The root posets and their rich antichains

Dedicated to Professor Liu Shao-Xue on his 90th Birthday

Claus Michael Ringel

Abstract. Let Δ be a (connected) Dynkin diagram of rank $n \geq 2$ and $\Phi_+ = \Phi_+(\Delta)$ the corresponding root poset (it consists of all positive roots with respect to a fixed root basis). The width of Φ_+ is n. We will show that Φ_+ is "conical": it is the disjoint union of n solid chains.

The rich antichains in Φ_+ are the antichains of cardinality $n-1$. It is well known that the number of rich antichains is equal to the cardinality of Φ_+. The set $\mathcal{R}(\Delta)$ of rich antichains in Φ_+ can itself be considered as a poset which is quite similar, but not always isomorphic, to Φ_+.

We will show that there always exists a unique rich antichain A such that any rich antichain is contained in the ideal generated by A. For $\Delta \neq \mathbb{E}_6$ all roots in A have the same length, namely e_2, where $e_1 \leq e_2 \leq \cdots \leq e_n$ are the exponents of Δ. For $\Delta = \mathbb{E}_6$, the antichain A consists of four roots of length $e_2 = 4$ and one root of length 5.

1. Introduction.

Let Φ be a root system of (connected) Dynkin type $\Delta = \mathbb{A}_n, \mathbb{B}_n, \dots, \mathbb{G}_2$ of rank n. The *root poset* $\Phi_+ = \Phi_+(\Delta)$ is the set of positive roots in Φ with respect to some fixed choice of a root basis; here, one takes the following partial ordering: $x \leq y$ provided $y - x$ is a non-negative linear combination of elements of the root basis. The root posets and its antichains play an important role in many parts of mathematics. The width of the root poset $\Phi_+(\Delta)$ is the rank of Δ (this is the number of vertices of Δ). If P is a poset of width n, the antichains of P of cardinality $n-1$ will be called the *rich* antichains. It is well known that the number of rich antichains is equal to number of positive roots and the paper is devoted to a study of the rich antichains of Φ_+.

Following Stanley [S], a finite poset P is said to be *graded* provided all maximal chains have cardinality m (the height of P). Given a graded poset P, we denote by P_t the set of elements of height t; note that P_t is an antichain. The cardinality of P_t will be denoted by $h_t(P)$. Of course, $h_1(P)$ is just the number of minimal elements of P.

Given a poset P, we call a subposet P' a *solid* subposet provided neighbors in P' are neighbors in P (if $x < y$ are neighbors in the subposet P', this interval cannot be refined in P). We say that P is *conical* provided P is a graded poset with a unique maximal element and is the disjoint union of solid subchains such that each of the subchains contains a minimal element (such a set of subchains will be called a *conical decomposition*). If $\mathcal{C}$ is a conical decomposition $\mathcal{C}$, the number of chains in $\mathcal{C}$ is equal to $h_1(P)$. The cardinalities of the chains in $\mathcal{C}$ will be called the *exponents of* P.

If P is conical, we have $h_t(P) \geq h_{t+1}(P)$ for all $t \geq 1$, thus $(h_1(P), h_2(P), \dots)$ is a (Young) partition and the dual partition $(e_1(P), e_2(P), \dots)$ is the sequence of exponents.

Finally, given a subset X of a poset P, we denote by $\Lambda(X) = \{w \in P \mid w \leq x \text{ for some } x \in X\}$ the ideal, by $V(X) = \{y \in P \mid x \leq y \text{ for some } x \in X\}$ the coideal generated by X.

Let us consider now the root posets. Since any root poset has a maximal element, the definition of the partial ordering of Φ_+ shows that all maximal chains in Φ_+ have the same cardinality, thus Φ_+ is a graded poset with a unique maximal element. For a positive root x, the height of x in Φ_+ is just the length of x, it is the sum of the coefficients when x is written as a linear combination with respect to the root basis (see for example [B]). We write Φ_t instead of $(\Phi_+)_t$.

Theorem 1. *A root poset is conical.*

Theorem 1 strengthens the well-known assertion that $h_t(\Phi_+) \geq h_{t+1}(\Phi_+)$ for all $t \geq 1$. Actually, the sequence of exponents of $\Phi_+(\Delta)$ as defined above is just the usual sequence of exponents as considered in the invariant theory of Weyl groups (this is the celebrated Shapiro-Kostant theorem, see for example [R2] Theorem 1.4.1.1). We also should mention that

$$h_t(\Phi_+) + h_{g-t+1}(\Phi_+) = n$$

for all t, where g is the Coxeter number for Δ (see Humphreys [H, Theorem 3.20] and Armstrong [Ar, Theorem 5.4.1]).

Note that $e_1 = 1$ and, for $n \geq 2$, we have $e_2 \geq 2$. It follows that all Φ_t with $2 \leq t \leq e_2$ are rich antichains, whereas any Φ_t with $t > e_2$ is an antichain of cardinality at most $n-2$.

The set of rich antichains of a poset can be considered as a poset $\mathcal{R}(\Delta)$ with $A \leq A'$ provided $V(A) \supseteq V(A')$ (recall that $V(A)$ is the coideal generated by A, see section 4). As we will see, for $\Delta \neq \mathbb{E}_6$, the antichain Φ_{e_2} is the largest element in $\mathcal{R}(\Delta)$, whereas for $\Delta = \mathbb{E}_6$, there is a single rich antichain which is larger than Φ_{e_2}.

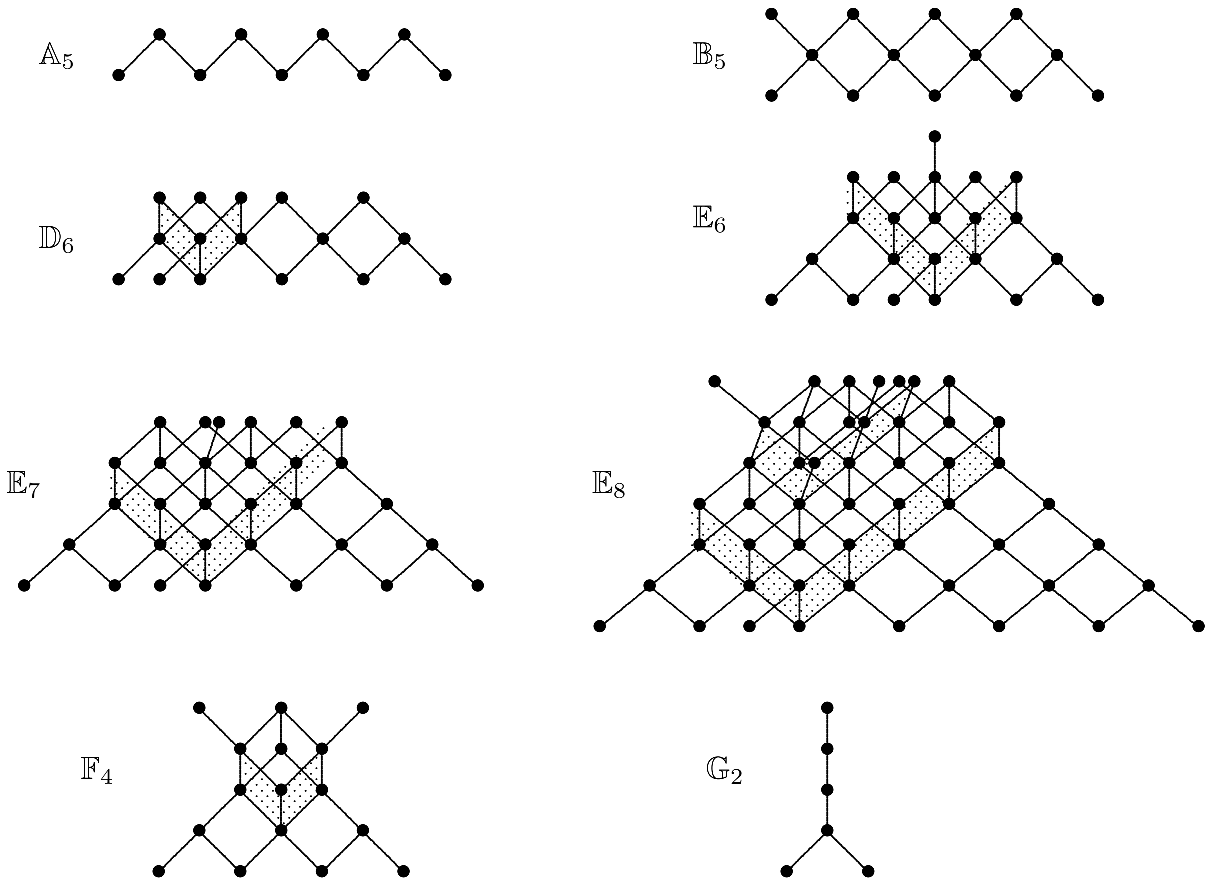
For any natural number t, let Φ'_t be the set consisting of join-irreducible elements of Φ_{t+1} and the elements $a \in \Phi_t$ which have no join-irreducible neighbor $a' > a$ in Φ_{t+1}. Since Φ_+ is conical, the cardinality of Φ'_t is equal to the cardinality of Φ_t. In particular, for $t = e_2$, Φ'_t is again a rich antichain.

Theorem 2. *Let Δ be a Dynkin diagram of rank at least 2. Then Φ_+ has a unique maximal rich antichain, namely Φ'_{e_2}. If $\Delta \neq \mathbb{E}_6$, then $\Phi'_{e_2} = \Phi_{e_2}$, whereas for $\Delta = \mathbb{E}_6$, the antichains Φ'_{e_2}, Φ_{e_2} differ by one element.*

Here is the value of $e_2 = e_2(\Delta)$:

Δ	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$e_2(\Delta)$	2	3	3	3	4	5	7	5	5

Let us reformulate Theorem 2. In oder to find the rich antichains of $\Phi_+(\Delta)$, it is sufficient to look at the ideal $\Lambda(\Phi'_{e_2})$, this is a poset with n minimal and $n-1$ maximal elements. Here are the posets $\Lambda(\Phi'_{e_3})$ for $\Delta = \mathbb{A}_5, \mathbb{B}_5, \mathbb{D}_6, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$.



It is well known that the number of rich antichains in Φ_+ is equal to the cardinality of Φ_+. It turns out that in general the poset structure of $\mathcal{R}(\Delta)$ is quite similar to $\Phi_+(\Delta)$, but these posets are not always isomorphic: For example, in case $\mathbb{F}_4$, the poset $\Lambda(\Phi'_{e_2})$ has a non-trivial symmetry, thus also $\mathcal{R}(\mathbb{F}_4)$, whereas $\Phi_+(\mathbb{F}_4)$ has no non-trivial symmetry.

Theorem 3. *The poset $\mathcal{R}(\Delta)$ of rich antichains in $\Phi_+(\Delta)$ is isomorphic to $\Phi_+(\Delta)$ if and only if $\Delta \neq \mathbb{E}_8,\ \mathbb{F}_4$.*

Acknowledgment. The paper is dedicated to Professor Liu Shao-Xue. His visit to Europe in 1985 was the start of a long lasting and very fruitful cooperation between China and Germany devoted to the representation theory of finite dimensional algebras. The core of this theory are the Dynkin algebras and their indecomposable representations, indexed by the elements of the corresponding root poset. It seems to be surprising that these root posets which look like innocent combinatorial creatures still provide a lot of mysteries. The present paper and its successor [R3] try to illuminate some of their features.

A first version [R1] of this paper was written in 2013 at SJTU Shanghai. Parts of the results have been presented at the 57th Annual Meeting of the Australian Mathematical Society 2013 in Sydney, and in my ICRA lectures 2014 at Sanya, see [R2]. The author is grateful to many mathematicians for comments. In particular, he has to thank H. Thomas

for pointing out the references Humphreys [H, Theorem 3.20] and Armstrong [Ar, Theorem 5.4.1]).

