# Minimal representation-infinite algebras

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$k$ algebraically closed field

$A$ finite dimensional $k$-algebra (associative, with 1), connected (no central idempotents except 0 and 1).

$\operatorname{mod} A$ the category of all (finite-dimensional) $A$-modules

$A$ is called *representation-infinite* provided there are infinitely many isomorphism classes of indecomposable $A$-modules, otherwise *representation-finite.*

$A$ is *minimal representation-infinite* provided $A$ is representation-infinite, but any proper factor algebra is representation-finite.

Definition: $A$ has a *good covering* $\widehat{A}$
provided $\widehat{A}$ is a Galois covering of $A$ with free Galois group
and $\widehat{A}$ is interval-finite.

Covering theory: Bongartz-Gabriel 1980

**Theorem (Bongartz, 2009).**
Let $A$ be minimal representation-infinite.
Then one of the following cases holds:

(1) $A$ is non-distributive
(i.e. the ideal lattice of $A$ is not distributive).

(2) $A$ has a good covering $\widehat{A}$ and
there is a full subcategory of $\widehat{A}$ which is tame concealed.

(3) $A$ has a good covering $\widehat{A}$ and
$\widehat{A}$ is locally representation finite
(i.e. any finite subcategory is representation-finite).

This result is a final step in a long series of investigations
Bautista-Gabriel-Roiter-Salmeron 1984

Fischbacher, ...

Let us discuss the three cases in more detail.

(1) *$A$ is non-distributive*

Non-distributive algebras $A$ have been studied already 1957 by Jans.

$A$ has a one-parameter family of modules (pairwise non-iso) modules which are both local and colocal ("diamonds").

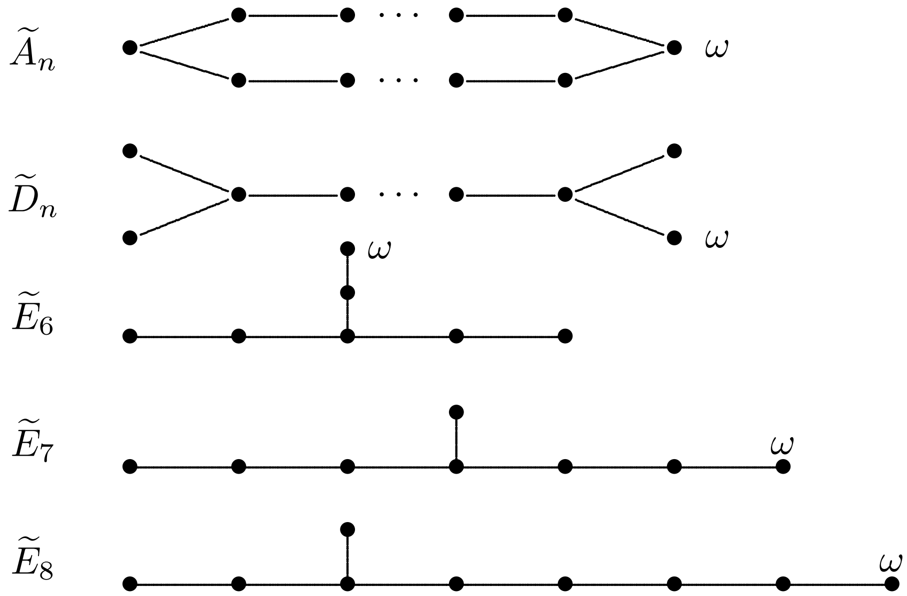
Thus, Brauer-Thrall II is obvious
(via Smalø's Brauer-Thrall $1\frac{1}{2}$, or direct construction)

(2) *$A$ has a good covering $\widehat{A}$ and*
*$\widehat{A}$ has a full subcategory which is tame concealed.*

The representation theory of tame concealed algebras $B$ is well-developed.
In particular, $B$ has a one-parameter family of pairwise non-isomorphic modules via the push-down functor, one obtains a one-parameter family of pairwise non-isomorphic $A$- modules.

Recall: Tame concealed algebras are of type $\widetilde{\mathbb{A}}_n,\ \widetilde{\mathbb{D}}_n,\ \widetilde{\mathbb{E}}_6,\ \widetilde{\mathbb{E}}_7,\ \widetilde{\mathbb{E}}_8,$
the indecomposable $B$-modules are described by the corresponding Kac-Moody root system.

## The extended Dynkin diagrams




A tame concealed algebra $B$ is derived equivalent to the path algebra $kQ$, where the quiver $Q$ has as underlying graph an extended Dynkin diagram.

