Lecture Notes

Introduction to Stochastic Analysis

Michael Röckner

Universität Bielefeld
Version: 2. März 2012
corrected until and including p. 130

This script has been written by Matthias Stephan on behalf of Prof. M. Röckner and has been revised by Dr. Judith Dohmann. Please note that despite several checks it may still contain mistakes and, therefore, there is no guarantee of correctness. A report of any kind of errors found in these notes is appreciated (mstephan@math.uni-bielefeld.de).

Inhaltsverzeichnis

1. Introduction to Pathwise Itô-Calculus

1.1. Preparation

Definition 1.1.1. *$X : [0,\infty[\to \mathbb{R}$ has* bounded variation*, if for all $t \geq 0$*

$$\operatorname{var}_t X := \sup_{\tau} \sum_{t_i \in \tau} \left| X\left(t_{i+1} \wedge t\right) - X\left(t_i \wedge t\right) \right| < \infty, \tag{1.1.1}$$

where τ is a partition $0 = t_0 < t_1 < \ldots < t_N < \infty$.

Notation: Since in Stochastics a process X also depends on ω, we write X_t and $X_t(\omega)$ instead of $X(t)$ and $X(t)(\omega)$ respectively.

As a reference see [dB03].

Definition 1.1.2. *Suppose X to be right-continuous and of bounded variation.*
Let $f \in \mathcal{C}(\mathbb{R})$ and $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of partitions, whose mesh

$$|\tau_n| := \sup_{1 \leq i \leq N_n} \left(t_{i+1}^{(n)} - t_i^{(n)} \right)$$

converges to zero as $n \to \infty$ and $t_{N_n} \overset{n\to\infty}{\longrightarrow} \infty$.
Then, since X is of bounded variation, there exists the Lebesgue-Stieltjes-Integral *defined by*

$$\int_0^t f_s \, dX_s := \lim_{n\to\infty} \sum_{t_i^{(n)} \in \tau_n} f_{t_i^{(n)}} \cdot \left(X_{t_{i+1}^{(n)} \wedge t} - X_{t_i^{(n)} \wedge t} \right). \tag{1.1.2}$$

Remark 1.1.3. *Note that for continuous X this definition is independent of the choice of $(\tau_n)_{n\in\mathbb{N}}$ (Exercise!).*

1.1.1. Quadratic Variation of Brownian Motion

We know (cf. [Röc11]) that a typical path of Brownian motion X on $\mathbb{R}^1$ is of unbounded variation, since for its quadratic variation we have $\langle X \rangle_t = t$ (see 1.1.4(i) below). Nonetheless, one wants to define

$$\int_0^t f_s \, \mathrm{d}X_s$$

for a typical path of the Brownian motion X. More generally, we want to do this for every continuous path with continuous quadratic variation $t \mapsto \langle X \rangle_t$.

$(X_t)_{t\geq 0}$ is a (continuous) $\mathbb{R}$-valued Brownian motion on $(\Omega, \mathcal{F}, P)$ if

(i) The increments $X_t - X_s$ are independent and $N(0, t-s)$ distributed ($t > s$).

(ii) $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$.

Theorem 1.1.4. *Let $(X_t)_{t\geq 0}$ be a (continuous) Brownian motion and $(\tau_n)_{n\in\mathbb{N}}$ a sequence of subdivisions with $|\tau_n| \overset{n\to\infty}{\longrightarrow} 0$, $\tau_n \subset \tau_{n+1}$, $t_{N_n}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty$. Then, for all $t \geq 0$*

(i)

$$\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 \overset{n\to\infty}{\longrightarrow} t \quad P\text{-a.s.}.$$

Here the zero set depends on the sequence $(\tau_n)_{n\in\mathbb{N}}$ (and so far it could depend on t.)

(ii) Moreover,

$$P\Bigg(\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 \overset{n\to\infty}{\longrightarrow} t, \quad \forall t\ge 0\Bigg) = 1.$$

Proof. (i) See [Röc11, Satz 9.4.5].

(ii) Exercise ("sandwich argument", notice that zero set in (i) a-priori depends on t).

□

1.2. Quadratic Variation and Itô's Formula

Fix a continuous and real-valued function $t\mapsto X_t$ on $[0,\infty[$ (in short $(X_t)_{t\ge0}$) with an existing sequence of subdivisions (τ_n) with $|\tau_n| \overset{n\to\infty}{\longrightarrow} 0$ and $t_{N_n}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty$ such that the quadratic variation (along (τ_n))

$$\langle X\rangle_t := \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 = \lim_{n\to\infty}\sum_{t_i^{(n)}\in\tau_n} (X_{t_{i+1}^{(n)}\wedge t} - X_{t_i^{(n)}\wedge t})^2, \quad t\ge 0, \tag{1.2.3}$$

exists for all $t\ge 0$ and such that $t\mapsto\langle X\rangle_t$ is continuous on $[0,\infty)$. Note that $\langle X\rangle_t$ in (1.2.3) could depend upon the choice of $(\tau_n)_{n\ge1}$ in contrast to the limit in (1.1.2) (cf. [RY99, (2.3)-(2.5) p. 27/28]). By definition it is obvious that $t\mapsto\langle X\rangle_t$ is increasing.

Remark 1.2.5. *(i) If $(X_t)_{t\ge0}$ is of bounded variation, then*

$$\sum_{t_i^{(n)}\in\tau_n} \left(X_{t_{i+1}^{(n)}\wedge t} - X_{t_i^{(n)}\wedge t}\right)^2 \le \underbrace{\max_{t_i^{(n)}\in\tau_n}\left|X_{t_{i+1}^{(n)}\wedge t} - X_{t_i^{(n)}\wedge t}\right|}_{<\varepsilon \text{ uniformly for large } n} \underbrace{\sum_{t_i^{(n)}\in\tau_n}\left|X_{t_{i+1}^{(n)}\wedge t} - X_{t_i^{(n)}\wedge t}\right|}_{<\infty} \to 0,$$

hence, $\langle X\rangle\equiv 0$.

Therefore, $\langle X\rangle\not\equiv 0$ implies that $(X_t)_{t\ge0}$ is not of bounded variation and the Lebesgue-Stieltjes-Integral can not be defined in the usual way.

(ii) If $t\mapsto\langle X\rangle_t$ is increasing, continuous and $\langle X\rangle_0 = 0$, hence, $t\mapsto\langle X\rangle_t$ is a distribution function of a measure μ (i.e. $d\mu = d\langle X\rangle_t$) on $([0,\infty),\mathcal{B}([0,\infty)))$, then (1.2.3) is equivalent to: The distribution function F_n of

$$\mu_n := \sum_{t_i^{(n)}\in\tau_n} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 \delta_{t_i^{(n)}}$$

converges pointwise to

$$F(t) := \mu(]-\infty, t]) = \int 1_{]-\infty,t]} \; d\mu.$$

But, since for continuous $\langle X\rangle_t$

$$\mu(\{t\}) = \lim_{n\to\infty} \mu\left(]-\infty, t]\right) - \mu\left(]-\infty, t-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\langle X\rangle_t - \langle X\rangle_{t-\frac{1}{n}}\right) = 0,$$

we have that the pointwise convergence of F_n *to* F *is equivalent to* $\mu_n \to \mu$ *weakly by Portemanteau.*

(iii) Note that if X *is an increasing function, then* X *is always of bounded variation:*

$$\operatorname{var}_t X = \sup_\tau \sum_{t_i\in\tau} |X_{t_{i+1}\wedge t} - X_{t_i\wedge t}| = \sup_\tau \sum_{t_i\in\tau} (X_{t_{i+1}\wedge t} - X_{t_i\wedge t}) = X_t - X_0 < \infty.$$

Therefore, we can define the integral with respect to $\langle X\rangle$ *as a Lebesgue-Stieltjes-Integral.*

Lemma 1.2.6 (Calculating integrals with respect to $\mathrm{d}\langle X\rangle_s$)**.** *Let* $g \in \mathcal{C}([0,\infty))$*. Then*

$$\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} g(t_i)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 \to \int_0^t g(s)\; d\langle X\rangle_s.$$

Proof. The left hand side is equal to $\int g1_{[0,t]}\;\mathrm{d}\mu_n$, whereas the right hand side equals $\int g1_{[0,t]}\;\mathrm{d}\mu$. But the integrand $g1_{[0,t]}$ is μ-a.e. continuous and bounded. Hence, convergence follows by the Portemanteau theorem and Remark 1.2.5 (ii). ☐

Theorem 1.2.7 (Pathwise Itô-formula)**:** Let $F \in \mathcal{C}^2(\mathbb{R})$. Then for all $t \ge 0$ the Itô-Formula is given by

$$F(X_t) - F(X_0) = \int_0^t F'(X_s)\;\mathrm{d}X_s + \frac{1}{2}\int_0^t F''(X_s)\;\mathrm{d}\langle X\rangle_s, \tag{1.2.4}$$

where

$$\int_0^t F'(X_s)\;\mathrm{d}X_s := \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} F'\left(X_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right).$$

$\int_0^t F'(X_s)\;\mathrm{d}X_s$ is called the *(pathwise) Itô-Integral* and depends on $(\tau_n)_{n\in\mathbb{N}}$.