Since the appendix of this paper has been used already quite frequently, we should stress that we have changed the labeling of the vertices of $\mathbb{B}_n$, $\mathbb{C}_n$, and $\mathbb{D}_n$ in order to focus the attention (for $\Delta \neq \mathbb{A}_n$) to a special vertex which now is always labeled c.

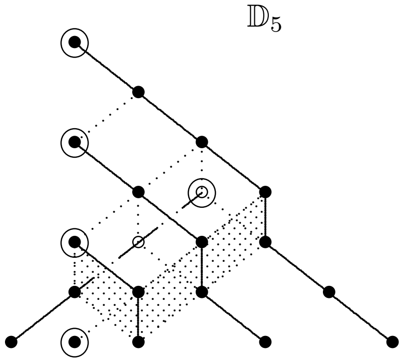
2. Solid subchain decompositions of Φ_+.

We are going to prove Theorem 1: *Any root poset has a conical decomposition.*

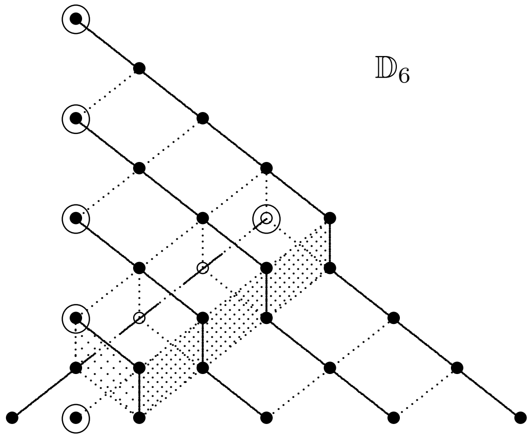
Proof of Theorem 1. The assertion is obvious for the cases $\mathbb{A}_n$, $\mathbb{B}_n$ and $\mathbb{G}_2$. Below we show solid subchain decompositions in the special cases $\mathbb{D}_5, \mathbb{D}_6$, and for $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4$. We hope that the cases $\mathbb{D}_5$ and $\mathbb{D}_6$ show nicely the general rule how to obtain a solid subchain decomposition in the cases $\mathbb{D}_n$ in general.

Always, we use solid lines in order to specify the solid subchains. The largest elements of the solid subchains are encircled.

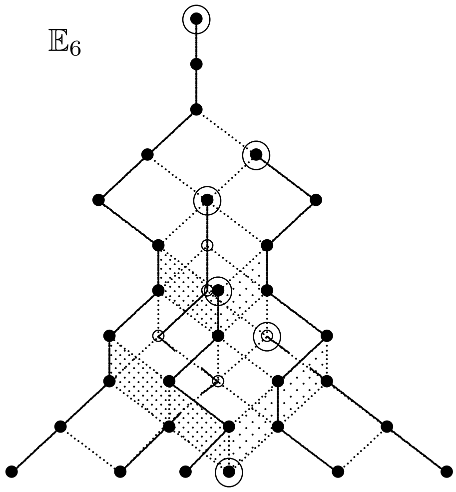
$\mathbb{D}_5$



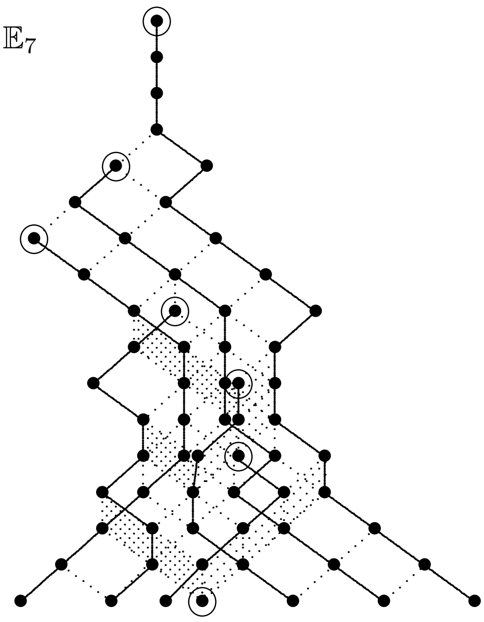
$\mathbb{D}_6$

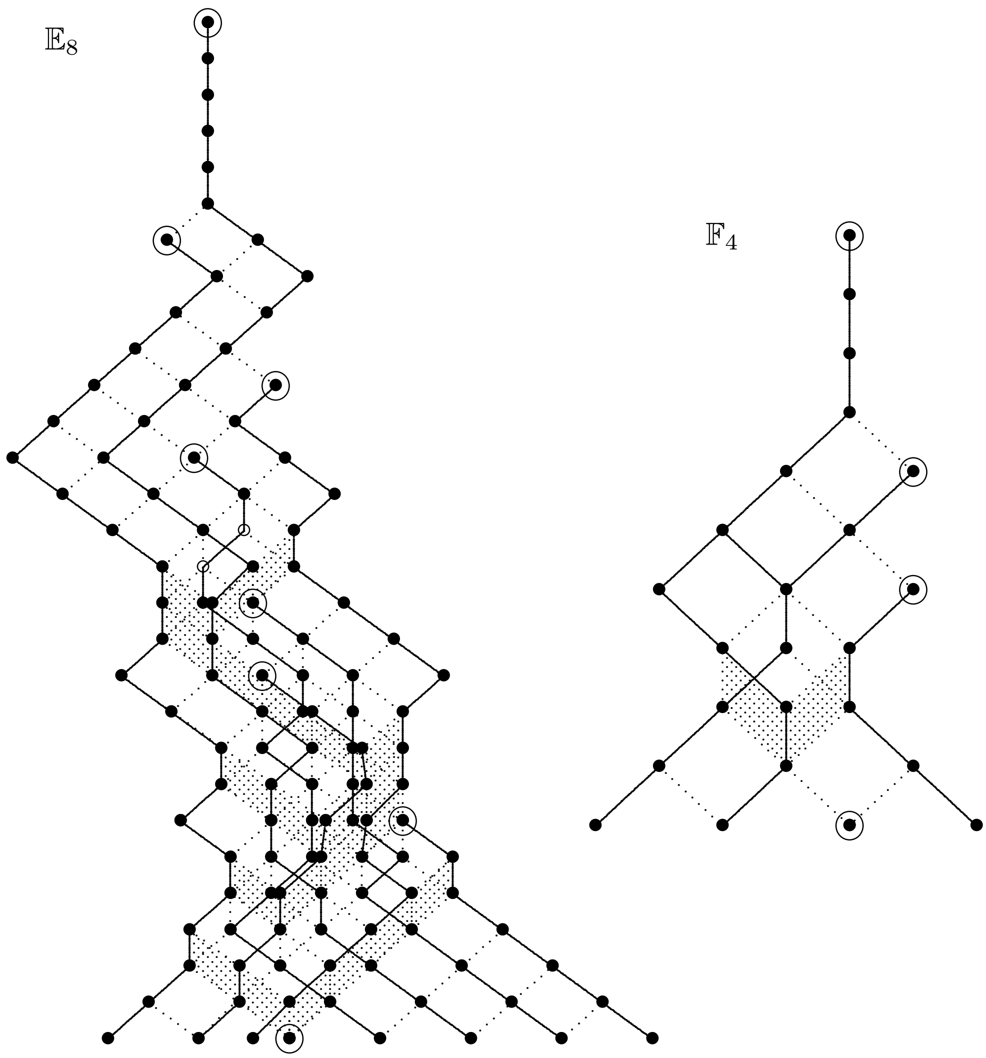


$\mathbb{E}_6$



$\mathbb{E}_7$





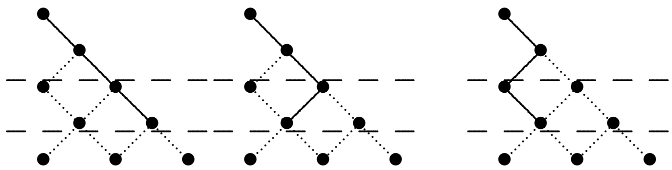
The largest elements of the solid subchains are always encircled (thus these are suitable roots of height ϵ_i, where $(\epsilon_1, \ldots, \epsilon_n)$ is the exponent partition). $\square$

Actually, the existence of a solid subchain decomposition concerns a local property, namely it concerns the bipartite subposets $\Phi_{t,t+1}$ of all roots of height t and $t+1$, we call these subposets the *steps* of Φ_+. As we know, we have $|\Phi_{t+1}| \leq |\Phi_t|$.

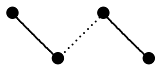
The essential assertion of Theorem 1 is that for any $t \geq 1$, there is a matching for $\Phi_{t,t+1}$ (an injective map $f : \Phi_{t+1} \to \Phi_t$ such that $f(y) < y$ for all $y \in \Phi_{t+1}$).

Namely, if we want to construct a conical decomposition, we may start at the top of the poset Φ_+ and go down. If the subchains have reached the layer Φ_{t+1}, we have to look at $\Phi_{t,t+1}$ and we have to continue the path downwards inside a matching. For example, in case $\mathbb{B}_3$, starting with the maximal element z, the next two choices for a solid chain containing z are arbitrary, but then in $\Phi_{2,3}$ we have to be careful:

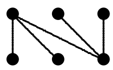
$\Phi_{2,3}$



The choice in the middle does not work, since $\Phi_{2,3}$ has just one matching namely:



Remark. As we have mentioned, for a conical poset P, the sequence $(h_1(P), h_2(P), \ldots)$ is a Young partition. Let us stress that there are graded posets with $(h_1(P), h_2(P), \ldots)$ being a Young partition, which cannot be written as the disjoint union of solid subchains such that each of the subchains contains a minimal element. Here is an example (note that its width is greater than $h_1(P)$):



3. The rich antichains of Φ_+.

This section is devoted to the proof of Theorem 2.

We say that a poset P is *bipartite* provided any element of P is minimal or maximal, but not both. If P is bipartite, let P_1 be the set of minimal elements and P_2 the set of maximal elements. A *matching* of P is an injective map $f\colon P_2 \to P_1$ with $f(y) < y$ for all $y \in P_2$.

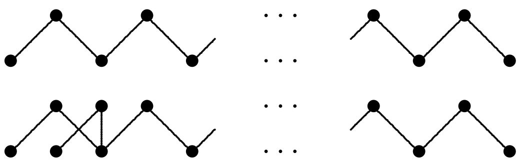
We say that P is an *M-poset* provided P is bipartite with $|P_1| = |P_2| + 1$ and such that for any $y \in P_2$ there is a matching f such that the unique element $x \in P_1$ which is not in the image of f satisfies $x < y$. Note that if P is an M-poset, then no element in P_2 is join-irreducible.

If T is a finite tree, we define $M(T)$ as the incidence poset of T (with $M(T)_1$ the set of vertices, $M(T)_2$ the set of edges, such that $x < y$ provided x is a vertex on the edge y).

For example:

$M(\mathbb{A}_n)$

$M(\mathbb{D}_n)$



Here, n is the number vertices of T, thus the number of minimal elements of the poset $M(T)$.

Proposition 1. *If T is a finite tree, then $M(T)$ is an M-poset.*

Proof. Assume that T has n vertices, thus $n-1$ edges. It follows that $M(T)$ is bipartite with $M(T)_2 = n-1$ and $M(T)_1 = n$. Let y be an edge, and x a vertex on y. For every edge y', let $f(y')$ be the vertex on y' with maximal distance to x. Then f is a matching, and x is not in the image of f. Of course, $x < y$.

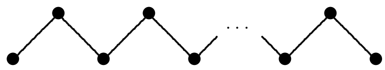
Corollary. *For any root poset $\Phi_+ = \Phi_+(\Delta)$, the poset Φ_{12} is an M-poset.*

Proof: Obviously, $\Phi_{12} = M(\Delta)$.