$A$ is given by identifying some vertices and some arrows of the quiver of $B$.

It remains to consider the case (3)
Bautista-Gabriel-Roiter-Salmeron singled out the cases (1) and (2), but were not able to treat (3).

(3) *$A$ has a good covering $\widehat{A}$ and*
*$\widehat{A}$ is locally representation finite.*

Questions:

How do we find one-parameter families of indecomposable $A$-modules?

Is it possible to determine the structure of such algebras $A$?

Is it possible to classify these algebras?

Is it possible to classify all the indecomposable $A$-modules ?

Using the new Bongartz result all these questions can be answered positively.

In fact, the algebras which arise and their module categories have been exhibited already in 2007.

**Theorem 1 (2010).** *Let $A$ be minimal representation-infinite and assume that $A$ has a good covering $\widehat{A}$ which is locally representation-finite. Then $A$ is a special-biserial algebra.*

This means: $A$ is given by identifying some vertices and some arrows of an algebra of type $\widetilde{\mathbb{A}}_n$.

**Theorem 2 (2007).** *Let $A$ be minimal representation-infinite, special biserial and after separation of the nodes, there are three possible cases:*

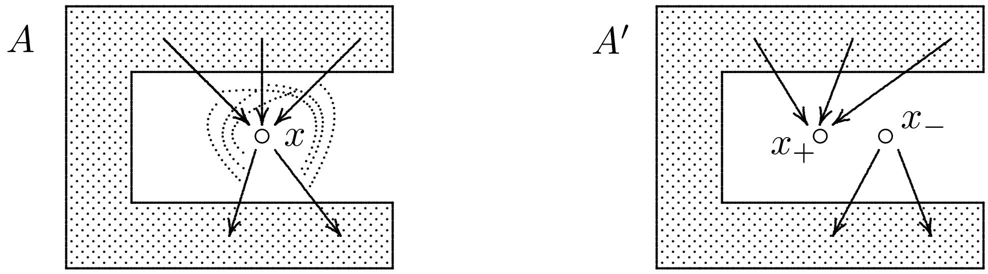
- *$A$ is hereditary of type $\widetilde{\mathbb{A}}_n$, or*
- *$A$ is a "barbell" with non-serial bar, or*
- *$A$ is a "wind wheel".*

We write $A = kQ/I$, where $Q$ is a finite quiver and $I$ is an admissible ideal.

**Separation of nodes.**

A vertex $x$ is a *node* provided
$\beta\alpha \in I$ for all arrows $\alpha$ ending in $s$ and $\beta$ starting in $s$.

Nodes can be separated:




There is a canonical functor $\operatorname{mod} A \to \operatorname{mod} A'$ which yields a bijection between the indecomposable $A$-modules and the indecomposable $A'$-modules different from the simple $A'$-module $S(x_-)$.

It is sufficient to deal with algebras without nodes:
$A$ is minimal representation-infinite if and only if
the algebra $A'$ obtained from $A$ by "separating" all the nodes is minimal representation-infinite.

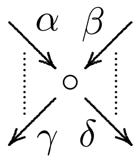
**Special biserial algebras.**

$A = kQ/I$ is *special biserial* provided

(1) Any vertex of $Q$ is endpoint of at most two arrows
Any vertex of $Q$ is starting point of at most two arrows.

(2) If arrows $\gamma$ and $\delta$ start in the endpoint of $\alpha$, then $\gamma\alpha$ or $\delta\alpha$ is in $I$.
If arrows $\alpha$ and $\beta$ end in the starting point of $\gamma$, then $\gamma\alpha$ or $\gamma\beta$ is in $I$.

The local shape is at most
(maybe less arrows or more relations)

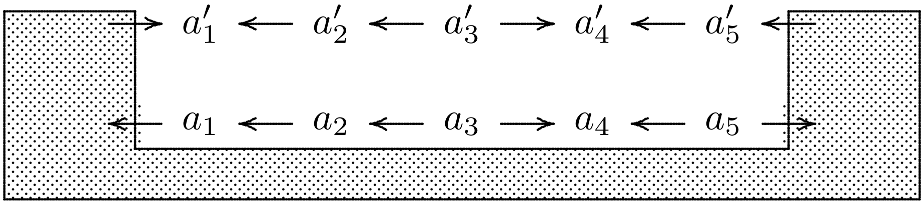



The representations theory of special biserial algebras is (essentially) due to Gelfand-Ponomarev (1968):