Proof. Consider $(\tau_n)_{n\in\mathbb{N}}$ such that $\langle X\rangle_t$ exists along $(\tau_n)_{n\in\mathbb{N}}$. We apply the Taylor formula to F. Hence, for all $n\in\mathbb{N}$ there exist $\theta_i^{(n)} \in]0,1[$ such that

$$\begin{aligned}
&\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} F\left(X_{t_{i+1}^{(n)}}\right) - F\left(X_{t_i^{(n)}}\right) \left(\overset{n\to\infty}{\longrightarrow} F(X_t) - F(X_0)\right) \\
&= \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} F'\left(X_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right) + \underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \frac{1}{2}F''\left(X_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2}_{\overset{1.2.6}{\to}\frac{1}{2}\int_0^t F''(X_s)\,\mathrm{d}\langle X\rangle_s} \\
&+ \underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \frac{1}{2}\left[F''\left(X_{t_i^{(n)}} + \theta_i^{(n)}\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right) - F''\left(X_{t_i^{(n)}}\right)\right]\cdot\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2}_{=:S}
\end{aligned}$$

Since F'' is locally uniformly continuous and X is uniformly continuous,

$$\left|F''\left(X_{t_i^{(n)}}+\theta_i^{(n)}\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)\right)-F''\left(X_{t_i^{(n)}}\right)\right|<\varepsilon$$

holds uniformly in i for n big enough. Therefore,

$$S\le\varepsilon\underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}}\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)^2}_{\overset{n\to\infty}{\longrightarrow}\langle X\rangle_t}\overset{\varepsilon\to 0}{\longrightarrow}0,$$

which finishes the proof. □

Remark 1.2.8. *If $\langle X\rangle_t\equiv 0$ (e.g. $(X_t)_{t\ge 0}$ is of bounded variation), then we are in the classical case:*

$$F(X_t)-F(X_0)=\int_0^t F'(X_s)\,dX_s$$

is an ordinary Lebesgue-Stieltjes-Integral. (If $X_t=t$, this is the "Fundamental Theorem of Calculus"). We introduce a short-hand notation for (1.2.4)

$$dF(X)=F'(X)\,dX+\frac{1}{2}F''(X)\,d\langle X\rangle, \tag{1.2.5}$$

in contrast to the classical case (i.e. $\langle X\rangle\equiv 0$), where

$$dF(X)=F'(X)\,dX. \tag{1.2.6}$$

Example 1.2.9. *Consider the differential equation*

$$dX^n=nX^{n-1}\,dX\quad\text{for }\ n\in\mathbb{N}\text{ fixed.}$$

If $\langle X\rangle=0$, then a solution is X^n.
If $\langle X\rangle\ne 0$, this is not a solution, since by Itô

$$dX^n=nX^{n-1}\,dX+\frac{1}{2}n(n-1)X^{n-2}\,d\langle X\rangle.$$

We would like to find a function $h_n:\mathbb{R}\to\mathbb{R}$ such that

$$dh_n(X)=nh_{n-1}(X)\,dX$$

in the general case $\langle X\rangle\not\equiv 0$. Later (cf. Example 1.3.25(ii) below), we shall see that the n-th Hermite polynomial h_n will provide a solution to this problem.

Remark 1.2.10. *We know that, if $f\in\mathcal{C}^1(\mathbb{R})$, then the Itô-integral $\int_0^t f(X_s)\,dX_s$, $t\ge 0$, is (well-)defined. (Simply take F as a primitive of f, i.e. $F'=f$, and apply Itô's formula.)*

Definition 1.2.11 (α-Integral)**.** *More generally we define for $\alpha\in[0,1]$ and $f\in\mathcal{C}^1(\mathbb{R})$*

$$\alpha\text{-}\int_0^t f(X_s)\,dX_s:=\lim_{n\to\infty}\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}}f\left(X_{t_i^{(n)}}+\alpha\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)\right)\cdot\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right). \tag{1.2.7}$$

Claim: This limit exists and

$$\alpha\text{-}\int_0^t f(X_s)\,\mathrm{d}X_s = \int_0^t f(X_s)\,\mathrm{d}X_s + \alpha \int_0^t f'(X_s)\,\mathrm{d}\langle X\rangle_s. \tag{1.2.8}$$

Proof. Exercise (Compare "α-sum" with "0-sum" (Itô-Integral) and use the mean-value-theorem for f). □

Special cases:

$\alpha = 0$: "Itô-integral"

$\alpha = 1$: "Backward Ito-integral".

$\alpha = \frac{1}{2}$: "Stratonovich-Fisk-Integral"

Notation: $\int \ldots \circ\ \mathrm{d}X_s := \frac{1}{2}\text{-}\int \ldots\ \mathrm{d}X_s$. Hence

$$\int_0^t f(X_s)\circ\ \mathrm{d}X_s \left(= \frac{1}{2} - \int_0^t f(X_s)\,\mathrm{d}X_s\right) = \int_0^t f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f'(X_s)\,\mathrm{d}\langle X\rangle_s$$

and we have by Itô the *Stratonovich-formula*

$$F(X_t) - F(X_0) = \int_0^t F'(X_s)\circ\ \mathrm{d}X_s. \tag{1.2.9}$$

Remark 1.2.12. *(i) An advantage of the Itô-integral is (see Section 1.4 below) that, if X is a martingale, then, again, $\int f(X_s)\ dX_s$ is a martingale.*

(ii) In the Stratonovich-fromula one only has to deal with derivatives of first order and, therefore, it can be used for manifold-valued X.

1.2.1. Supplement on the Quadratic Variation

Lemma 1.2.13. *(i) Let $F \in \mathcal{C}^1(\mathbb{R})$. Then $t \mapsto F(X_t)$ has (finite) quadratic variation (along fixed $(\tau_n)_{n\in\mathbb{N}}$)*

$$\langle F(X)\rangle_t = \int_0^t (F'(X_s))^2\ d\langle X\rangle_s \quad \textit{(automatically continuous in t).}$$

(ii) If $M_t := X_t + A_t, t \geq 0$, for some $t \mapsto A_t$ continuous and $\langle A\rangle \equiv 0$ (again $\langle A\rangle$ calculated along (τ_n)), then

$$\langle M\rangle_t = \langle X\rangle_t.$$

(iii) The Itô-integral $t \mapsto \int_0^t f(X_s)\ dX_s =: M_t$ with $f \in \mathcal{C}^1(\mathbb{R})$, has quadratic variation (along (τ_n)) and

$$\langle M\rangle_t = \left\langle \int_0^t f(X_s)\ dX_s \right\rangle_t = \int_0^t f(X_s)^2\ d\langle X\rangle_s.$$

Proof. (i) We first apply Taylor up to order 1, then take the square on both sides and finally apply the Binomial formula to get

$$\begin{aligned}
&\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(F(X_{t_{i+1}^{(n)}}) - F(X_{t_i^{(n)}})\right)^2 \\
&= \underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} F'(X_{t_i^{(n)}})^2 \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2}_{\overset{n\to\infty}{\longrightarrow}\int_0^t (F'(X))^2 \,\mathrm{d}\langle X\rangle_s \text{ by Lemma 1.2.6}} \\
&+ \sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \underbrace{\left(F'\left(X_{t_i^{(n)}} + \theta_i^{(n)}\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right) - F'\left(X_{t_i^{(n)}}\right)\right)^2}_{\substack{<\varepsilon \text{ for large n, since } F' \text{ is uniformly continuous}\\ \text{on the compact set } \{X_s | s\in[0,t+1]\}}} \cdot \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2 \\
&+ 2\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} F'(X_{t_i^{(n)}})\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right) \\
&\cdot \left(F'\left(X_{t_i^{(n)}} + \theta_i^{(n)}\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right) - F'(X_{t_i^{(n)}})\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right).
\end{aligned}$$

Since the second term goes to zero as $n\to\infty$, so, by Cauchy-Schwartz, does the third.

(ii)

$$\begin{aligned}
\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(M_{t_{i+1}}^{(n)} - M_{t_i}^{(n)}\right)^2 &= \underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2}_{\overset{n\to\infty}{\longrightarrow}\langle X\rangle_t} + \underbrace{\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(A_{t_{i+1}}^{(n)} - A_{t_i}^{(n)}\right)^2}_{\overset{n\to\infty}{\longrightarrow}0 \text{ by assumption}} \\
&+ \underbrace{2\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\left(A_{t_{i+1}}^{(n)} - A_{t_i}^{(n)}\right).}_{\overset{n\to\infty}{\longrightarrow}0 \text{ by Cauchy-Schwartz}}
\end{aligned}$$

(iii) Let $F\in\mathcal{C}^2(\mathbb{R})$ such that $F' = f$ and apply Itô to get

$$M_t = F(X_t) - \underbrace{\left(F(X_0) + \frac{1}{2}\int_0^t F''(X_s)\,\mathrm{d}\langle X\rangle_s\right)}_{=:A_t}.$$

But A_t can be written as a difference of increasing functions. Therefore, A_t is of bounded variation, hence, $\langle A\rangle = 0$. Thus, by (ii) and (i)

$$\langle M\rangle \overset{(ii)}{=} \langle F(X)\rangle_t \overset{(i)}{=} \int_0^t (F'(X_s))^2\,\mathrm{d}\langle X\rangle_s.$$

□

1.3. d-Dimensional Itô-Formula and Covariation

Fix $X, Y : [0, \infty) \to \mathbb{R}$ continuous with bounded quadratic variation $\langle X \rangle, \langle Y \rangle$ (along the same $(\tau_n)_{n \in \mathbb{N}}$) (cf. [RY99, (2.3)-(2.5) (p.27/28)]).

Definition 1.3.14. *If*

$$\langle X, Y \rangle_t := \lim_{n \to \infty} \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \leq t}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}} \right) \left(Y_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}} \right), \quad t \geq 0,$$

exists, then it is called the covariation of X and Y (along (τ_n)).

Lemma 1.3.15. *The following assertions are equivalent:*

(i) $\langle X, Y \rangle$ *exists and is continuous.*

(ii) $\langle X + Y \rangle$ *exists and is continuous. In this case the* Polarization identity *holds:*

$$\langle X, Y \rangle = \frac{1}{2} \left(\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle \right).$$

In particular, $\langle X, Y \rangle$ *is the distribution function of a signed measure on* $\mathbb{R}_+$

$$d\langle X, Y \rangle = \frac{1}{2}\, d\langle X + Y \rangle - \frac{1}{2}\, d\langle X \rangle - \frac{1}{2}\, d\langle Y \rangle.$$

Furthermore, if $\langle X, Y \rangle$ *exists, we have*

$$|\langle X, Y \rangle| \leq \langle X \rangle^{\frac{1}{2}} \langle Y \rangle^{\frac{1}{2}},$$

i.e. a Cauchy-Schwartz inequality.