Proposition 2. *Assume that Δ has rank at least 3, let $\Phi = \Phi(\Delta)$ and $e = e_2(\Delta)$. If $\Delta \neq \mathbb{D}_4,\ \mathbb{E}_6$, then $\Phi_{e,e+1}$ is an M-poset.*

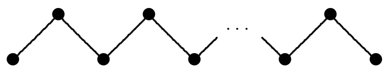
We claim that $\Phi_{e,e+1}$ is obtained from a poset of the form $P = M(T)$ by adding for some pairs $x \in P_1,\ y \in P_2$ the relation $x < y$. Since $M(T)$ is an M-poset, also $\Phi_{e,e+1}$ has to be an M-poset. Here are drawings of the various cases. On the right, we exhibit $P = M(T)$. In the drawings, the solid lines show the relations in $M(T)$, the added relations are dashed.

$\Phi_{2,3}(\mathbb{A}_n)$ $(n \geq 2)$



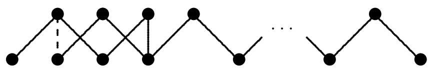
$M(\mathbb{A}_{n-1})$

$\Phi_{3,4}(\mathbb{B}_n)$ $(n \geq 2)$



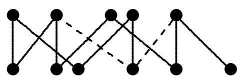
$M(\mathbb{A}_{n-1})$

$\Phi_{3,4}(\mathbb{D}_n)$ $(n \geq 5)$



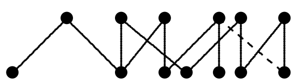
$M(\mathbb{E}_{n-1})$

$\Phi_{5,6}(\mathbb{E}_7)$



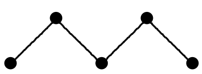
$M(\mathbb{A}_6)$

$\Phi_{7,8}(\mathbb{E}_8)$



$M(\mathbb{E}_7)$

$\Phi_{5,6}(\mathbb{F}_4)$



$M(\mathbb{A}_3)$

Proof of Theorem 2. If the rank n of Δ is equal to 2, then $\mathcal{R}(\Delta) = \Phi_+$. Thus, we can assume that $n \geq 3$. For $\Delta = \mathbb{D}_4$, $\Phi_4(\Delta)$ is just the maximal element of Φ_+, and this element cannot belong to any antichain of cardinality greater than 1. Thus, we assume now that $\Delta \neq \mathbb{D}_4$.

The case $\mathbb{E}_6$ is special (since $\Phi_5(\mathbb{E}_6)$ contains a join-irreducible element; in particular, $\Phi_{45}(\mathbb{E}_6)$ is not an M-poset) and has to be treated separately. Thus, we assume that Δ has rank at least 3 and is different from $\mathbb{D}_4,\ \mathbb{E}_6$. This means that we can apply Proposition 2.

Let z be a root of length $t \geq e+1$. Let $\Phi(z)$ be the set of positive roots which are not comparable with z. We claim that $\Phi(z)$ has width at most $n-3$ (this shows that z cannot belong to an antichain of cardinality $n-1$).

Let $\mathcal{C}$ be a conical decomposition of Φ_+. Let $C \in \mathcal{C}$ with $z \in C$. Since the length of z is at least $e+1$, the chain C contains a root y of length $e+1$. Since $\Phi_{e,e+1}$ is an M-poset, there is a matching $f\colon \Phi_{e+1} \to \Phi_e$ such that the unique element $x \in \Phi_e$ which does not belong to the image of f satisfies $x < y$.

Let v be the root of length 2 which belongs to C. Since Φ_{12} is an M-poset, there is a matching $g\colon \Phi_2 \to \Phi_1$ such that the simple root u which does not belong to the image of g, satisfies $u < v$. Let $\mathcal{D}$ be obtained from the decomposition $\mathcal{C}$ by replacing in any chain $C' \in \mathcal{C}$ of cardinality at least 2 its simple root by the simple root $g(v')$, where v' is the root of length 2 in C' and using the singleton $\{u\}$ as the unique chain of cardinality 1 in $\mathcal{D}$. Thus, $\mathcal{D}$ is again a conical decomposition of Φ_+.

Now, let $y' \in \Phi_{e+1}$. If y' belongs to $D \in \mathcal{D}$, let $J_{y'}$ be the set of elements $z' \in D$ with $y' \leq z'$.

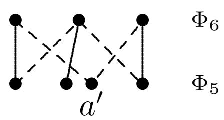
Similarly, assume that $x' \in \Phi_e$ belongs to $D \in \mathcal{D}$ and let $I_{x'}$ be the set of elements $w \in D$ with $w \leq x'$. Since $f(y') < y'$, we see that the union $I_{f(y')} \cup J_{y'}$ is a chain.

We have decomposed Φ_+ into the chains $I_{x'}, J_{y'}$ with $x' \in \Phi_e$ and $y' \in \Phi_{e+1}$, as well as the singleton $\{u\}$. We know that the elements in J_y, $I_{f(y)}$, I_x as well as the element u all are comparable with z, thus they belong to $\Phi(z)$.

It follows that $\Phi \setminus \Phi(z)$ is covered by the chains $I_{f(y')} \cup J_{y'}$, where the elements y' are the elements of $\Phi_{e+1} \setminus \{y\}$. The cardinality of Φ_{e+1} is $n-2$, thus there are $n-3$ elements of the form y'. This shows that $\Phi \setminus \Phi(z)$ is covered by $n-3$ chains.

It remains to look at the case $\Delta = \mathbb{E}_6$. We denote by a' the join-irreducible element of Φ_5, and by a its neighbor in Φ_4. Thus Φ'_4 is obtained from Φ_4 by replacing a by a'.

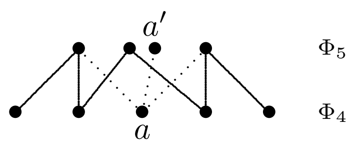
Let us look at $\Phi_{56}(\mathbb{E}_6)$. There is the following matching of $\Phi_{56}(\mathbb{E}_6)$:



It follows that there is a conical decomposition $\mathcal{C}$ of Φ_+ such that a' is the maximal element of one of the chains. Also, we can assume that the simple root corresponding to the branching vertex c is the singleton in $\mathcal{C}$.

Let z be a root of length at least 5 and different from a'. We want to show that z does not belong to a rich antichain. As above, we denote by $\Phi(z)$ the set of positive roots which are not comparable with z. We claim that $\Phi(z)$ has width at most 3 $(= n-3)$. Note that the simple root corresponding to the branching vertex c belongs to $\Phi(z)$. Thus, only the five chains in $\mathcal{C}$ which are not singletons have to be considered.

Here is Φ_{45}. Any element of $\Phi_5 \setminus \{a'\}$ has been connected to two elements of Φ_4.



As above, we use $\mathcal{C}$ in order to define chains J_y for $y \in \Phi_5$ (with $J_{a'} = \{a'\}$) as well as chains I_x for $x \in \Phi_4$. It follows that the elements outside of $\Phi(z)$ belong to three chains which are combined from three of the chains of the form J_y and three of the chains of the form I_x.

This completes the proof. $\square$

4. The poset $\mathcal{R}(\Delta)$ of rich antichains of $\Phi_+(\Delta)$.

If P is a finite poset and t a non-negative integer, let $\mathcal{A}_t(P)$ be the set of antichains in P of cardinality t. For a poset P of width n, we call the antichains of cardinality $n-1$ the rich antichains. Thus, for Δ a Dynkin diagram of rank n, we write $\mathcal{R}(\Delta) = \mathcal{A}_{n-1}(\Phi_+(\Delta))$.

Given a Dynkin diagram Δ of rank n with root poset Φ_+, one knows that $|\mathcal{A}_t(\Phi_+)| = |\mathcal{A}_{n-t}(\Phi_+)|$, for $0 \le t \le n$, see [At]. In particular, since $A_1(\Phi_+) = \Phi_+$, we always have

$$|\mathcal{R}(\Delta)| = |\Phi_+|,$$

and one may ask whether it is possible to recover also the partial ordering of $\Phi_+(\Delta)$ by looking at the set $\mathcal{R}(\Delta)$ of rich antichains in $\Phi_+(\Delta)$.

Let us point out, that *it is not possible in general to recover the partial ordering of* Φ_+ *by looking at the set of rich antichains.* Namely, consider the Dynkin type $\mathbb{F}_4$. According to Theorem 2, $\mathcal{R}(\mathbb{F}_4)$ is the poset of rich antichains in $\Lambda(\Phi_5(\mathbb{F}_4))$. Now the poset $\Lambda(\Phi_5(\mathbb{F}_4))$ has an automorphism ϕ of order 2, and ϕ induces a non-trivial automorphism on $\mathcal{R}(\mathbb{F}_4)$ (since obviously there are rich antichains which are not invariant under ϕ), whereas $\Phi_+(\mathbb{F}_4)$ itself has no non-trivial automorphisms.

Note that given a poset P, there are several ways to consider $\mathcal{A}_t(P)$ as a poset: given two antichains x, y, we write $x \le_\Lambda y$ provided x lies in the ideal generated by y; similarly, we write $x \le_V y$ provided y lies in the coideal generated by x. Of course, there is also the possibility to combine the two partial orderings, namely to set $x \le y$ provided both $x \le_\Lambda y$ and $x \le_V y$, thus provided both $\Lambda(x) \subseteq \Lambda(y)$ and $V(x) \supseteq V(y)$. Actually, it turns out that for $\mathcal{R}(\Delta)$, there is the following observation:

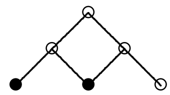
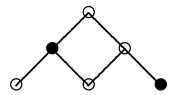
Proposition. *Let A and B be rich antichains of Φ_+, with $A \le_V B$. Then we can label the elements of A and B as $A = \{a_1, \dots, a_{n-1}\}$, $B = \{b_1, \dots, b_{n-1}\}$ with $a_i \le b_i$ for $1 \le i \le n-1$. As a consequence, $A \le_\Lambda B$.*

Proof. Let $\Phi_+ = \Phi_+(\Delta)$, where Δ is a (connected) Dynkin diagram of rank n. We assume that A, B are rich antichains of Φ_+ with $A \le_V B$. Let z be a simple root which does not belong to A. Since $B \subseteq V(A)$, it follows that z does not belong to B. According to the Addendum of Theorem 1, there is a conical decomposition $\mathcal{C}$ such that $\{z\}$ is one of the elements of $\mathcal{C}$. Let $C_1, \dots, C_{n-1}$ be the remaining chains in $\mathcal{C}$. Since z does not belong to A, the elements of A belong to the chains C_i with $1 \le i \le n-1$, the elements a of A can be labeled as $a_1, \dots, a_{i-1}$ with $a_i \in C_i$, for $1 \le i \le n-1$. Similarly, the elements of B can be labeled as $b_1, \dots, b_{i-1}$ with $b_i \in C_i$, for $1 \le i \le n-1$. We claim that $a_i \le b_i$ for all i. Otherwise, there is some i with $a_i > b_i$. Since $B \subseteq V(A)$, there is some $a_j \in A$ with $b \ge a_j$. But $a_i > b_i \ge a_j$ is impossible, since A is an antichain. This is the first assertion.

Of course, if $A = \{a_1, \ldots, a_{n-1}\}$, $B = \{b_1, \ldots, b_{n-1}\}$ with $a_i \le b_i$ for all i, then $A \le_\Lambda B$ (as well as $A \le_V B$). $\square$

We always consider $\mathcal{R}(\Delta)$ as a poset using the partial ordering $\le$ (having in mind that this is the same as the partial ordering $\le_V$).

Remark. If A and B are rich antichains of Φ_+ with $A \le_\Lambda B$, then we may not have $A \le_V B$. Here is the first example: $\Delta = \mathbb{A}_3$ and

$A =$  $B =$ 

Proof of Theorem 3. We are going to present only the essential steps. But we hope that the arguments which we provide shed some further light on the structure of the individual root posets. In case Δ has rank 2, the rich antichains of Φ_+ are just the singletons, thus in this case $\Phi_+(\Delta) = \mathcal{R}(\Delta)$. In particular, we do not have to consider the case $\mathbb{G}_2$.