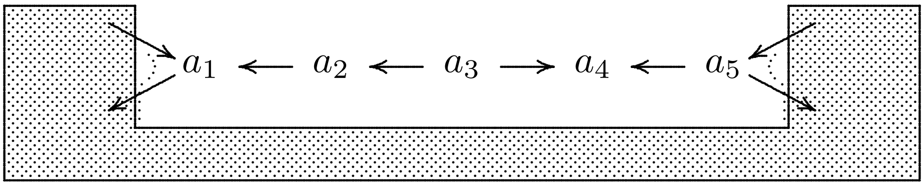
*Special biserial algebras are representation-finite or tame,*
*there are two kinds of indecomposable representations, string modules and band modules.*

**Barification.** We start with a special biserial algebra wit quiver $Q$.

Consider two disjoint maximal isomorphic paths in $Q$ which are not involved in relations.
For example




The paths are now identified in order to form a *bar*. This barification yields a quiver of the following form (the dotted box is not changed):




Of importance are the new zero relations on the left and on the right!

**Barbells with non-serial bar.** Example: start with

$$
\begin{tikzcd}
1' \arrow[r, "\beta_1'"] \arrow[d, bend right, "\alpha"'] & 3' & 2' \arrow[l, "\beta_2'"'] \arrow[d, bend left, "\gamma"] \\
1 \arrow[r, "\beta_1"'] & 3 & 2 \arrow[l, "\beta_2"]
\end{tikzcd}
$$

Barification yields:

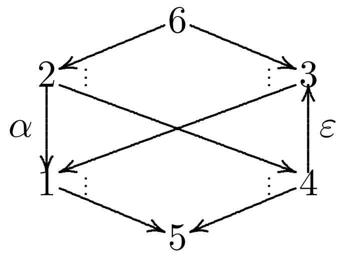
$$
\begin{tikzcd}
1 \arrow[loop above, "\alpha"] \arrow[dr, "\beta_1"'] & & 2 \arrow[loop above, "\gamma"] \arrow[dl, "\beta_2"] \\
& 3 &
\end{tikzcd}
$$

Here the bar (given by the arrows $1 \to 3 \leftarrow 2$) is not serial.

**Theorem.** *The barbells with non-serial bars are minimal representation-infinite. They are tame and of non-polynomial growth* (in particular, not domestic), they are Gorenstein algebras of Gorenstein dimension 1.

**Wind wheels:**
obtained from cycles by a sequence say of $t$ **serial** barifications (for serial barifications, a further zero relation is added).

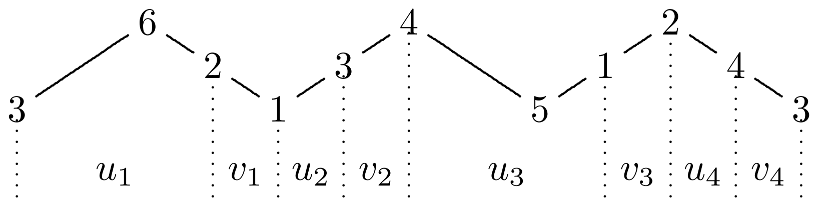
Here is an example:

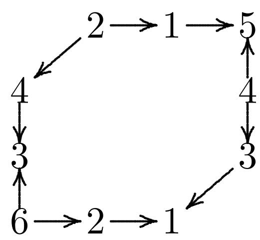



with the further relations
$6 \to 2 \to 1 \to 5$ and $2 \to 4 \to 3 \to 1$.

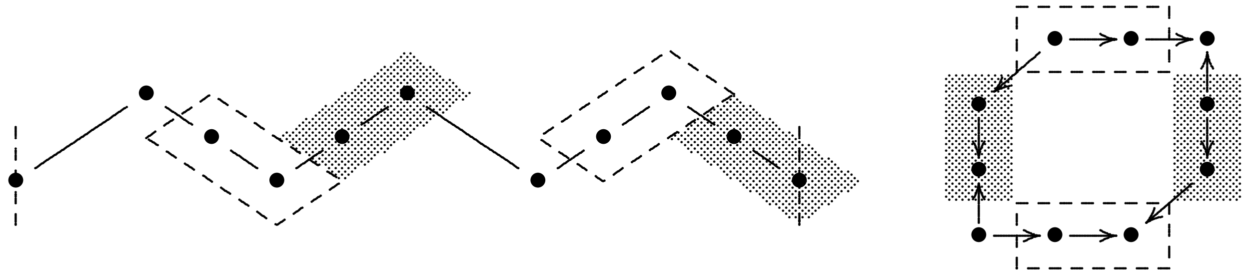
There are two bars, namely $\alpha$ and $\epsilon$.