Proof. Exercise. □

Remark 1.3.16. $|\langle X, Y \rangle| \leq \langle X \rangle^{\frac{1}{2}} \langle Y \rangle^{\frac{1}{2}}$ *is a special case of the Kunita-Watanabe-inequality (cf. 2.2.16 below).*

Example 1.3.17. *(i) Let* $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$ *be independent Brownian motions on* $(\Omega, \mathcal{F}, P)$. *Then there exists* $\langle X, Y \rangle(\omega)$ *for* P*-a.e.* $\omega \in \Omega$ *and*

$$\langle X, Y \rangle(\omega) = 0 \quad P\text{-a.e. } \omega \in \Omega.$$

Proof. We know that

$$Z_t := \frac{1}{\sqrt{2}} (X_t + Y_t), \quad t \geq 0,$$

is a Brownian motion. Hence, (cf. Propostition 1.1.4(i)) $\langle Z \rangle_t = t$, so there exists $\langle X + Y \rangle_t = 2\langle Z \rangle_t = 2t$. Then it follows by Lemma 1.3.15 applied to P-a.e. $\omega \in \Omega$ that

$$\langle X, Y \rangle = \frac{1}{2} (\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle) = 0.$$

□

(ii) Let $f, g \in \mathcal{C}(\mathbb{R})$ and

$$Y_t := \int_0^t f(X_s)\ d\langle X\rangle_s, \quad Z_t := \int_0^t g(X_s)\ d\langle X\rangle_s.$$

Then (again with respect to our (τ_n)) there exists

$$\langle Y, Z\rangle_t = \int_0^t f(X_s)g(X_s)\ d\langle X\rangle_s.$$

Proof. By 1.2.13(iii) the quadratic variation of

$$Y_t + Z_t = \int_0^t (f+g)(X_s)\ \mathrm{d}X_s$$

along our (τ_n) exists. Hence, by the polarization identity and Lemma 1.2.13(iii) we get

$$\begin{aligned} 2\langle Y, Z\rangle &= \langle Y+Z\rangle - \langle Y\rangle - \langle Z\rangle \\ &= 2\int f(X_s)g(X_s)\ \mathrm{d}\langle X\rangle_s + \int f(X_s)^2\ \mathrm{d}\langle X\rangle_s + \int g(X_s)^2\ \mathrm{d}\langle X\rangle_s - \langle Y\rangle - \langle Z\rangle \\ &= 2\int f(X_s)g(X_s)\ \mathrm{d}\langle X\rangle_s. \end{aligned}$$

□

Proposition 1.3.18 (Itô's product rule)**.** *Let X, Y be as above such that there exists $\langle X, Y\rangle$ (with respect to (τ_n)) and is continuous in $t \geq 0$. If there exists either*

$$\lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\leq t}} X_{t_i^{(n)}} \left(Y_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}} \right) =: \int_0^t X_s\ dY_s$$

or

$$\lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\leq t}} Y_{t_i^{(n)}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}} \right) =: \int_0^t Y_s\ dX_s,$$

then both of these limits exist and

$$X_tY_t = X_0Y_0 + \int_0^t X_s\ dY_s + \int_0^t Y_s\ dX_s + \langle X, Y\rangle_t.$$

Proof. We have

$$X_tY_t = \frac{1}{2}\left((X_t+Y_t)^2 - X_t^2 - Y_t^2 \right).$$

Furthermore, by Lemma 1.3.15 $\langle X+Y\rangle$ exists and is continuous, since by Itô with $F(X) = \frac{1}{2}X^2$ we know that

$$\begin{aligned} X_tY_t = &\underbrace{\frac{1}{2}\left((X_0+Y_0)^2 - X_0^2 - Y_0^2 \right)}_{=X_0Y_0} + \int_0^t (X+Y)_s\ \mathrm{d}(X+Y)_s \\ &- \int_0^t X_s\ \mathrm{d}X_s - \int_0^t Y_s\ \mathrm{d}Y_s + \underbrace{\frac{1}{2}\left(\langle X+Y\rangle_t - \langle X\rangle_t - \langle Y\rangle_t \right)}_{=\langle X,Y\rangle_t}. \end{aligned}$$

By definition we have

$$\begin{aligned}&\int_0^t (X+Y)_s \,\mathrm{d}(X+Y)_s\\ &= \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \underbrace{\left(X_{t_i^{(n)}} + Y_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}} + Y_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}}\right)}_{\substack{=X_{t_i^{(n)}}\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)+Y_{t_i^{(n)}}\left(Y_{t_{i+1}^{(n)}}-Y_{t_i^{(n)}}\right)\\ +X_{t_i^{(n)}}\left(Y_{t_{i+1}^{(n)}}-Y_{t_i^{(n)}}\right)+Y_{t_i^{(n)}}\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)}}\end{aligned}$$

Therefore, we get

$$X_tY_t = X_0Y_0 + \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(X_{t_i^{(n)}}\left(Y_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}}\right) + Y_{t_i^{(n)}}\left(X_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}}\right)\right) + \langle X, Y\rangle_t,$$

which implies the assertion.

□

Remark 1.3.19. *If X or Y has bounded variation, e.g. X, then there already exists $\int Y_s\, dX_s$ and* all *assumptions in Proposition 1.3.18 are fulfilled. In this case $\langle X, Y\rangle = 0$ and, by substituting $dY_s = Y_s' ds$, we are in the classical case of integration by parts:*

$$X_tY_t - X_0Y_0 = \int X_sY_s'\; ds + \int Y_sX_s'\; ds.$$

Example 1.3.20. *Suppose $t \mapsto Y_t$ is of bounded variation, hence, $\langle Y\rangle \equiv 0$ and $\langle X, Y\rangle = 0$ (by Hölder). Then by Proposition 1.3.18*

$$\int_0^t Y_s\; dX_s = -\int_0^t X_s\; dY_s + X_tY_t - X_0Y_0.$$

Here, we can define the left hand side by the right hand side since $\int_0^t X_s\; dY_s$ is a usual Lebesgue-Stieltjes ingegral. This approach was used by Paley-Wiener to define stochastic integrals, if X is a Brownian motion:
Let $X(\omega)$ be a typical Brownian path (hence, $X_0(\omega) = 0$) and $h(s)(= Y_s)$ continuous, of bounded variation and independent of ω with $h(1) = 0$. Define

$$\int_0^1 h(s)\; dX_s(\omega) := -\int_0^1 X_s(\omega)\; dh(s).$$

One can show that

$$E\left[\left(\int_0^1 h(s)\; dX_s\right)^2\right] = \int_0^1 h(s)^2\; ds,$$

hence,

$$\begin{array}{ccc} \mathcal{L}^2([0,1], ds) & \to & \mathcal{L}^2(\Omega, \mathcal{F}, P)\\ \Cup & & \Cup\\ h & \mapsto & \int h(s)\; dX_s(\omega) \end{array}$$

is an isometry. It is first defined for a dense subset of functions h in $\mathcal{L}^2([0,1], ds)$, and then extended to the closure, i.e. for all $h \in \mathcal{L}^2([0,1], ds)$, by this isometry.

Fix now $X_t = \left(X_t^1, \dots, X_t^d\right) : [0,\infty) \to \mathbb{R}^d$ continuous with continuous $\langle X^i \rangle, \langle X^i, X^j \rangle$, $i,j \in \{1,\dots,d\}$, $i \neq j$ (along our (τ_n)).

Proposition 1.3.21 (d-dimensional Itô-formula)**.** *Let $F \in \mathcal{C}^2(\mathbb{R}^d)$. Then with $(\cdot,\cdot)_t$ as Euklidean inner product on $\mathbb{R}$*

$$F(X_t) - F(X_0) = \int_0^t (\nabla F(X_s),\ dX_s) + \frac{1}{2}\int_0^t \sum_{k,l=1}^d \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s)\ d\langle X^k, X^l\rangle_s, \tag{1.3.10}$$

where

$$\int_0^t (\nabla F(X_s),\ dX_s) := \lim_{n\to\infty} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \left(\nabla F(X_{t_i^{(n)}}), X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)$$

is called multidimensional Itô-integral*.*

Proof. Obviously, we have

$$\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} F\left(X_{t_{i+1}^{(n)}}\right) - F\left(X_{t_i^{(n)}}\right) \overset{n\to\infty}{\longrightarrow} F(X_t) - F(X_0).$$

Furthermore, by d-dimensional Taylor forumla we obtain

$$\begin{aligned}
&\sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} F\left(X_{t_{i+1}^{(n)}}\right) - F\left(X_{t_i^{(n)}}\right) \\
&= \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \left(\nabla F\left(X_{t_i^{(n)}}\right), X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right) \\
&\underbrace{+ \frac{1}{2} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \left(A\left(X_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right), \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right)}_{=:S} \\
&+ \frac{1}{2} \sum_{\substack{t_i^{(n)}\in\tau_n \\ t_i^{(n)}\le t}} \left(\left(A\left(X_{t_i^{(n)}} + \theta_i^n\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right) - A\left(X_{t_i^{(n)}}\right)\right)\right. \\
&\qquad \left.\cdot \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right), \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)\right),
\end{aligned}$$

where

$$A(x) = \left(\frac{\partial^2}{\partial x_i \partial x_j} F(x)\right)_{i,j}.$$

The third summand vanishes analogously to the 1-dimensional case. Moreover, since we can interchange the sums, we have by polarization

$$\begin{aligned}
S &= \sum_{j,k=1}^{d} \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \le t}} \frac{\partial^2 F}{\partial x_j \partial x_k} \left(X_{t_i^{(n)}} \right) \left(X^{(j)}_{t_{i+1}^{(n)}} - X^{(j)}_{t_i^{(n)}} \right) \left(X^{(k)}_{t_{i+1}^{(n)}} - X^{(k)}_{t_i^{(n)}} \right) \\
&= \sum_{j,k=1}^{d} \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \le t}} \frac{\partial^2 F}{\partial x_j \partial x_k} \left(X_{t_i^{(n)}} \right) \\
&\quad \cdot \frac{1}{2} \left(\left(\left(X^{(j)}_{t_{i+1}^{(n)}} + X^{(k)}_{t_{i+1}^{(n)}} \right) - \left(X^{(j)}_{t_i^{(n)}} + X^{(k)}_{t_i^{(n)}} \right) \right)^2 - \left(X^{(j)}_{t_{i+1}^{(n)}} - X^{(j)}_{t_i^{(n)}} \right)^2 - \left(X^{(k)}_{t_{i+1}^{(n)}} - X^{(k)}_{t_i^{(n)}} \right)^2 \right) \\
&\overset{n\to\infty}{\longrightarrow} \frac{1}{2} \sum_{j,k=1}^{d} \left(\int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k}(X_s) \, \mathrm{d}\langle X^{(j)} + X^{(k)} \rangle_s - \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k}(X_s) \, \mathrm{d}\langle X^{(j)} \rangle_s \right. \\
&\qquad \left. - \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k}(X_s) \, \mathrm{d}\langle X^{(k)} \rangle_s \right) \\
&= \int_0^t \sum_{k,l=1}^{d} \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s) \, \mathrm{d}\langle X^k, X^l \rangle_s, \quad P\text{-a.s..}
\end{aligned}$$