If Δ is a Dynkin diagram and $t \ge 1$, we denote by $\Phi(\Delta, t)$ the set of positive roots with coefficients bounded by t such that at least one coefficient is equal to t. Of course, Φ_+ is the disjoint union of the subsets $\Phi(\Delta, t)$ with $t \ge 1$. Note that $\Phi(\Delta, 1)$ is always an ideal of Φ_+ (the ideal generated by the minimal sincere root: all its coefficients are equal to 1), whereas the remaining non-empty subposets $\Phi(\Delta, t)$ are intervals. It should be stressed that for Δ equal to $\mathbb{B}_n$ and $\mathbb{C}_n$, the subsets $\Phi(\Delta, 2)$ of Φ_+ are the same despite the fact that for any element x of Φ_+ its coefficients may depend on whether we deal with $\mathbb{B}_n$ or $\mathbb{C}_n$.

We are trying to exhibit maps $R = R(\Delta, t) \colon \Phi(\Delta, t) \to \mathcal{R}(\Delta)$ which combine to a bijection $\Phi_+ \to \mathcal{R}(\Delta)$. All our maps $R(\Delta, t)$ will be injective and the images of $R(\Delta, t)$ and $R(\Delta, t')$ will be disjoint for $t \ne t'$. Thus, if for some Δ, the maps $R(\Delta, t)$ are defined for all t (this will be the case for $\Delta = \mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6$ and $\mathbb{F}_4$), then the union of the images has to be all of $\mathcal{R}(\Delta)$, since it is well-known that $|\mathcal{R}(\Delta)| = |\Phi_+|$. Thus, we do not bother to verify surjectivity assertions.

The bijections $R = R(\Delta, 1) \colon \Phi(\Delta, 1) \to \mathcal{R}(\Phi_{12})$.

Let r be in $\Phi(\Delta, 1)$, thus r is uniquely determined by its support $\operatorname{supp} r$, this is the set of simple roots x with $x \le r$, it is a connected subdiagram of Δ. We define $R(r)$ as follows:

$$R(r) = R(r)_1 \cup (r)_2, \quad \text{where} \quad R(r)_1 = \{x \in \Phi_1 \mid x \not\le r\},\ R(r)_2 = \{y \in \Phi_2 \mid y \le r\}.$$

It is easy to see that $R(r)$ is an antichain: both $R(r)_1$ and $R(r)_2$ are antichains, and if $x \in R(r)_1$ and $y \in R(r)_2$, then $x \not< y$ (since $x < y$ would imply $x < y \le r$, but $x \not\le r$). Also, $R(r)$ is rich (namely, if r has length m and Δ has rank n, then $R(r)_1$ has cardinality $n - m$ and $R(r)_2$ has cardinality $m - 1$).

Finally, we claim: *R preserves and reflects the partial orderings* (if r, r' are roots in $\Phi(\Delta, 1)$, then $r \le r'$ if and only if $R(r') \le_V R(r)$).

Here is the proof. Of course, $r \le r'$ if and only if $\operatorname{supp} r \subseteq \operatorname{supp} r'$. First, assume that $\operatorname{supp} r \subseteq \operatorname{supp} r'$. We want to show that $R(r') \subseteq V(R(r))$. We have $R(r')_1 \subseteq R(r)_2$, thus, it remains to show that any element $y \in R(r')_2$ belongs to $V(R(r))$. Let a, b be the support of y. If a is not in the support of r, then $a \in R(r)$, thus $y \in V(R(r))$. Similarly, if b is not in the support of r, then $y \in V(R(r))$. Finally, if both a, b are in the support of r, then $y \in R(r)_2 \subseteq V(R(r))$.

Second, assume that $R(r') \subseteq V(R(r))$. We have to show that $\operatorname{supp} r \subseteq \operatorname{supp} r'$. Let x belong to $\operatorname{supp} r$ and assume that x does not belong to $\operatorname{supp} r'$. Then $x \in R(r')_1 \subseteq V(R(r))$. Since x is a minimal element, it follows that $x \in R(r)$, thus x belongs to $R(r)_1$ and therefore $x \notin \operatorname{supp} r$, a contradiction. $\square$

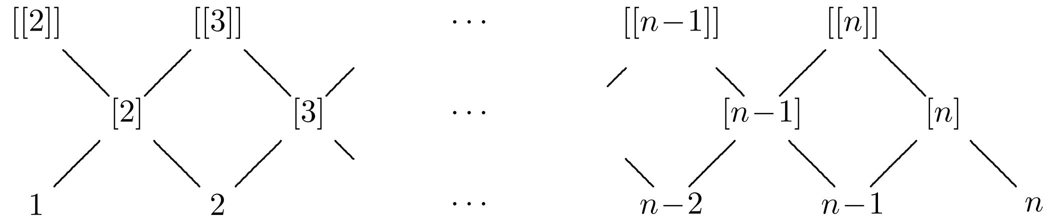
Case $\Delta = \mathbb{A}_n$. In this case, $\Phi_+ = \Phi(\Delta, 1)$, thus $R = R(\Delta, 1)$ provides an isomorphism between $\Phi_+(\mathbb{A}_n)$ and $\mathcal{R}(\mathbb{A}_n)$.

Let us now consider the posets $\Phi(\Delta, t)$ with $t \ge 2$. Whereas for $t = 1$, we were able to provide a general recipe, we now have to work case by case.

Case $\Delta = \mathbb{B}_n$. The set $\Phi(\mathbb{B}_n, 2)$ is just the coideal of Φ_+ generated by the maximal root of $\mathbb{B}_2$ (considered as a subdiagram of $\mathbb{B}_n$).

We start with the poset $P = \Phi(\mathbb{B}_n)_{13}$, since the rich antichains of $\Phi_+(\mathbb{B}_n)$ are contained in P.

P



The rich antichains in P_{12} form the poset $\mathcal{R}(\mathbb{A}_n)$; its elements are of the form $R(r)$ where r is a root with coefficients 0 and 1. We have to determine the additional rich antichains (it is obvious that such an antichain has to contain the vertex $[[2]]$).

We consider the roots of $\mathbb{B}_n$ of the form

$$r = r(i, j) = (2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$$

with i coefficients being 2, then $j - i$ coefficients being 1 and finally $n - j$ coefficients equal to 0, where $0 \le i < j \le n$. The roots of the form $r(i, j)$ with $1 \le i < j \le n$ are just the elements of $\Phi(\mathbb{B}_n, 2)$.

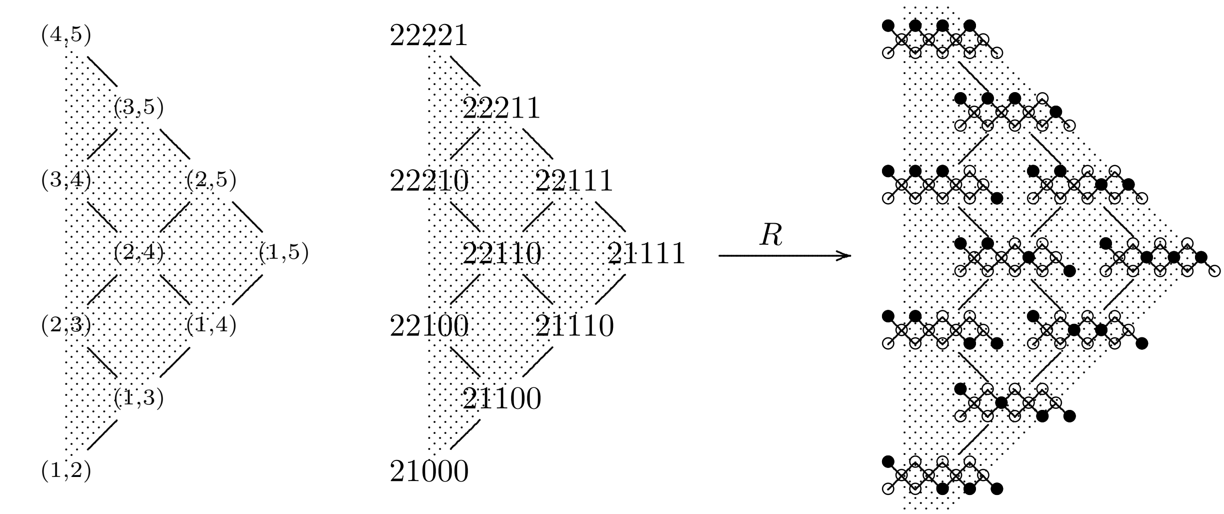
For $0 \le i < j \le n$, we may also consider the subset

$$R(i, j) = \{[[2]], \dots, [[i+1]], [i+2], \dots, [j], j+1, \dots, n\}.$$

of $\Phi_+(\mathbb{B}_n)$ (with i roots of length 2, $j - i - 1$ roots of length 1, and $n - j$ roots of length 1). It is obvious that $R(i, j)$ is a rich antichain.

We define $R = R(\mathbb{B}_n, 2)\colon \Phi(\mathbb{B}_n, 2) \to \mathcal{R}(\mathbb{B}_n)$ by $R(r(i,j)) = R(i,j)$.

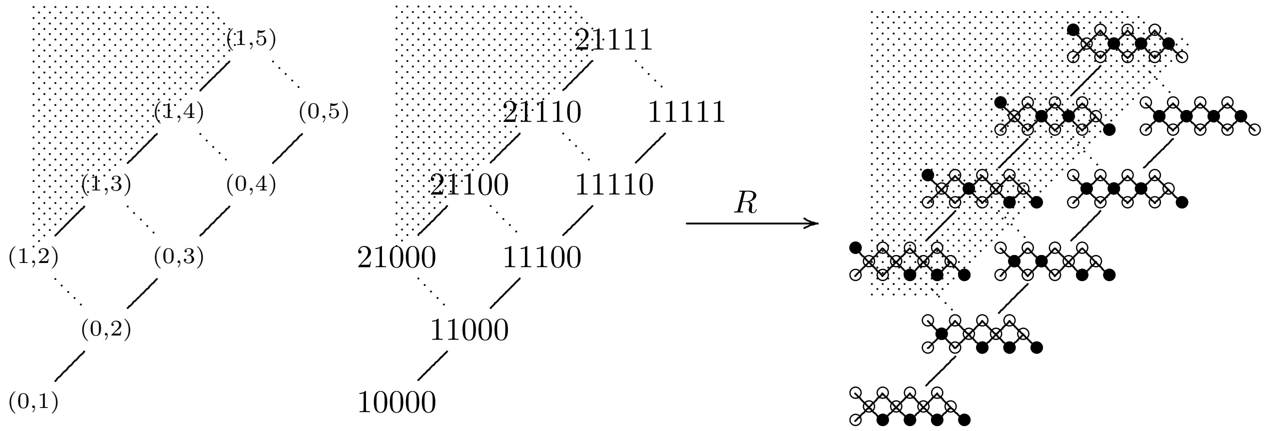
Here is the case $n = 5$. On the left, we draw the set of indices (i,j). The map R provides the following bijection:



The labels (i,j) $\qquad r(i,j) \qquad R(i,j)$

One checks without difficulties that the map R preserves and reflects the respective partial orderings.

It remains to see in which way the partial ordered sets $\Phi(\mathbb{B}_n, 1)$ and $\Phi(B_n, 2)$ as well as the images of $R(\mathbb{B}_n, 1)\colon \Phi(\mathbb{B}_n, 1) \to \mathcal{R}(\mathbb{B}_n)$ and $R(\mathbb{B}_n, 2)\colon \Phi(\mathbb{B}_n, 2) \to \mathcal{R}(\mathbb{B}_n)$ are connected. This concerns the following neighbors, drawn by dotted lines:



The labels (i,j) $\qquad r(i,j) \qquad R(i,j)$

It turns out that the bijections $R(\mathbb{B}_n, 1)$ and $R(\mathbb{B}_n, 2)$ combine to provide an isomorphism $\Phi_+(\mathbb{B}_n) \to \mathcal{R}(\mathbb{B}_n)$.