The original cycle is seen by looking at the only primitive cyclic word



Thus, we start with a quiver which can be drawn either as a zigzag (with arrows pointing downwards), where the left end and the right end have to be identified, or else as a proper cycle:



and we barify on the one hand the two subquivers which are enclosed in rectangular boxes, on the other hand also the two subquivers with shaded background.

In both cases, the barification yields an identification of a projective serial module of length 2 with an injective serial module of length 2.

**Theorem.** *The wind wheels are minimal representation-infinite.*

*Wind wheels are* 1*-domestic*
(there is a unique primitive 1-parameter-family of indecomposable modules)
*and there are precisely $t$ non-periodic, but biperiodic $\mathbb{Z}$-words.*

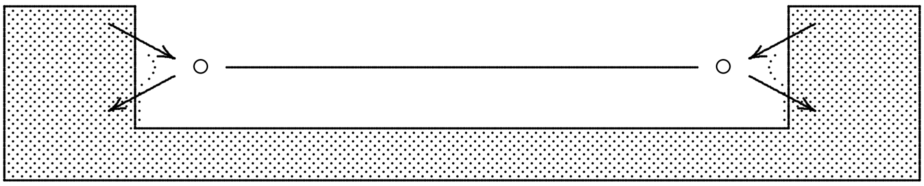
**Sketch of the proof of theorem 2.** Let $A$ be a biserial algebra which is minimal representation-infinite

Let $x$ be a vertex. In case there are $a$ arrows ending in $x$ and $b$-arrows starting in $x$, then we say that $x$ is an $(a+b)$-vertex.

**(1)** *Assume $A$ is biserial and minimal representation-infinite. If the vertex $x$ is a 4-vertex, then $x$ is a node.*

Since we may assume that there is no node, we easily see that there are only 2-vertices and 3-vertices. In case all the vertices are 2-vertices, then we deal with a cycle.

Thus we can assume that there are 3-vertices,
there will be a subquiver of the following form
(with only 2-vertices between the two given 3-vertices):



It follows: $A$ is obtained from a cycle by a sequence of barifications, adding, if necessary, further relations. It remains to analyze these barifications!

**(2)** Assume that $A$ is obtained from a cycle by barification with bar $B$. In case $B$ is not serial, any further barification yields an algebra which no longer is minimal representation-infinite.

Barifications with a single non-serial bar yield the barbells.

Thus we can assume that $A$ is obtained from a cycle by a sequence of serial barifications, say identifying the serial paths $w_i$ and $w_i'$, for $1 \leq i \leq t$.

**(3)** One of $M(w_i), M(w_i')$ is projective, the other injective, say $M(w_i)$ is projective.

**(4)** The radicals of all $M(w_i')$ lie in one exceptional tube, the socle factor module of all $M(w_i)$ lie in the other exceptional tube.

The algebras obtained in this way are the wind wheels.

Finally, let us look at Theorem 1.

**Theorem 1.** *Let $A$ be minimal representation-infinite and assume that $A$ has a good covering $\widehat{A}$ which is locally representation-finite. Then $A$ is a special-biserial algebra with only zero relations.*

Note the following converse: If $A$ is a special-biserial algebra with only zero relations, then $A$ has a good covering $\widehat{A}$ which is locally representation-finite.

Let $\widehat{Q}$ be the quiver of $\widehat{A}$ and $Q$ the quiver of $A$,
let $\pi\colon \widehat{Q} \to Q$ be the covering map.

The essential steps of the proof. One shows:

(1) There is a path in $\widehat{A}$ starting with an arrow $\alpha$ and ending in the inverse $\beta^{-1}$ of an arrow, such that $\pi(\alpha) \neq \pi(\beta)$ but $\pi(h(\alpha)) = \pi(h(\beta)$, with no subpath involved in a relation.

Here one uses the Bongartz classification of the large indecomposable representations of directed algebras: all have large subquivers of type $\mathbb{A}_n$.

(2) Serial $\widehat{A}$-modules of length 3 with isomorphic radical, or with isomorphic socle factor module are isomorphic.

(3) There are no local or colocal $\widehat{A}$-modules of length 4 with Loewy length 2.

For the proof of (2) and (3) one shows that otherwise $\widehat{A}$ would contain a subquiver of type $\widetilde{\mathbb{E}}_7$.