□

Remark 1.3.22. *(i) We have defined*

$$\int_0^t (\nabla F(X_s),\ dX_s) := \lim_{n\to\infty} \sum_{k=1}^{d} \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \le t}} \frac{\partial F}{\partial x_k} \left(X_{t_i^{(n)}} \right) \left(X^k_{t_{i+1}^{(n)}} - X^k_{t_i^{(n)}} \right),$$

but we cannot *interchange* $\sum_{k=1}^d$ *with the limit, since we do* not *know whether for every* k *the limit exists.*

(ii) For $F \in \mathcal{C}^1(\mathbb{R}^d)$ *the function* $t \mapsto F(X_t)$ *has continuous quadratic variation and*

$$\langle F(X) \rangle_t = \sum_{k,l=1}^{d} \int_0^t \frac{\partial F}{\partial x_k}(X_s) \frac{\partial F}{\partial x_l}(X_s) \; d\langle X^k, X^l \rangle_s.$$

Proof. By Taylor

$$\begin{aligned}
&\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(F\left(X_{t_{i+1}^{(n)}}\right)-F\left(X_{t_i^{(n)}}\right)\right)^2\\
&=\underbrace{\sum_{k,l=1}^{d}\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \frac{\partial F}{\partial x_k}\left(X_{t_i^{(n)}}\right)\frac{\partial F}{\partial x_l}\left(X_{t_i^{(n)}}\right)\left(X^k_{t_{i+1}^{(n)}}-X^k_{t_i^{(n)}}\right)\left(X^l_{t_{i+1}^{(n)}}-X^l_{t_i^{(n)}}\right)}_{\xrightarrow{1.2.6}\sum_{k,l=1}^{d}\int_0^t \frac{\partial F}{\partial x_k}(X_s)\frac{\partial F}{\partial x_l}(X_s)\,\mathrm{d}\langle X^k,X^l\rangle_s \text{ by polarization}}\\
&+\sum_{k,l=1}^{d}\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \underbrace{\left(\frac{\partial F}{\partial x_k}\left(X_{t_i^{(n)}}+\theta_i^n\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)\right)-\frac{\partial F}{\partial x_k}\left(X_{t_i^{(n)}}\right)\right)}_{<\varepsilon \text{ uniformly in } i \text{ for } n \text{ big enough}}\\
&\quad\cdot\underbrace{\left(\frac{\partial F}{\partial x_l}\left(X_{t_i^{(n)}}+\theta_i^n\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)\right)-\frac{\partial F}{\partial x_l}(X_{t_i^{(n)}})\right)}_{<\infty}\left(X^k_{t_{i+1}^{(n)}}-X^k_{t_i^{(n)}}\right)\left(X^l_{t_{i+1}^{(n)}}-X^l_{t_i^{(n)}}\right)\\
&+2\sum_{k,l=1}^{d}\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \left(\frac{\partial F}{\partial x_k}\left(X_{t_i^{(n)}}+\theta_i^n\left(X_{t_{i+1}^{(n)}}-X_{t_i^{(n)}}\right)\right)-\frac{\partial F}{\partial x_k}\left(X_{t_i^{(n)}}\right)\right)\left(X^k_{t_{i+1}^{(n)}}-X^k_{t_i^{(n)}}\right)\\
&\quad\cdot\sum_{\substack{t_i^{(n)}\in\tau_n\\ t_i^{(n)}\le t}} \frac{\partial F}{\partial x_l}\left(X_{t_i^{(n)}}\right)\left(X^l_{t_{i+1}^{(n)}}-X^l_{t_i^{(n)}}\right).
\end{aligned}$$

By Cauchy Schwartz the last two double sums converge to 0 as n goes to ∞. ☐

(iii) Since

$$\frac{1}{2}\int_0^t \sum_{k,l=1}^{d}\frac{\partial^2 F}{\partial x_k\partial x_l}(X_s)\ d\langle X^k,X^l\rangle_s$$

is of bounded variation, its quadratic variation is 0*. But then, by 5.2.7 and 1.2.13 (ii), we can conclude that*

$$\left\langle \int_0^{\cdot} (\nabla F(X_s),dX_s)\right\rangle_t = \langle F(X)-F(X_0)\rangle_t = \langle F(X)\rangle_t \overset{(ii)}{=} \sum_{k,l=1}^{d}\int_0^t \frac{\partial F}{\partial x_k}(X_s)\frac{\partial F}{\partial x_l}\ d\langle X^k,X^l\rangle_s.$$

1.3.1. Important special cases

Brownian motion in $\mathbb{R}^d$ and Laplace-operator Δ

The components of a d-dimensional Brownian motion are independent, hence, (by 1.3.17(i))

$$\langle X^k,X^l\rangle_t(\omega)=\delta_{kl}\cdot t \quad (\text{“covariation reflects independence”}),$$

which implies by Itô for $F\in\mathcal{C}^2(\mathbb{R})$

$$F(X_t)-F(X_0)=\int_0^t(\nabla F(X_s),dX_s)+\frac{1}{2}\int_0^t \Delta F(X_s)\,\mathrm{d}s.$$

In particular, if F is harmonic (i.e. $\Delta F = 0$), then

$$F(X_t) = F(0) + \int_0^t (\nabla F(X_s), dX_s),$$

which is an Itô-integral of a Brownian motion. Hence, a harmonic function preserves the martingale property since the Itô-integral again is a martingale (see Section 1.4 below).

Itô-formula for time dependent functions

Proposition 1.3.23. *Let $F \in \mathcal{C}^2(\mathbb{R}^2)$ and $X : [0,\infty) \to \mathbb{R}$ be continuous with continuous $\langle X \rangle$ (along $(\tau_n)_{n\in\mathbb{N}}$). Then*

$$F(X_t, \langle X\rangle_t) = F(X_0, 0) + \int_0^t \frac{\partial F}{\partial x}(X_s, \langle X\rangle_s)\ dX_s + \int_0^t \left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial y}\right)(X_s, \langle X\rangle_s)\ d\langle X\rangle_s.$$

Proof. Apply Itô for $d = 2$ with $(X_t, \langle X\rangle_t)$ to get

$$\begin{aligned} &F(X_t, \langle X\rangle_t) - F(X_0, 0) \\ &= \underbrace{\int_0^t \frac{\partial F}{\partial x}(X_s, \langle X\rangle_s)\,\mathrm{d}X_s + \int_0^t \frac{\partial F}{\partial y}(X_s, \langle X\rangle_s)\,\mathrm{d}\langle X\rangle_s}_{=:S} + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, \langle X\rangle_s)\,\mathrm{d}\langle X\rangle_s \\ &\quad \underbrace{+\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial y^2}(X_s, \langle X\rangle_s)\,\mathrm{d}\langle\langle X\rangle_s, \langle X\rangle_s\rangle + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x \partial y}(X_s, \langle X\rangle_s)\,\mathrm{d}\langle X_s, \langle X\rangle_s\rangle}_{=0 \text{ since } \langle X\rangle_s \text{ is of bounded variation and by 1.3.20}}. \end{aligned}$$

The second summand of S exists since $\langle X\rangle_t$ is of bounded variation (along $(\tau_n)_{n\in\mathbb{N}}$). As the whole sum S exists by Itô ($d = 2$), so does the first summand. □

Remark 1.3.24. *(i) If X_t is a Brownian motion, then $\langle X\rangle_t$ represents the time, i.e.*

$$F(X_t, \langle X\rangle_t) = F(X_t, t).$$

Therefore, $\langle X\rangle_t$ is also called inner clock *of X_t.*

(ii) If F is a solution to the backward heat equation, i.e.

$$\frac{1}{2}\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial t} = 0,$$

then we have by above (for X as above)

$$F(X_t, \langle X\rangle_t) = F(X_0, 0) + \int_0^t \frac{\partial F}{\partial x}(X_s, \langle X\rangle_s)\ dX_s = F(X_0, 0) + \text{“Itô-integral”}.$$

Later we shall see that the Itô-integral is a local martingale, if X is one. Therefore, every solution to the backward heat equation provides a local martingale!

Example 1.3.25. *(i) Let $F(x,t) := \exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right)$. Then F solves the backward heat equation. Thus, if $X_0 = 0$, then*

$$G_t := F(X_t, \langle X\rangle_t) = \exp\left(\alpha X_t - \frac{1}{2}\alpha^2 \langle X\rangle_t\right)$$

solves the differential equation

$$\begin{aligned} G_0 &= 1, \\ dG &= \alpha G \; dX, \end{aligned}$$

i.e.

$$G_t = 1 + \int_0^t \alpha G_s \; dX_s.$$

Recall, if $\langle X \rangle \equiv 0$, *then* $G_t = \exp(\alpha X_t)$. *Hence, in the Itô-calculus* G_t *as above is the right analogon to the exponential function* $\exp(\alpha t)$ *in the usual case.*

Application to Brownian motion:
For X *with* $\langle X \rangle_t = t$

$$G_t(\omega) := G_0 e^{\beta t} \exp\left(\alpha X_t(\omega) - \frac{1}{2}\alpha^2 t\right), \quad t \geq 0,$$

by Proposition 1.3.23 solves the linear SDE

$$dG = \alpha G \; dX + \beta G \; dt.$$

Classical $(\alpha = 0)$*:*

$$dG = \beta G \; dt \quad \Rightarrow \quad G_t = G_0 e^{\beta t}.$$

Here, for $\beta > 0$ G_t *tends to* ∞ *as* $t \to \infty$.