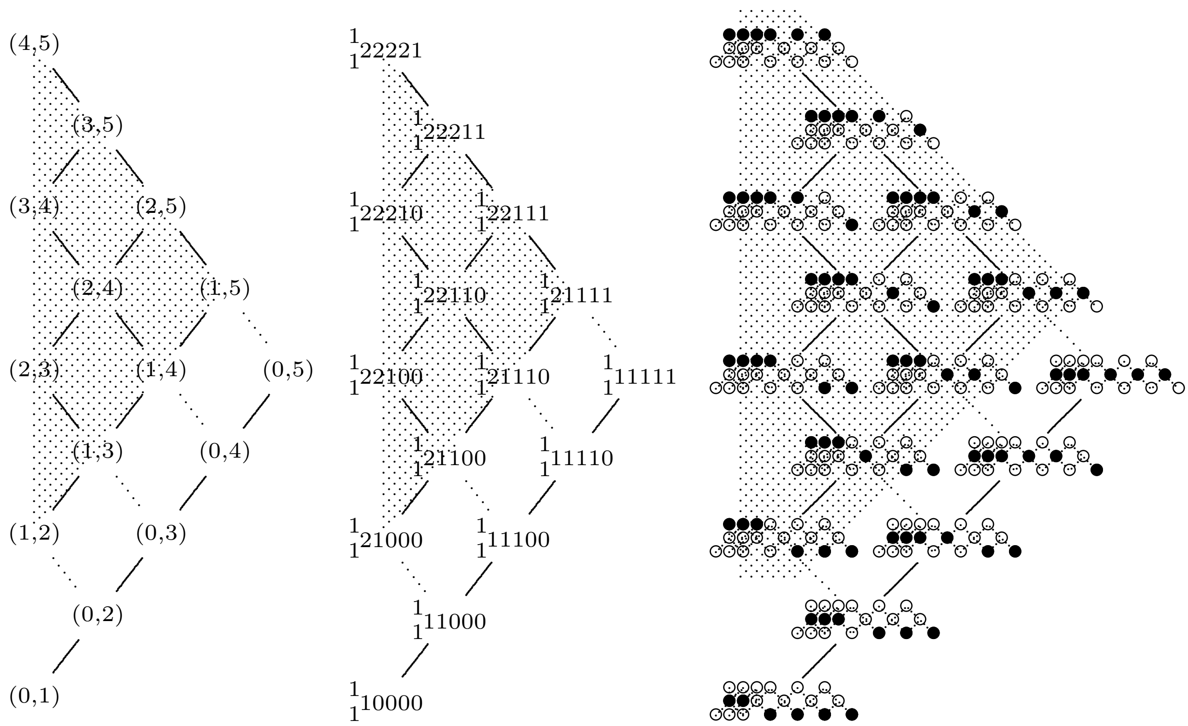
Case $\mathbb{D}_n$. This case is similar to $\mathbb{B}_n$. The set $\Phi(\mathbb{D}_n, 2)$ is just the coideal of Φ_+ generated by the maximal root of $\mathbb{D}_4$. For $0 \le i < j \le n-2$, let $r = r(i,j)$ be the root of $\mathbb{D}_n$ of the form

$$r = r(i,j) = \begin{pmatrix} 1 & & & & & & & & \\ & 2 & \cdots & 2 & 1 \cdots & 1 & 0 \cdots & 0 \\ 1 & & & & & & & & \end{pmatrix}$$

with i coefficients being 2 and $j-i+2$ coefficients being 1 (to be precise: with coefficients 1 on the short arms, and $n-j-2$ coefficients equal to 0 on the long arm). The roots of the form $r(i,j)$ with $1 \le i < j \le n-2$ are just the elements of $\Phi(\mathbb{D}_n, 2)$.

For $0 \le i < j \le n-2$, we also consider a corresponding subset $R(i,j)$ of $\Phi_+(\mathbb{D}_n)$. For $i = 0$, $R(i,j)$ is the rich antichain which contain the roots $\begin{pmatrix} 1 & \\ & 1\ 0 \cdots 0 \\ 0 & \end{pmatrix}$ and $\begin{pmatrix} 0 & \\ & 1\ 0 \cdots 0 \\ 1 & \end{pmatrix}$, $j-1$ additional roots of height 2 and $n-j-2$ simple roots. For $i > 0$, $R(i,j)$ contains the $\Phi(\mathbb{D}_4)_3$, $j-i-1$ roots of height 2 and $n-j-2$ simple roots. We define $R\colon \Phi(\mathbb{D}_n, 2) \to \mathcal{R}(\mathbb{D}_n)$ by $R(r(i,j)) = R(i,j)$.

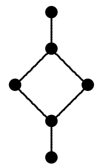
Here is the case $n = 7$. On the left, we draw the set of indices (i,j). The map R provides the following bijection:



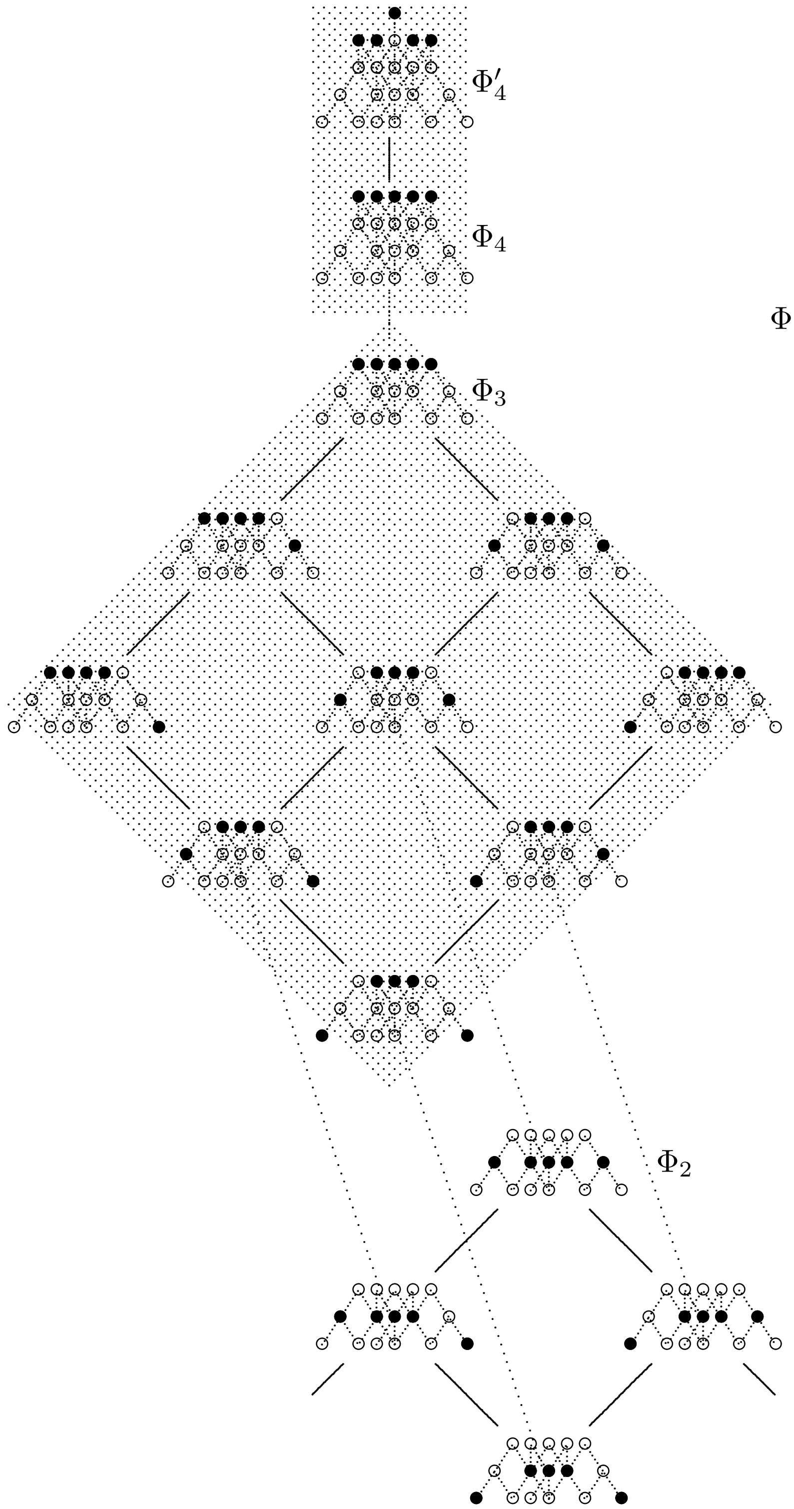
The labels (i,j) $\qquad r(i,j) \quad \overset{R}{\longmapsto} \quad R(i,j)$

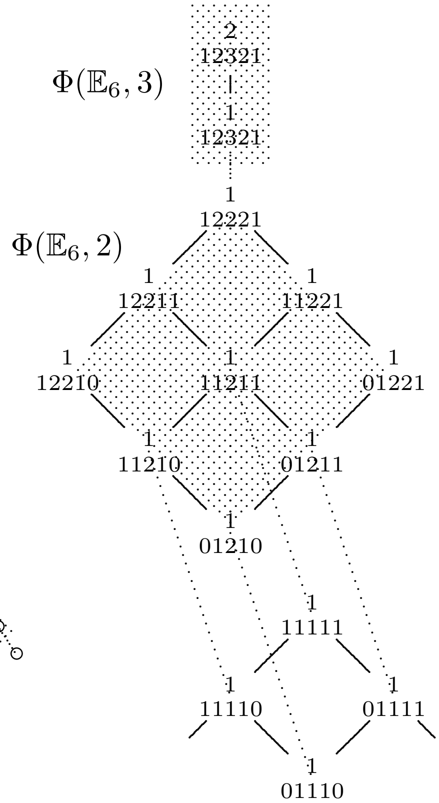
As in the case $\mathbb{B}_n$, we also see in the case $\mathbb{D}_n$ that the bijections $R(\mathbb{D}_n, 1)$ and $R(\mathbb{D}_n, 2)$ combine to provide an isomorphism $\Phi_+(\mathbb{D}_n) \to \mathcal{R}(\mathbb{D}_n)$.

The cases $\Delta = \mathbb{E}_6$ and $\Delta = \mathbb{E}_7$. Here, we have to consider (besides $\Phi(\Delta, 1)$) the subsets $\Phi(\Delta, 2)$ and $\Phi(\Delta, 3)$, and finally also $\Phi_(\mathbb{E}_7, 4)$. Note that the posets $\Phi(\mathbb{E}_6, 2)$, $\Phi(\mathbb{E}_7, 2)$ as well as $\Phi(\mathbb{E}_7, 3)$ are products of the form $[3] \times [3]$, $[3] \times W(4)$, and $[2] \times [4]$, respectively, where $[m]$ denotes the chain of cardinality m and where $W(4)$ is the poset



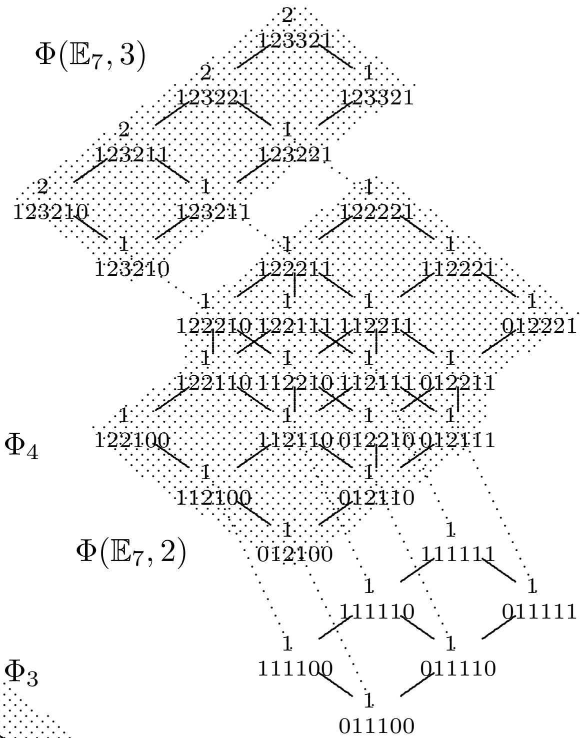
The case $\mathbb{E}_6$.

