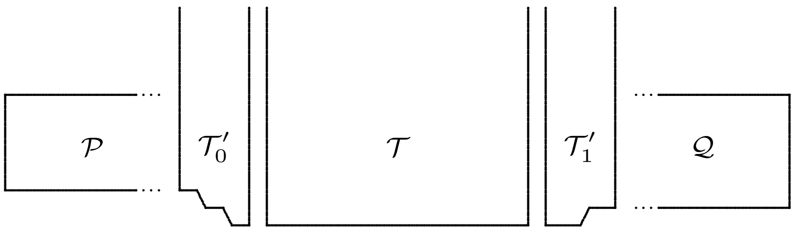
(with two commutativity relations and one zero relation).

The corresponding algebra Θ can be shown to be "tubular", and in this way we obtain a description of nearly all of the category $\mathcal{S}(\widetilde{6})$.

The shape of the category of Θ-modules is as follows:

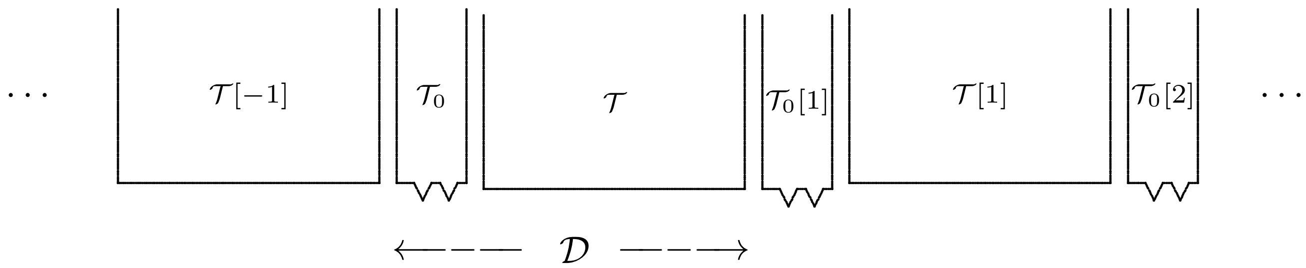


The shape of the category $\operatorname{mod}\Theta$:



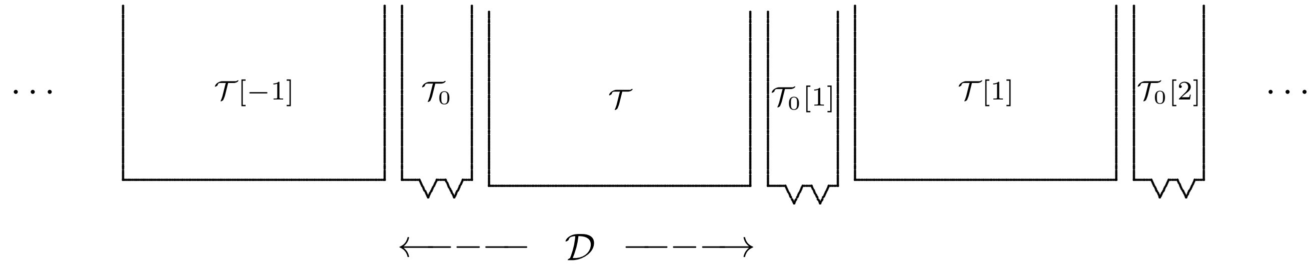
The subcategories $\mathcal{T}_0'$ and $\mathcal{T}_1'$ will be partly identified in $\mathcal{S}(6)$.

The category $\mathcal{S}(\widetilde{6})$ can be described in the following way:



$\mathcal{D}$ is the fundamental domain for $\mathcal{S}(\widetilde{6})$ under the shift σ.

The category $\mathcal{S}(\widetilde{6})$ can be described in the following way:



$\mathcal{D}$ is the fundamental domain for $\mathcal{S}(\widetilde{6})$ under the shift σ.

Since $\mathcal{S}(\widetilde{6})$ is “locally bounded”, it follows that $\mathcal{S}(6) = \mathcal{S}(\widetilde{6})/\sigma$ (= the grading theorem).

Of essential importance is the central part $\mathcal{T}$,
there are **countably many stable tubular families** $\mathcal{T}_\gamma$ indexed by $\gamma \in \mathbb{Q}^+$,
each $\mathcal{T}_\gamma$ is **a** $\mathbb{P}_1(k)$**-family of tubes of type** $(6,3,2)$**.**