Stochastic $(\alpha \neq 0)$***:***

$$G_t = G_0 \exp\left(\alpha X_t + \left(\beta - \frac{1}{2}\alpha^2\right) t\right).$$

By law of iterated logarithm

$$\frac{X_t}{t} \overset{t\to\infty}{\longrightarrow} 0, \;\; a.s.,$$

hence, for large t

$$\left|\alpha \frac{X_t}{t}\right| < \frac{1}{2}\left(\frac{1}{2}\alpha^2 - \beta\right).$$

Thus, for $\beta < \frac{1}{2}\alpha^2$

$$G_t \leq G_0 e^{-\left(\frac{1}{2}\alpha - \beta\right)\frac{t}{2}} \overset{t\to\infty}{\longrightarrow} 0, \quad (a.s.\ pathwise\ stable).$$

But G_t *is not uniformly integrable, since by the Residue theorem*

$$E[e^{\alpha X_t}] = e^{\frac{1}{2}\alpha^2 t},$$

and therefore,

$$E[G_t] = G_0 e^{\beta t} E\left[e^{\alpha X_t}\right] e^{-\frac{1}{2}\alpha^2 t} = G_0 e^{\beta t} \to \infty, \;\; if\ \beta > 0 \quad (unstable\ in\ the\ mean).$$

(ii) "Hermite- polynomials" (cf. Example 1.2.9):
Define $h_n(x,t)$ *by*

$$e^{\alpha x - \frac{1}{2}\alpha^2 t} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} h_n(x,t), \tag{1.3.11}$$

where the left hand side is analytic in α, *i.e.*

$$h_n(x,t) = \frac{\partial^n}{\partial\alpha^n}\left(e^{\alpha x-\frac{1}{2}\alpha^2 t}\right)\Big|_{\alpha=0} \left(= e^{\alpha x}\sum_{k=0}^n \binom{n}{k} x^k \frac{\partial^{n-k}}{\partial\alpha^{n-k}}\left(e^{-\frac{1}{2}\alpha^2 t}\right)\Big|_{\alpha=0}\right). \tag{1.3.12}$$

Recall, that for all $n \in \mathbb{N}$

$$H_n(\cdot,t) := \frac{1}{\sqrt{n! t^n}} h_n(\cdot,t)$$

is an orthonormal basis of $\mathcal{L}^2(\mathbb{R}, N(0,t))$.

Proof. Clearly, (by monotone classes) $\operatorname{span}\{H_n, n \in \mathbb{N}\}$ is a dense subset of $\mathcal{L}^2(\mathbb{R}, N(0,t))$. Hence, it is sufficient to show that it is an ONS:

$$\sum_{n,m} \frac{\alpha^n}{n!} h_n(x,t) \frac{\beta^m}{m!} h_m(x,t) \overset{(1.3.11)}{=} e^{(\alpha+\beta)x-\frac{1}{2}(\alpha^2+\beta^2)t} = \exp((\alpha+\beta)x)e^{-\frac{1}{2}(\alpha+\beta)^2 t} e^{\alpha\beta t}.$$

By integration with $N(0,t)$ in x, since sums interchange with integration and

$$\int e^{(\alpha+\beta)x} N(0,t)(\,\mathrm{d}x) = e^{\frac{1}{2}(\alpha+\beta)^2 t},$$

we get by (1.3.12) that

$$\sum_{n,m} \alpha^m \beta^m \int \frac{1}{n!} h_n(x,t) \frac{1}{m!} h_m(x,t) N(0,t)(\,\mathrm{d}x) = e^{\alpha\beta t} = \sum_n \alpha^n \beta^n \frac{t^n}{n!}, \quad \forall \alpha, \beta.$$

□

We have for $n \geq 0$ *by* (1.3.12) *and by interchanging order of differentiation*

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} h_n + \frac{\partial}{\partial t} h_n = 0.$$

Additionally, for $n \geq 1$ *we have*

$$\frac{\partial}{\partial x} h_n(\cdot,t) \overset{(1.3.11)}{=} n \cdot h_{n-1}(\cdot,t).$$

Therefore, if $X, \langle X\rangle$ *are continuous and* $X_0 = 0$, *(since* $h_n(0,0) = 0$*), we get*

$$\begin{aligned} h_n(X_t, \langle X\rangle_t) &= \int_0^t \frac{\partial}{\partial x} h_n(X_{t_1}, \langle X\rangle_{t_2})\, dX_{t_1} = n \int_0^t h_{n-1}(X_{t_1}, \langle X\rangle_{t_1})\, dX_{t_1} \\ &= \ldots = n! \int_0^t dX_{t_1} \int_0^{t_1} dX_{t_2} \ldots \int_0^{t_{n-1}} dX_{t_n}. \end{aligned}$$

1.4. Itô-Integrals as (Local) Martingales

Recall: If $(\mathcal{F}_t)_{t\geq 0}$ is not right-continuous, then define

$$\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s, \quad t \geq 0.$$

Then $(\mathcal{F}_{t+})_{t\geq 0}$ is right-continuous.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s.$$

Let $T : \Omega \to [0, \infty]$ and let $X : \{(t, \omega) \in]0, \infty[\times \Omega \ : \ t < T(\omega)\} \cup \{0\} \times \Omega \to \mathbb{R}$ be a map such that X is a *continuous local martingale* (up to T) i.e. there exists a localizing sequence $T_1 \leq T_2 \leq \ldots \leq T$ and $< T$ on $\{T > 0\}$, i.e. they are $(\mathcal{F}_t)$-stopping times such that

(i) $(X_{t\wedge T_n})_{t\geq 0}$ is a continuous $(\mathcal{F}_t)$-martingale for all $n \in \mathbb{N}$,

(ii) $\sup_{n\in\mathbb{N}} T_n = T \quad P$-a.s..

Assume in addition, that X has a continuous quadratic variation $[0, T(\omega)[\ni t \to \langle X\rangle_t(\omega)$ along a sequence of partitions (τ_n) as above for P-a.e. $\omega \in \Omega$.

Remark 1.4.26. *Later we shall see that for any (τ_n) there exists (τ_{n_k}) such that X as above has continuous quadratic variation $\langle X\rangle_t$ along (τ_{n_k}) for P-a.e. $\omega \in \Omega$.*

Proposition 1.4.27. *Let $f \in \mathcal{C}^2(G \times \mathbb{R}_+)$, $G \subset \mathbb{R}^1$ open (or $f \in \mathcal{C}^1(G)$). Assume there exists a compact set $K \subset G$ such that*

$$X_0(\omega) \subset K \quad \textit{for} \quad P\textit{-a.e.}\, \omega \in \Omega.$$

Define $M_0(\omega) = 0$ for all $\omega \in \Omega$ and

$$M_t(\omega) := \int_0^t f(X_s(\omega), \langle X\rangle_s(\omega))\, dX_s(\omega), \quad 0 < t < T(\omega) \textit{ for } \omega \in \{T > 0\}\,. \textit{ (Itô-integral!)}$$

Then M is a continuous local martingale up to

$$S := \underbrace{\inf\{t > 0 | X_t \notin G\}}_{=:\,\sigma_{G^c}} \wedge T.$$

Proof. **Step 1:** Assume that $T = \infty$, X is a bounded martingale, $G = \mathbb{R}^1$ and f is bounded. Then $M_t(\omega)$ is defined for all $t \geq 0$, since $S = +\infty$.
Claim: $(M_t)_{t\geq 0}$ is a continuous martingale.

To see this define for $n \in \mathbb{N}$ fixed

$$M_t^{(n)} := \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \leq t}} f\left(X_{t_i^{(n)}}, \langle X\rangle_{t_i^{(n)}}\right) \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right).$$

To see that $(M_t^{(n)})_{t\geq 0}$ is a martingale, (but not nessassarily continuous), let $s < t$ and take $t_k^{(n)}$, $t_l^{(n)} \in \tau_n$ such that $t_k^{(n)} \leq t < t_{k+1}^{(n)}$, $t_l^{(n)} \leq s < t_{l+1}^{(n)}$. Then, we have $M_t^{(n)} = M_{t_k}^{(n)}$ and

$M_s^{(n)} = M_{t_l}^{(n)}$ and therefore,

$$\begin{aligned}
E[M_t^{(n)} - M_s^{(n)}|\mathcal{F}_s] &= E[M_{t_k}^{(n)} - M_{t_l}^{(n)}|\mathcal{F}_s] \\
&= \sum_{i=l+1}^{k} E\Bigg[E\Bigg[\underbrace{f\left(X_{t_i^{(n)}}, \langle X\rangle_{t_i^{(n)}}\right)}_{\mathcal{F}_{t_i^{(n)}}-\text{measurable}} \left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)|\mathcal{F}_{t_i^{(n)}}\Bigg]|\mathcal{F}_s\Bigg] \\
&= \sum_{i=l+1}^{k} E\Bigg[f\left(X_{t_i^{(n)}}, \langle X\rangle_{t_i^{(n)}}\right) \underbrace{E\Bigg[\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)|\mathcal{F}_{t_i^{(n)}}\Bigg]}_{=0 \text{ since } X_t \text{ is a martingale}}|\mathcal{F}_s\Bigg] = 0
\end{aligned}$$