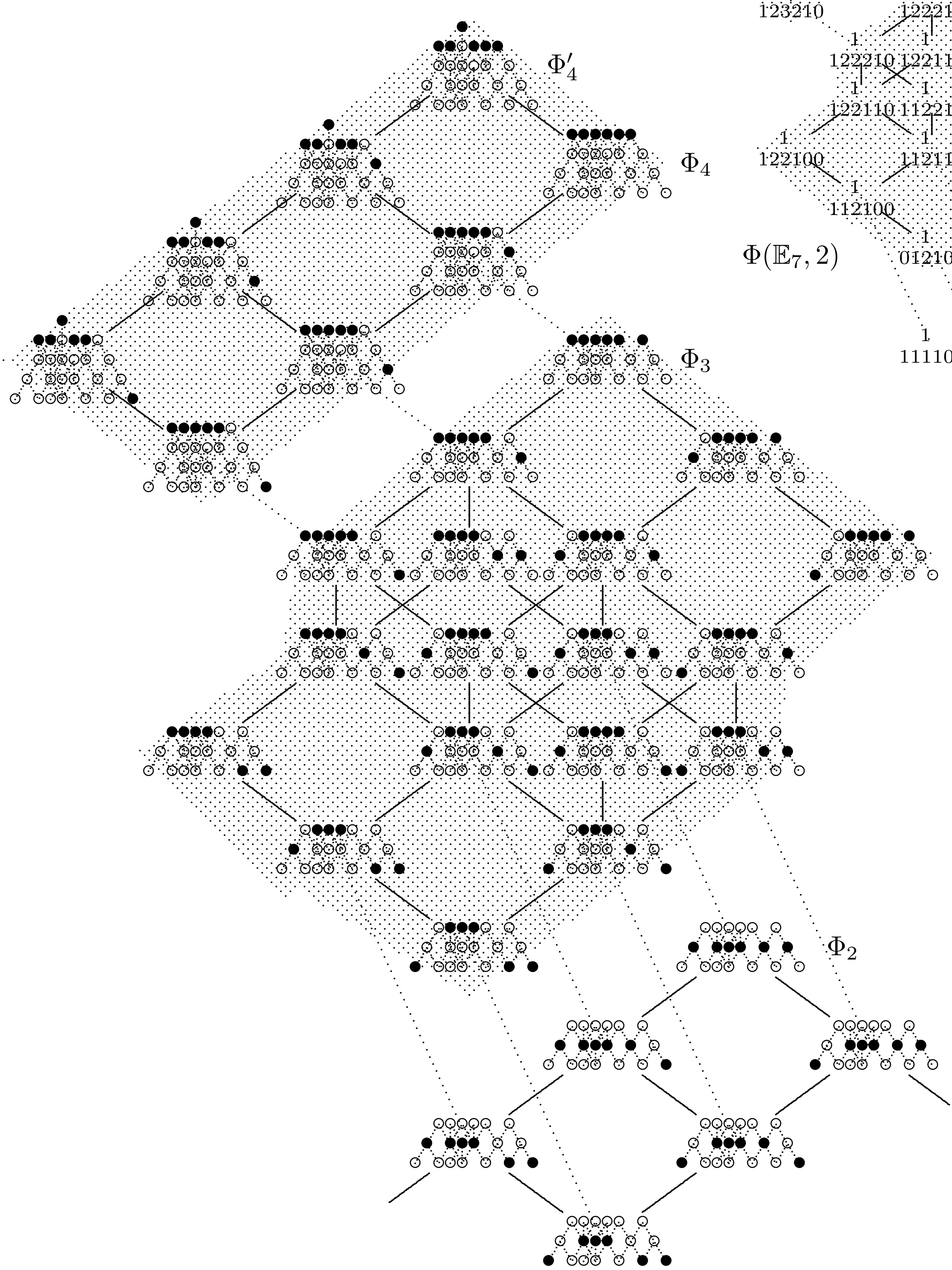




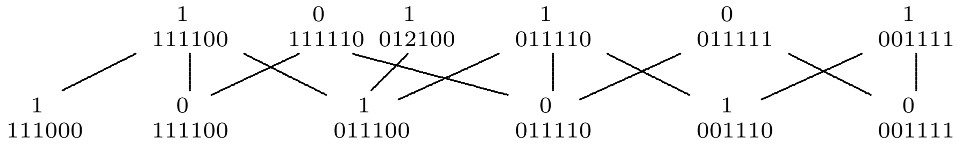
We see that there is a poset isomorphism between $\Phi_+(\mathbb{E}_5)$ and $\mathcal{R}(\mathbb{E}_6)$.

The case $\mathbb{E}_7$.

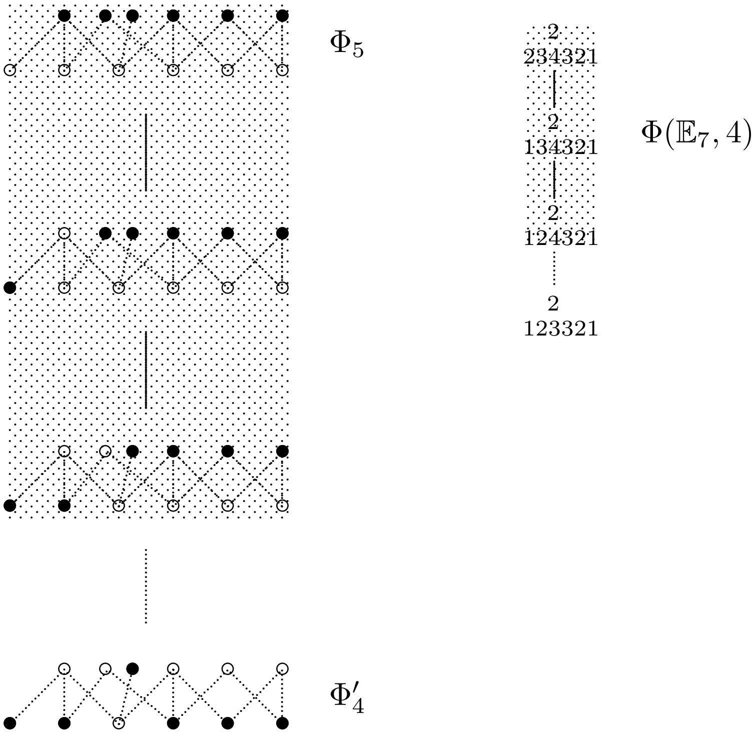




It remains to present the rich antichains between Φ'_4 and Φ_5. They are contained in $\Phi_{4,5}$:

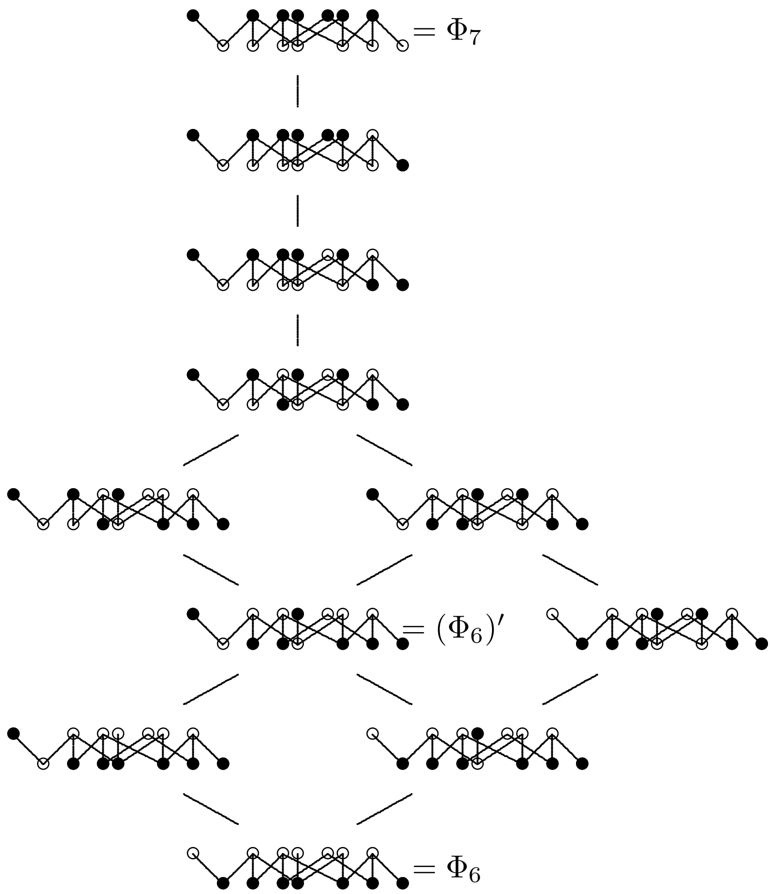


Here are the rich antichains between Φ'_4 and Φ_5:



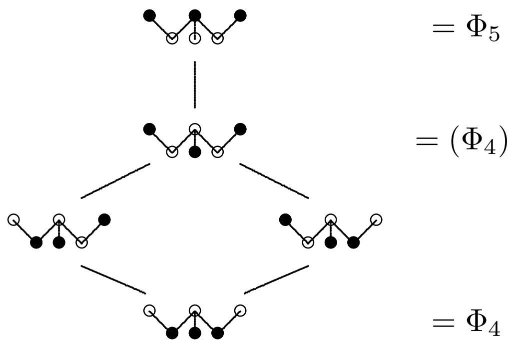
Altogether we see that there is a poset isomorphism between $\Phi_+(\mathbb{E}_7)$ and $\mathcal{R}(\mathbb{E}_7)$.

Case $\mathbb{E}_8$. Let $\Phi_+ = \Phi_*(\mathbb{E}_8)$. Let us exhibit a solid subposet of $\mathcal{R}(\mathbb{E}_8)$ consisting of rich antichains in Φ_{67}:



According to Theorem 1, Φ_7 is the maximal element of $\mathcal{R}(\mathbb{E}_8)$. Thus, we see that the dual poset $(\mathcal{R}(\mathbb{E}_8))^*$ has two different elements of height 5, whereas $(\Phi_+)^*$ has just one element of height 5. This shows that $\mathcal{R}(\mathbb{E}_8)$ is not isomorphic to Φ_+.

Case $\mathbb{F}_4$. Let $\Phi_+ = \Phi_*(\mathbb{F}_4)$. Let us exhibit a solid subposet of $\mathcal{R}(\mathbb{F}_4)$ consisting of rich antichains in Φ_{45}:



According to Theorem 1, Φ_5 is the maximal element of $\mathcal{R}(\mathbb{F}_4)$. Thus, we see that the dual poset $(\mathcal{R}(\mathbb{F}_4))^*$ has two different elements of height 3, whereas $(\Phi_+)^*$ has just one element of height 3. This shows that $\mathcal{R}(\mathbb{F}_4)$ is not isomorphic to Φ_+.

This completes the proof of Theorem 3. □

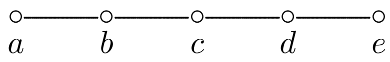
Appendix: Some pictures of the root posets.

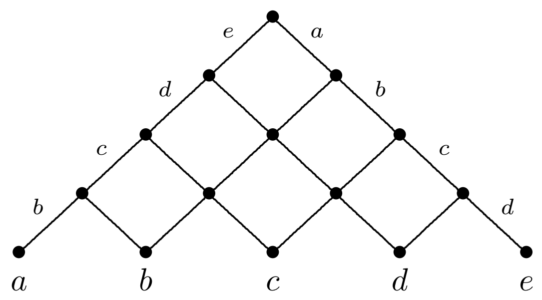
We exhibit a visualization of the root posets $\mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and $\mathbb{F}_4$ which we found useful when preparing this note. Indeed, the pictures which we found in the literature and in the web did not seem to be quite convincing (for us). The pictures shown below draw the attention to the fact that for any pair of elements $x, z \in \Phi_+$, the interval $\{y \in \Phi_+ \mid x \le y \le z\}$ is a distributive lattice which can be constructed in a convenient way using segments, squares and (3-dimensional) cubes. A drawback of our visualization is that it does not take into account the diversity of the various edges (which otherwise could be indicated by using different slopes). As a remedy, we label the edges by the corresponding basis vectors (it is sufficient to do this at the boundary, since the modularity transfers this information to the remaining edges).