Hence by Proposition 2.1.4 below it is sufficient to show that $M_t^n \to M_t$ in $\mathcal{L}^1$ for all $t \geq 0$. We know that by definition,

$$M_t^{(n)} \to M_t, \quad P\text{-a.s.}\,\forall t \geq 0.$$

Furthermore, for all $t \geq 0$,

$$E\left[\left(M_t^{(n)}\right)^2\right] = \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \leq t}} E\left[f^2\left(X_{t_i^{(n)}}, \langle X\rangle_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}\right)^2\right],$$

since all nondiagonal terms vanish by the martingale property of X_t. Furthermore,

$$\begin{aligned}
E\left[\left(M_t^{(n)}\right)^2\right] &\leq \sup_{\mathbb{R}\times\mathbb{R}_+} f^2 \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \leq t}} E\Big[\underbrace{E\Big[X^2_{t_{i+1}^{(n)}} - 2X_{t_{i+1}^{(n)}}X_{t_i^{(n)}} + X^2_{t_i^{(n)}}\Big|\mathcal{F}_{t_i^{(n)}}\Big]}_{=E\left[X^2_{t_{i+1}^{(n)}}\middle|\mathcal{F}_{t_i^{(n)}}\right] - X^2_{t_i^{(n)}}}\Big] \\
&= \sup_{\mathbb{R}\times\mathbb{R}_+} f^2 \sum_{\substack{t_i^{(n)} \in \tau_n \\ t_i^{(n)} \leq t}} E\left[X^2_{t_{i+1}^{(n)}} - X^2_{t_i^{(n)}}\right] \\
&= \sup_{\mathbb{R}\times\mathbb{R}_+} f^2 \underbrace{E\left[X^2_{t^n_{N_n}} - X_0^2\right]}_{\leq \sup_{t,\omega} X_t^2(\omega)}.
\end{aligned}$$

Hence, $\sup_n E\left[\left(M_t^{(n)}\right)^2\right] < \infty$ and thus $\left(M_t^{(n)}\right)_{n\in\mathbb{N}}$ is uniformly integrable. By the generalized Lebesgue dominated convergence theorem the claim follows. Therefore $(M_t)_{t\geq 0}$ is a martingale. It still remains to show that M_t has P-a.s. continuous sample paths:
In order to see that consider

$$\tilde{M}_t^{(n)} := \sum_{i=0}^{n} f\left(X_{t_i^{(n)}}, \langle X\rangle_{t_i^{(n)}}\right)\left(X_{t_{i+1}^{(n)}\wedge t} - X_{t_i^{(n)}\wedge t}\right).$$

Then as above one shows that $\tilde{M}_t(\omega)$ is a martingale. Note that $\left(\tilde{M}_t^{(n)}\right)_{t\geq 0}$ has P-a.s. continuous sample paths and that

$$M_t^{(n)} - \tilde{M}_t^{(n)} \overset{n\to\infty}{\longrightarrow} 0 \quad P\text{-a.s.}$$

and in $\mathcal{L}^q$ (by the same argument as above). Hence, $\tilde{M}_t^{(n)} \overset{n\to\infty}{\longrightarrow} M_t$ in $\mathcal{L}^q$ for all $q \in [1,2)$. Then by Doob's maximal inequality one can show that $(M_t)_{t\geq 0}$ has P-a.s. continuous sample paths

(cf. below Proposition 2.1.4).
Step 2 ("Localization by stopping times"):
Let (T_n) be a localizing sequence for X. For $n \in \mathbb{N}$ define

$$\tilde{T}_n := \inf\{t > 0 | \langle X\rangle_t > n\} \wedge T$$

and

$$S_n := T_n \wedge \sigma_{G_n^c} \wedge \tilde{T}_n,$$

where $G_n \nearrow G$, G_n relatively compact and open and $\bar{G}_n \subset G_{n+1}$ for all $n \in \mathbb{N}$. Without loss of generality $K \subset G_n \; \forall n \in \mathbb{N}$. Then $\sup_n \sigma_{G_n^c} = \sigma_{G^c}$ and S_n are stopping times such that $S_n \leq T_n \leq T$ and $S_n \leq T_n < T$ on $\{T > 0\}$. Furthermore, $\tilde{T}_n \nearrow T$, hence, $S_n \nearrow T \wedge \sigma_{G^c}$. By optional stopping $(X_{t\wedge S_n})_{t\geq 0}$ is a continuous martingale taking values in G_n, hence $(X_{t\wedge S_n})_{t\geq 0}$ is bounded in (t, ω). Furthermore, (exercise)

$$M_{t\wedge S_n(\omega)}(\omega) \overset{!}{=} \int_0^t f\left(X_{s\wedge S_n(\omega)}(\omega), \langle X\rangle_{s\wedge S_n(\omega)}\right) \,\mathrm{d}X_{s\wedge S_n(\omega)}(\omega). \tag{1.4.13}$$

Take $\chi_n \in \mathcal{C}_0^2(G \times \mathbb{R}_+)$ such that

$$\chi_n = 1 \text{ on } G_n \times [0, n].$$

Then we can replace f in (1.4.13) by $\chi_n f \in \mathcal{C}_b^2(G \times \mathbb{R}_+)$. Therefore, the representation for $(M_{t\wedge S_n})_{t\geq 0}$ in (1.4.13) and Step 1 imply that $(M_{t\wedge S_n})_{t\geq 0}$ is a continuous martingale. □

Corollary 1.4.28. *Let X be a continuous local martingale up to T with continuous quadratic variation (later proved to always be the case). Then*

(i) $X^2 - \langle X\rangle$ is a continuous local martingale up to T.

(ii) If $\langle X\rangle = 0$ (which is particularly true if X has bounded variation), then for P-a.s. $\omega \in \Omega$

$$X_t(\omega) \equiv X_0(\omega) \quad \forall t \in [0, T(\omega)[. \tag{1.4.14}$$

Proof. Without loss of generality assume that $X_0 \equiv 0$. (Otherwise, consider $X_t - X_0$, $t \leq T$.)

(i) By Itô

$$X_t^2 = 2 \cdot \int_0^t X_s \,\mathrm{d}X_s + \langle X\rangle_t \quad \text{ on } \{t < T\}.$$

But the first term on the right hand side is a continuous local martingale up to T by Proposition 5.3.9.

(ii) By (i) it also follows that X^2 is a continuous local martingale up to T if $\langle X\rangle = 0$. Hence, if $T_n \nearrow T$ is a localising sequence for X^2, then

$$E\left[X_{t\wedge T_n}^2\right] = E\left[X_0^2\right] = 0 \quad \forall t \geq 0.$$

Therefore, $1_{\{t<T\}} X_t = 0$ P-a.s. $\forall t \geq 0$, (with zero set depending on t). Hence,

$$P\left[1_{\{t<T\}} X_t = 0 \quad \forall t \in \mathbb{Q}\right] = 1,$$

and by P-a.s. continuity in t it follows that $X_t = 0$ on $\{t < T\}$ for all $t \geq 0$ P-a.s..

□

Proposition 1.4.29 (d-dimensional version of Proposition 5.3.9)**.** *Let $X = (X^1, \dots, X^d)$ with $X^1, \dots, X^d$ continuous local martingales up to T such that $\langle X_i, X_j \rangle$ exist for all $1 \leq i, j \leq d$ and are continuous up to T. Let $F \in \mathcal{C}^2(D)$, $D \subset \mathbb{R}^d$, D open, with $X_0(\omega) \subset K$ for P-a.e. $\omega \in \Omega$ for some compact $K \subset D$. Define $M_0 := 0$ and*

$$M_t := \int_0^t (\nabla F(X_s), dX_s)_{\mathbb{R}^d} \quad \text{on } \{t < T\} \quad (d\text{-dimensional Itô-integral})$$

Then M is a continuous local martingale up to

$$S := T \wedge \underbrace{\inf\{t > 0 | X_t \notin D\}}_{=:\sigma_{D^c}}.$$

Proof. Exercise (Proceed as for Proposition 5.3.9). □

Remark 1.4.30. *By Remark 1.3.22 (iii) for M as in Proposition 1.4.29 we have*

$$\langle M \rangle_t = \sum_{k,l=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s) \frac{\partial F}{\partial x_l}(X_s) \, d\langle X^k, X^l \rangle_s \quad \text{on } \{t < S\}.$$

Hence, if $S \equiv \infty$, then M is even a martingale provided $\frac{\partial F}{\partial x_k}$, $1 \leq k \leq d$, are all bounded and X is a Brownian motion. This is a consequence of Corollary 1.4.32 below.

Proposition 1.4.31. *Let M be a continuous local martingale up to T (with continuous $\langle M \rangle$ up to T) and let $T_0 \leq T$ with $T_0 < T$ on $\{T > 0\}$ be a stopping time such that*

$$E[\langle M \rangle_{T_0}] < \infty.$$

Then $(M_{t \wedge T_0})_{t \geq 0}$ is a continuous martingale such that $\sup_{t \geq 0} E[M^2_{T_0 \wedge t}] \leq E[\langle M \rangle_{T_0}]$. Furthermore,

$$E[M_{T_0}] = E[M_0].$$

Note that T_0 doesn't need to be bounded.

Proof. Without loss of generality assume $M_0 \equiv 0$. (Otherwise, consider $M_t - M_0$.) Let $T_n \nearrow T$ be a localizing sequence for M. Then $(M_{T_0 \wedge T_n \wedge t})_{t \geq 0}$ is a continuous martingale. Hence

$$\lim_{n \to \infty} M_{T_0 \wedge T_n \wedge t} = M_{T_0 \wedge t} \quad P\text{-a.s. } \forall t \geq 0$$

and

$$E\left[M^2_{T_0 \wedge T_n \wedge t}\right] \overset{\text{Cor. 5.3.10}}{=} E[\langle M \rangle_{T_0 \wedge T_n \wedge t}] \leq E[\langle M \rangle_{T_0}] < \infty.$$

Hence, $\{M_{T_0 \wedge T_n \wedge t} | n \in \mathbb{N}\}$ is uniformly integrable $\forall\, t \geq 0$. In particular, by Lebesgue

$$\lim_{n \to \infty} M_{T_0 \wedge T_n \wedge t} = M_{T_0 \wedge t} \quad \text{in } \mathcal{L}^1 \, \forall t \geq 0.$$

Therefore, $(M_{T_0 \wedge t})_{t \geq 0}$ is a continuous martingale. In addition,

$$E\left[M^2_{T_0 \wedge t}\right] \leq \liminf_{n \to \infty} E\left[M^2_{T_0 \wedge T_n \wedge t}\right] \leq E\left[\langle M \rangle_{T_0}\right] < \infty \quad \forall t \geq 0.$$

So, by the $\mathcal{L}^2$-martingale convergence theorem

$$\lim_{t \to \infty} M_{T_0 \wedge t} = M_{T_0} \text{ in } \mathcal{L}^2,$$

In particular,

$$E\left[M_{T_0}\right] = \lim_{t \to \infty} E\left[M_{T_0 \wedge t}\right] = E\left[M_0\right] = 0.$$

□

Corollary 1.4.32. *Let M be a continuous local martingale up to $T \equiv \infty$ (with continuous $\langle M\rangle$) such that*

$$E[\langle M\rangle_t] < \infty \quad \forall t \geq 0.$$

Then M is a square integrable martingale.

Proposition 1.4.33. *Let T be a stopping time and X, A continuous processes with continuous $\langle X\rangle$ and $\langle A\rangle = 0$ up to T. Assume $A_0 = 0$ and $X_0 = 0$. Then the following are equivalent:*

(i) X is a local martingale up to T with $\langle X\rangle = A$.

(ii) For all $\alpha \geq 0$

$$G_t^\alpha := \begin{cases} 1 & \text{if } t = 0 \\ \exp[\alpha X_t - \frac{1}{2}\alpha^2 A_t] & \text{on } \{t < T\} \end{cases}$$

is a local martingale up to T.

Proof. (i) ⇒ (ii): By Itô's formula for time dependent functions (cf. Proposition 1.3.24)

$$G_t^\alpha = G_0^\alpha + \int_0^t \alpha G_s^\alpha \, \mathrm{d}X_s + 0.$$

But the right hand side is a local martingale up to T by Proposition 5.3.9 for $f(X_t, \langle X\rangle_t) = \exp(\alpha X_t - \frac{1}{2}\alpha^2 \langle X\rangle_t)$.
(ii) ⇒ (i): By Itô's formula we get

$$\begin{aligned} X_t &= \frac{1}{\alpha}\big(\log G_t^\alpha \underbrace{- \log G_0^\alpha}_{=0}\big) + \frac{1}{2}\alpha A_t \\ &\overset{\text{Itô}}{=} \frac{1}{\alpha}\bigg(\int_0^t \frac{1}{G_s^\alpha}\,\mathrm{d}G_s^\alpha - \frac{1}{2}\int_0^t \frac{1}{(G_s^\alpha)^2}\,\mathrm{d}\langle G^\alpha\rangle_s\bigg) + \frac{1}{2}\alpha A_t \\ &\overset{1.3.22(ii)}{=} \frac{1}{\alpha}\bigg(\int_0^t \frac{1}{G_s^\alpha}\,\mathrm{d}G_s^\alpha - \frac{1}{2}\int_0^t \frac{1}{(G_s^\alpha)^2}\,(\alpha G_s^\alpha)^2\,\mathrm{d}\langle X\rangle_s\bigg) + \frac{1}{2}\alpha A_t \\ &= \frac{1}{\alpha}\int_0^t \frac{1}{G_s^\alpha}\,\mathrm{d}G_s^\alpha + \frac{\alpha}{2}(A_t - \langle X\rangle_t) \quad \forall \alpha \in \mathbb{R}\backslash\{0\}. \qquad (1.4.15) \end{aligned}$$

Here, $\frac{1}{\alpha}\int_0^t \frac{1}{G_s^\alpha}\,\mathrm{d}G_s^\alpha$ is a local martingale up to T by Proposition 5.3.9 with $G =]0,\infty[$. Consider $\alpha \neq \alpha'$ and take the difference of the two corresponding equalities to get

$$0 = M + \frac{\alpha - \alpha'}{2}(A_t - \langle X\rangle_t),$$

where M is a local martingale up to T. By Corollary 5.3.10(ii) it follows that $A_t - \langle X\rangle_t = A_0 - \langle X\rangle_0 = 0$ on $\{t < T\}$ P-a.s., and then (1.4.15) implies that X is a local martingale up to T. □

1.5. Levy's Characterization of Brownian Motion

Let $X = (X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ with continuous sample paths (and continuous quadratic variation $\langle X\rangle$).

Proposition 1.5.34 (Levy)**.** *Assume that X is a continuous local martingale up to ∞ with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$ such that X is $(\mathcal{F}_t)$-adapted. If $\langle X\rangle_t = t$ holds for all $t \geq 0$, then X is a Brownian motion.*

Remark 1.5.35. *Every continuous martingale can be transformed by a time change into a Brownian motion provided its quadratic variation is strictly increasing (see later). Therefore, Brownian motion is the only local martingale, where the quadratic variation is the usual time.*

Proof of 1.5.34. Corollary 1.4.32 implies that X is a martingale. For $u \in \mathbb{R}$ we apply the Itô-formula to

$$F(x) = e^{iux} = \cos(ux) + i\sin(ux),\ x \in \mathbb{R}.$$

Then, for all $s < t$

$$e^{iuX_{t\wedge T_n}} - e^{iuX_{s\wedge T_n}} = \int_0^{t\wedge T_n} iue^{iuX_r}\,\mathrm{d}X_r - \int_0^{s\wedge T_n} iue^{iuX_r}\,\mathrm{d}X_r + \frac{1}{2}\int_{s\wedge T_n}^{t\wedge T_n}(-u^2)e^{iuX_r}\,\underbrace{\mathrm{d}\langle X\rangle_r}_{=\,\mathrm{d}r},$$

where $T_n \nearrow \infty$ are stopping times such that

$$\left(\int_0^{t\wedge T_n} iue^{iuX_r}\,\mathrm{d}X_r\right)_{t\geq 0}$$

is a martingale for all n. Now take $E[\cdot|\mathcal{F}_s]$ of the equality above:

$$E\left[e^{iuX_{t\wedge T_n}} - e^{iuX_{s\wedge T_n}}\middle|\mathcal{F}_s\right] = \frac{1}{2}E\left[\int_{s\wedge T_n}^{t\wedge T_n}(-u^2)e^{iuX_r}\,\mathrm{d}r\middle|\mathcal{F}_s\right].$$

Hence, letting $n \to \infty$ we obtain

$$\frac{1}{2}E\left[\int_s^t(-u^2)e^{iuX_r}\,\mathrm{d}r\middle|\mathcal{F}_s\right] = E[e^{iuX_t} - e^{iuX_s}|\mathcal{F}_s].$$

Multiplication by e^{-iuX_s} yields

$$E[e^{iu(X_t-X_s)}|\mathcal{F}_s] - 1 = \frac{1}{2}E\left[\int_s^t(-u^2)e^{iu(X_r-X_s)}\,\mathrm{d}r\middle|\mathcal{F}_s\right].$$

Therefore, for all $A \in \mathcal{F}_s$

$$\underbrace{E[e^{iu(X_t-X_s)}, A]}_{=:\varphi(t)} - P(A) \overset{\text{Fubini}}{=} -\frac{1}{2}u^2\int_s^t \underbrace{E[e^{iu(X_r-X_s)}, A]}_{=\varphi(r)}\,\mathrm{d}r.$$

Then, $\varphi \in \mathcal{C}^1$ (since the right hand side is $\mathcal{C}^1$) and by differentiation

$$\dot{\varphi}(t) = -\frac{1}{2}u^2\varphi(t) \quad \forall t \geq s.$$

Solving this equation we get

$$\varphi(t) = C\cdot e^{-\frac{1}{2}u^2 t},\ t \geq s.$$

Substituting s for t yields

$$P(A) = \varphi(s),$$

which implies

$$C = P(A)e^{\frac{1}{2}u^2 s},$$

hence,

$$E[e^{iu(X_t-X_s)}; A] = e^{-\frac{1}{2}u^2(t-s)}P(A) \quad \forall A \in \mathcal{F}_s. \tag{1.5.16}$$

Taking $A := \Omega$ implies that $X_t - X_s$ is $N(0,(t-s))$ distributed by uniqueness of the Fourier-transform. By a monotone class argument, (1.5.16) implies that $X_t - X_s$ is independent of $\mathcal{F}_s$ (Exercise!). In particular,

$$X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_2} - X_{t_1}$$

are independent for all $0 \leq t_1 < t_2 < \ldots < t_n < \infty$. □

2. (Semi-)Martingales and Stochastic Integration

In this chapter we want to define

$$\int g(t, \cdot)\, \mathrm{d}M_t,$$

where g is defined "more general" and M_t is an arbitrary semimartingale. This is more general than so far, since we could only define

$$\int f(t, \mu_t)\, \mathrm{d}M_t, \quad \forall f \in \mathcal{C}^1.$$

2.1. Review of Some Facts from Martingale Theory

Fix a probability space $(\Omega, \mathcal{F}, P)$ and let $(\mathcal{F}_t)_{t\geq 0}$ be a right-continuous filtration. (Sometimes assume in addition that $N \in \mathcal{F}$, $P(N) = 0 \Rightarrow N \in \mathcal{F}_0$.) These conditions are also called *"usual conditions"*.

Proposition 2.1.1. *Let $(M_t)_{t\geq 0}$ be a martingale. Then there exists a version*

$$(\tilde{M})_{t\geq 0} \quad (\textit{i.e. } M_t = \tilde{M}_t \textit{ on } N(t)^c \textit{ such that } P(N(t)) = 0),$$

such that $t \mapsto \tilde{M}_t(\omega)$ is càdlàg *(i.e. is right-continuous and has left limits) for all $\omega \in \Omega$.*

Proof. Doob's upcrossing lemma (cf. [vWW90]). ☐

Proposition 2.1.2 (Optional stopping theorem)**.** *Let $(M_t)_{t\geq 0}$ be a martingale and T be a stopping time. Then $(M_{t\wedge T})_{t\geq 0}$ is a martingale.*

Proof. See [Röc11, Proposition VIII.4.2, Corollary VIII.4.3]. ☐

Proposition 2.1.3 (Doob's inequality)**.** *Let $p > 1$. Then*

$$\left\| \sup_{s\leq t} |M_s| \right\|_p \leq \frac{p}{p-1} \|M_t\|_p ,$$

where $\|\cdot\|_p = \|\cdot\|_{\mathcal{L}^p}$. In particular, if

$$M^* := \sup_{t\geq 0} |M_t|,$$

then

$$\|M^*\|_p \leq \frac{p}{p-1} \sup_{t\geq 0} \|M_t\|_p .$$

Proposition 2.1.4. *Let $p \geq 1$ and let $\left(M_t^{(n)}\right)_{t\geq 0}$, $n \in \mathbb{N}$, be a sequence of martingales such that*

$$M_t^{(n)} \overset{n\to\infty}{\longrightarrow} M_t \quad \textit{in } \mathcal{L}^p, \forall t \geq 0,$$

Then

(i) $(M_t)_{t\geq 0}$ *is again a martingale in* $\mathcal{L}^p$.