For the sake of completeness, we also include the cases $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n$ and $\mathbb{G}_2$. It is well-known that the root posets $\mathbb{B}_n$ and $\mathbb{C}_n$ do not differ as long as we do not refer to the labels of the edges. But since we present the cases $\mathbb{D}_n, \mathbb{E}_m$ and $\mathbb{F}_4$ with labels, we do it also for $\mathbb{B}_n$ and $\mathbb{C}_n$, thus we have to present these types separately.

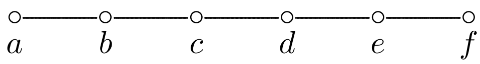
The cubical pictures for $\mathbb{D}_n, \mathbb{E}_m$ and $\mathbb{F}_4$ stress a division of the positive roots into "levels" which seems to be a kind of measure of the complexity of a positive root. The roots belonging to a fixed level form a planar graph, often a rectangle. All levels have a unique maximal element. Level 1 has $n-1$ minimal elements, all other levels have a unique minimal element. We list the minimal and the maximal elements, as well as the number of roots belonging to the level. A further analysis of the root posets will be provided in [R3].

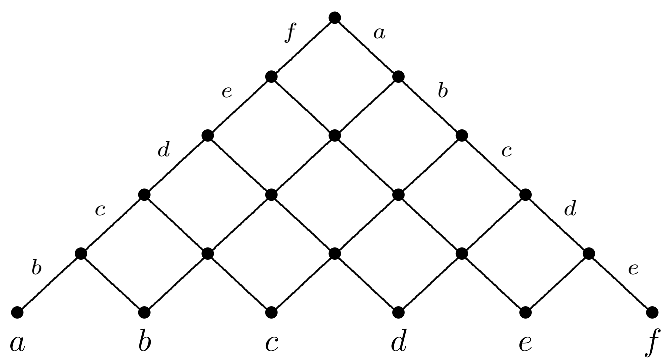
The root poset $\mathbb{A}_5$.



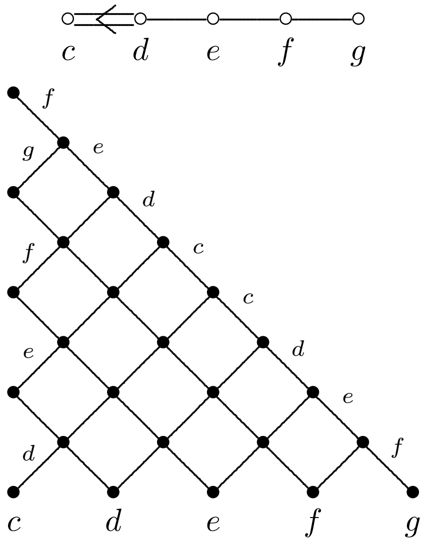


The root poset $\mathbb{A}_6$.

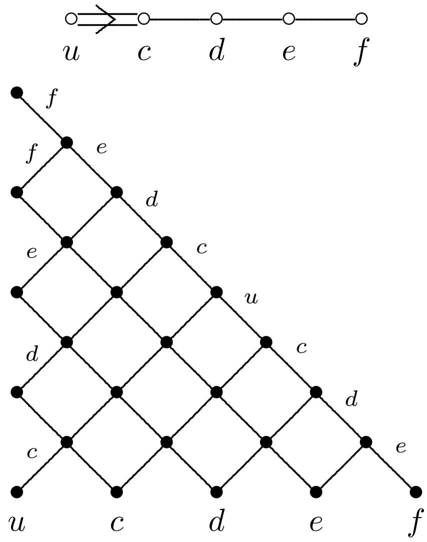




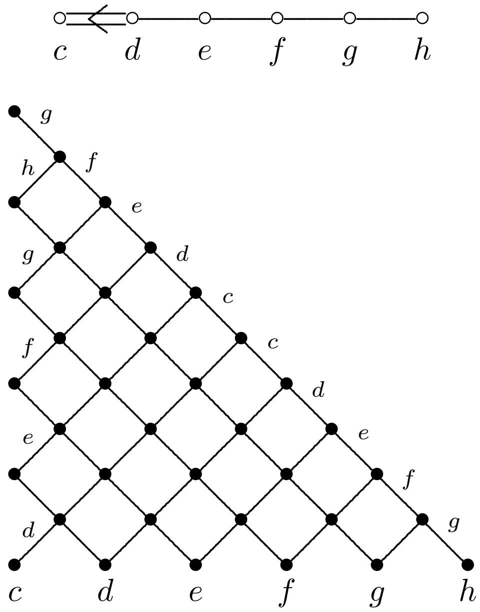
The root poset $\mathbb{B}_5$.



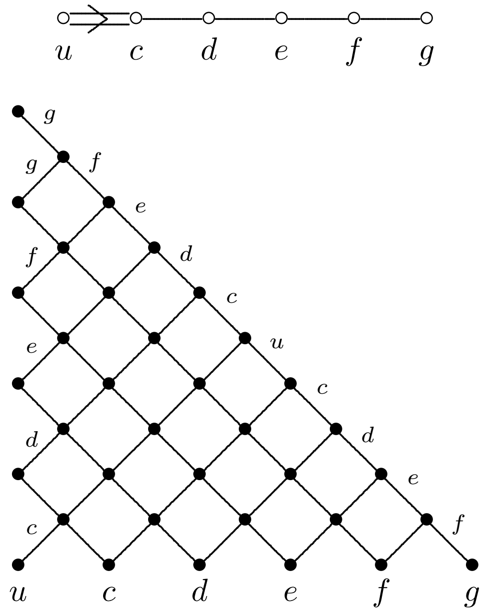
The root poset $\mathbb{C}_5$.



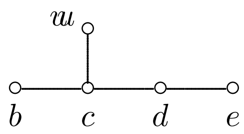
The root poset $\mathbb{B}_6$.

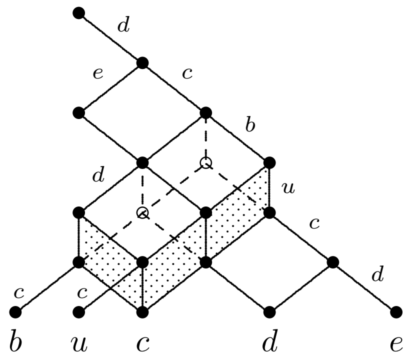


The root poset $\mathbb{C}_6$.

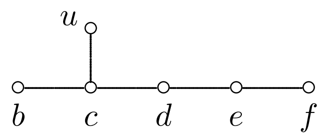


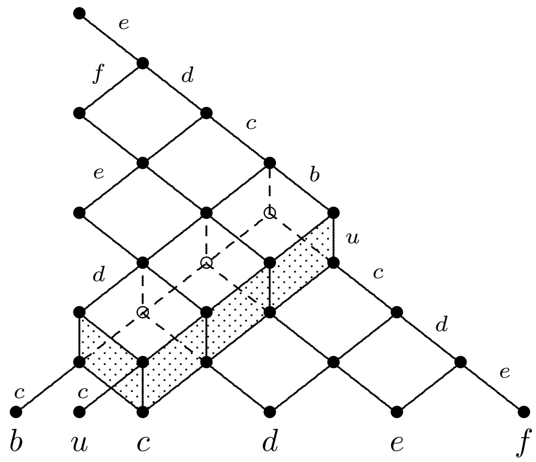
The root poset $\mathbb{D}_5$.





The root poset $\mathbb{D}_6$.

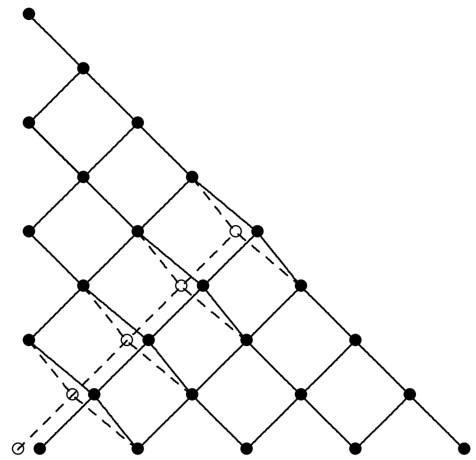




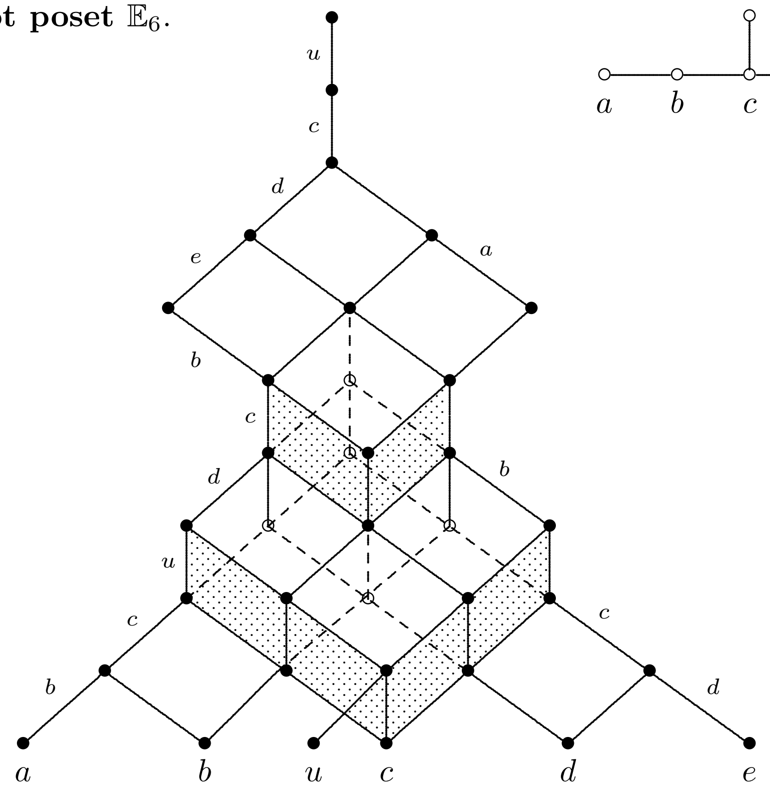
The levels for $\mathbb{D}_n$.

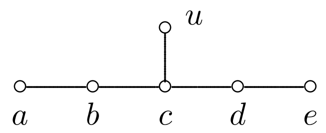
Level	Conditions	Minimal elements	Maximal element	number
1	$u = 0$	$n-1$ simple roots	$\begin{smallmatrix} & 0 & & & & \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{smallmatrix}$	$\binom{n}{2}$
2	$u = 1$	$\begin{smallmatrix} & 1 & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & & & \\ 1 & 2 & \cdots & 2 & 2 & 1 \end{smallmatrix}$	$\binom{n}{2}$

Unfortunately, the cubical pictures hides the symmetry given by the automorphism of order 2. Thus, one may want to “squeeze” the picture slightly. Here is the case $n = 6$:



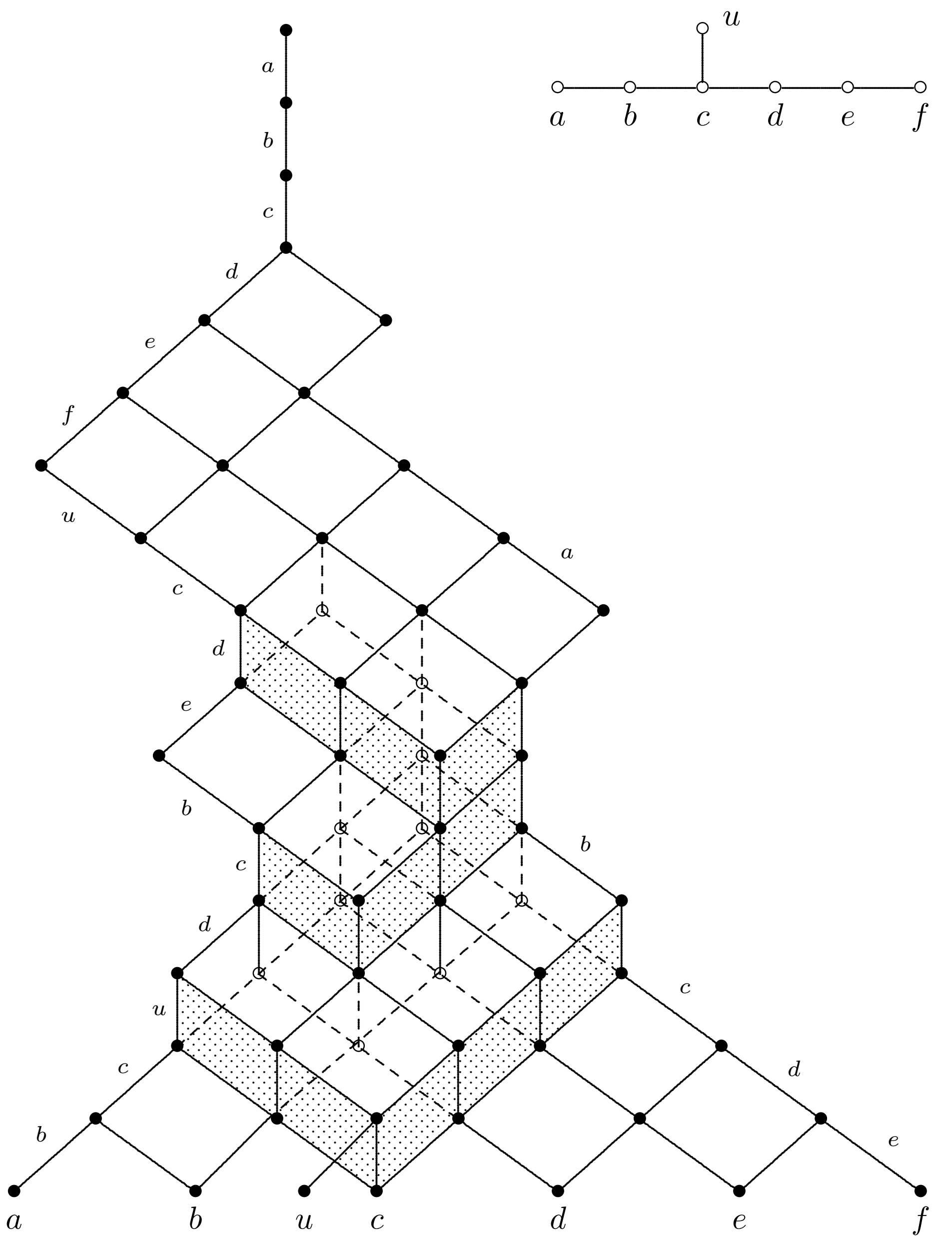
The root poset $\mathbb{E}_6$.





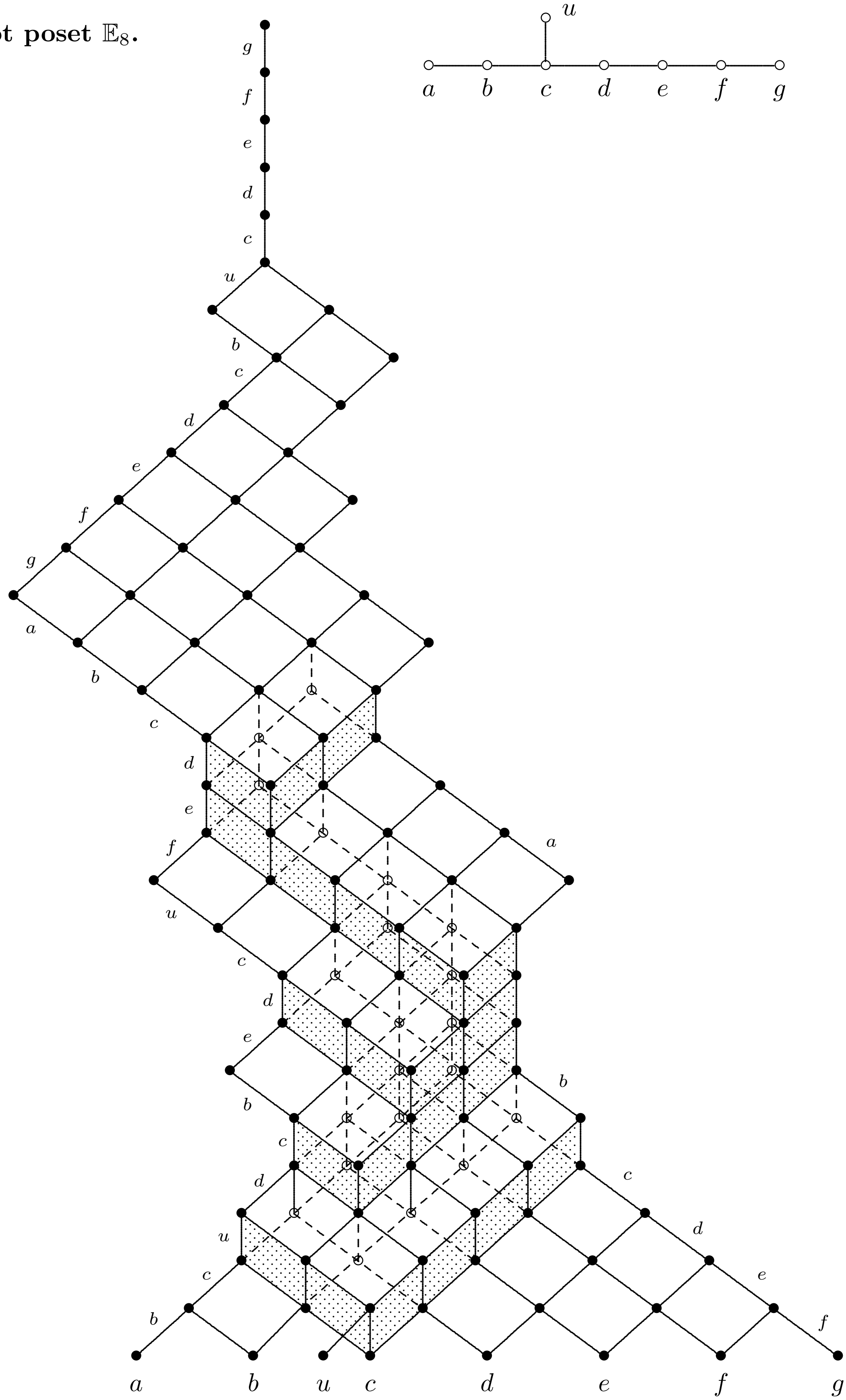
Level	Conditions	Minimal elements	Maximal element	number
1	$u = 0$	5 simple roots	$\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	15
2	$u = 1$	$\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	$3 \times 3 + 1$
3	$c \geq 2$	$\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$	$3 \times 3 + 2$

The root poset $\mathbb{E}_7$.



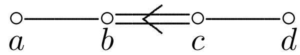
Level	Conditions	Minimal elements	Maximal element	number
1	$u=0$	6 simple roots	0 1 1 1 1 1 1	21
2	$u=1,\ c\le 1$	1 0 0 0 0 0 0	1 1 1 1 1 1 1	$3\times 4+1$
3	$c=2,\ d=1$	1 0 1 2 1 0 0	1 1 2 2 1 1 1	3×3
4	$d\ge 2$	1 0 1 2 2 1 0	2 2 3 4 3 2 1	$5\times 3+5$

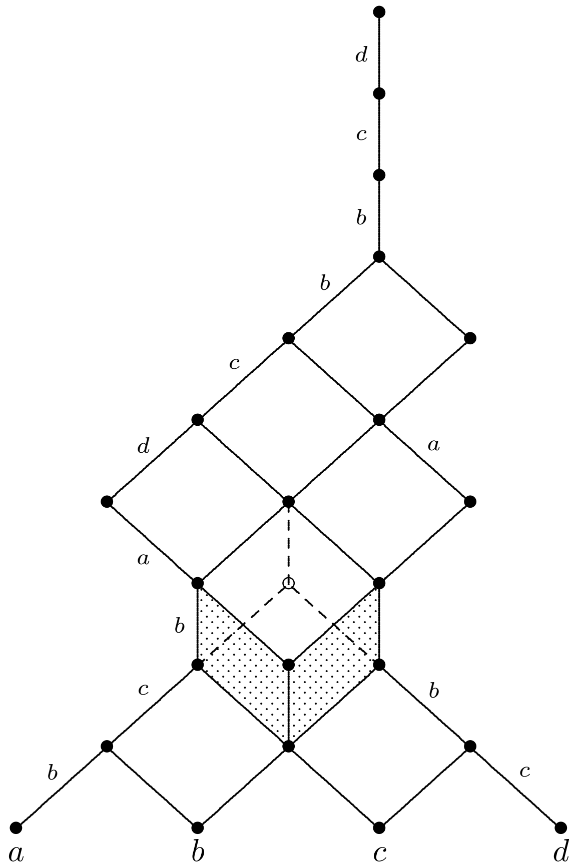
The root poset $\mathbb{E}_8$.



Level	Conditions	Minimal elements	Maximal element	number
1	$u=0$	7 simple roots	0 1 1 1 1 1 1 1	28
2	$u=1,\ c\le 1$	1 0 0 0 0 0 0 0	1 1 1 1 1 1 1 1	$3\times 5+1$
3	$c=2,\ d=1$	1 0 1 2 1 0 0 0	1 1 2 2 1 1 1 1	3×4
4	$d=2,\ e=1$	1 0 1 2 2 1 0 0	2 1 2 3 2 1 1 1	5×3
5	$d=2,\ e=3$	1 0 1 2 2 2 1 0	2 1 2 3 2 2 2 1	5×3
6	$d\ge 3$	1 1 2 3 3 2 1 0	3 2 4 6 5 4 3 12	$5\times 4+14$

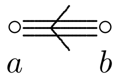
The root poset $\mathbb{F}_4$.

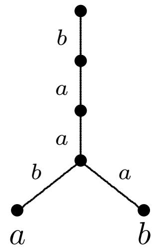




Level	Conditions	Minimal elements	Maximal element	number
1	$b\le 1$	4 simple roots	1 1 1 1	10
2	$b\ge 2$	0 2 1 0	2 4 3 2	$3\times 3+5$

The root poset $\mathbb{G}_2$.





References.

[Ar] Armstrong, D.: Generalized noncrossing partitions and combinatorics of Coxeter groups. Memoirs of the Amer. Math. Soc. 949 (2009).

[At] Athanasiadis, C. A.: On a refinement of the generalized Catalan numbers for Weyl groups. Trans. Amer. Math. Soc. 357 (2004), 179-196.

[B] Bourbaki, N.: Groupes et Algebres de Lie. IV-VI. Hermann (1968).

[H] Humphreys, J. E.: Reflection groups and Coxeter groups. Cambridge University Press (1990).

[S] Richard Stanley, Enumerative Combinatorics, vol.1, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, 1995,

[R1] Ringel, C. M.: The $(n-1)$-antichains in a root poset of width n. arXiv:1306.1593v1

[R2] Ringel, C. M.: The Catalan combinatorics of the hereditary artin algebras. In: Recent Developments in Representation Theory, Contemp. Math., 673, Amer. Math. Soc., Providence, RI, 2016.

[R3] Ringel, C. M.: Root posets and hammocks. In preparation.

Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld
POBox 100 131, 33 501 Bielefeld, Germany
`e-mail: ringel@math.uni-bielefeld.de`