(ii) *If* $p > 1$ *and* $(\mathcal{F})_{t\geq 0}$ *is such that all* P*-zero sets in* $\mathcal{F}_t$ *are in* $\mathcal{F}_0$ *and each* $\left(M_t^{(n)}\right)_{t\geq 0}$ *has* P*-a.s. (right-) continuous ((càdlàg)) sample paths, then* $(M_t)_{t\geq 0}$ *has a (right-) continuous ((càdlàg))* $(\mathcal{F}_t)$ *-adapted version again denoted by* $(M_t)_{t\geq 0}$ *and*

$$M_t^{(n)} \overset{n\to\infty}{\longrightarrow} M_t$$

in $\mathcal{L}^p(\Omega, \mathcal{F}, P;\, C([0,t];\mathbb{R}))$ *and hence locally uniformly in* t *and in* $\mathcal{L}^p$ *and has a locally uniformly in* t P*-a.s. convergent subsequence.*

Proof. (i) Obvious.

(ii) Fix $t > 0$. Since $M^n - M^m$ is a martingale, we have by Doob

$$\left\| \sup_{s\leq t} |M_s^{(n)} - M_s^{(m)}| \right\|_p \leq \frac{p}{p-1} \left\| M_t^{(n)} - M_t^{(m)} \right\|_p .$$

Since they are $\mathcal{L}^p$-convergent, for some subsequence $(n_k)_{k\in\mathbb{N}}$

$$\sum_{k=1}^{\infty} \left\| \sup_{s\leq t} \left| M_s^{(n_{k+1})} - M_s^{(n_k)} \right| \right\|_p \leq \frac{p}{p-1} \sum_{k=1}^{\infty} \left\| M_t^{(n_{k+1})} - M_t^{(n_k)} \right\|_p < \infty.$$

Therefore,

$$P\left(\sum_{k=1}^{\infty} \sup_{s\leq t} \left| M_s^{(n_{k+1})} - M_s^{(n_k)} \right| < \infty \right) = 1.$$

Hence, $M_s^{(n_k)} \overset{k\to\infty}{\to} M_s$ uniformly on $[0,t]$ P-a.s. (cf. Proof of Riesz-Fischer!). Then up to a P-zero set M_s is $\mathcal{F}_s$-measurable, since so is each $M_s^{(n_k)}$. But $\mathcal{F}_s$ contains all P-zero sets in $\mathcal{F}_0$. Therefore, M_s is $\mathcal{F}_s$-measurable.

□

Remark 2.1.5 (Localization)**.** *Let* $(M_t)_{t\geq 0}$ *be a continuous local martingale (up to* ∞*) such that* $M_0 = 0$. *Set*

$$R_n(\omega) := \inf\left\{ t > 0 \middle| \, |M_t(\omega)| > n \right\} \left(\overset{n\to\infty}{\longrightarrow} \infty \right).$$

Claim: *For all* $n \in \mathbb{N}$ $(M_{t\wedge R_n})_{t\geq 0}$ *is a continuous bounded martingale.*

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a localizing sequence for M, such that $T_k \leq k$. (Otherwise consider $T_k \wedge k$.) Then for $s \leq t$ and n fixed

$$E[M_{t\wedge R_n \wedge T_k}; A_s] = E[M_{s\wedge R_n\wedge T_k}; A_s] \quad \forall A_s \in \mathcal{F}_s.$$

By Lebesgue, for all $A_s \in \mathcal{F}_s$,

$$E[M_{t\wedge R_n\wedge T_k}; A_s] \overset{k\to\infty}{\longrightarrow} E[M_{t\wedge R_n}; A_s]$$

and, since $|M_{t\wedge R_n\wedge T_k}| \leq n$ for fixed n,

$$E[M_{s\wedge R_n\wedge T_k}; A_s] \overset{k\to\infty}{\longrightarrow} E[M_{s\wedge R_n}; A_s].$$

So, we get $E[M_{t\wedge R_n}; A_s] = E[M_{s\wedge R_n}; A_s]$. □

2.2. Quadratic Variation and Covariation for Continuous Local Martingales

Let $(M_t)_{t\geq 0}$ be a càdlàg martingale.
Approach to stochastic integration:
Assume initially $M_t \in \mathcal{L}^2$ for all $t \geq 0$. (If $M_t \notin \mathcal{L}^2$, then localize.) By Jensen's inequality $(M_t^2)_{t\geq 0}$ is a submartingale. Then one can show (by the Doob-Meyer decomposition) that there exists a unique adapted process $(\langle M\rangle_t)_{t\geq 0}$ with $\langle M\rangle_0 = 0$, increasing, right-continuous, predictable (see below) such that

$$\left(M_t^2 - \langle M\rangle_t\right)_{t\geq 0} \tag{$\star$}$$

is a martingale. Then, $\langle M\rangle$ is the variance process of M, i.e.

$$\begin{aligned}
&E[(M_t - M_s - \overbrace{E[M_t - M_s]}^{=0})^2|\mathcal{F}_s] \quad \text{(conditioned variance of } M_t - M_s \text{ given } \mathcal{F}_s\text{)}\\
=&E[M_t^2 - M_s^2|\mathcal{F}_s] \overset{(\star)}{=} E[\langle M\rangle_t - \langle M\rangle_s|\mathcal{F}_s].
\end{aligned}$$

In [Röc11] we proved the Doob-Meyer decomposition in discrete time. In continuous time for càdlàg martingales this is much more difficult (cf. [Kry]).
We are going to prove the Doob-Meyer decomposition only for *continuous* martingales. In the stochastic integration theory below, we shall, however, allow càdlàg martingales, simply assuming Doob-Meyer without proof, and refering to [vWW90, Cor. 6.6.3] instead.
For a continuous martingale M we shall construct the process $\langle M\rangle$ such that it pathwise coincides P-a.s. with the quadratic variation of M along $(\tau_n)_{n\in\mathbb{N}}$ of the previous chapter. Let M be a continuous local martingale (up to ∞) and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,\infty)$ such that

$$|\tau_n| \overset{n\to\infty}{\longrightarrow} 0$$

and

$$t_{N_n}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty.$$

Definition 2.2.6. *Let*

$$V_t^{(n)} := \sum_{s\in\tau_n} \left(M_{s'\wedge t} - M_{s\wedge t}\right)^2, \quad n\in\mathbb{N},\, t\geq 0$$

be the quadratic variation of M along τ_n on $[0,t]$. *Here, s' denotes the successor of s in τ_n. Note that $t\mapsto V_t^{(n)}$ is P-a.s. continuous for all $n\in\mathbb{N}$.*

Remark 2.2.7. *Since $(a-b)^2 = a^2 - 2b(a-b) - b^2$,*

$$\begin{aligned}
V_t^{(n)} &= M_t^2 - M_0^2 - 2\sum_{u\in\tau_n} M_{u\wedge t}\left(M_{u'\wedge t} - M_{u\wedge t}\right)\\
&= M_t^2 - M_0^2 - 2\underbrace{\sum_{u\in\tau_n,\, u\leq t} M_u\left(M_{u'\wedge t} - M_u\right)}_{\text{local martingale}}.
\end{aligned}$$

Proposition 2.2.8. *Assume that $\mathcal{F}_0$ contains all P-zero sets. Let M be a continuous (for all $\omega\in\Omega$), local martingale (up to ∞). Fix (τ_n) as above and let $(V^{(n)})_{n\in\mathbb{N}}$ be as in Definition 2.2.6. Then there exists a continuous increasing $(\mathcal{F}_t)$-adapted process $\langle M\rangle$ with $\langle M\rangle_0 = 0$ such that*

(i) *$\langle M\rangle_t = \lim_{n\to\infty} V_t^{(n)}$ in P-measure locally uniformly in $t \geq 0$.*
(Convergence is even in $\mathcal{L}^2(\Omega, \mathcal{F}, P; \mathcal{C}([0,t];\mathbb{R}))$ for all $t \geq 0$, if M is a bounded martingale.)

(ii) *$M^2 - \langle M\rangle$ is a continuous local martingale.*

(iii) *If $M_t \in \mathcal{L}^2$ for all $t \geq 0$ and if M_t is a martingale, then $M_t^2 - \langle M\rangle_t$ is a martingale. In particular,*

$$E[M_t^2] = E[M_0^2] + E[\langle M\rangle_t] \quad \forall t \geq 0.$$

(Warning: At this stage $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ depends on (τ_n)!)

Proof. (i) **Case 1:** M is bounded (i.e. $\sup_{t\geq 0, \omega\in\Omega} |M_t(\omega)| < \infty$) and a martingale.
Claim: $(V_t^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2$ for all $t \geq 0$.
Take $m, n \geq N, N$ large so that $t \leq t_{N_n}^{(n)}, t \leq t_{N_m}^{(m)}$. Define

$$V_t^{(m,n)} := M_t^2 - M_0^2 - 2 \sum_{s\in\tau_n\cup\tau_m} M_s\left(M_{s'\wedge t} - M_{s\wedge t}\right) \left(= V_t^{(n,m)}\right).$$

Here, s' denotes the successor of s in $\tau_n \cup \tau_m$. Then by Remark 2.2.7

$$V_t^{(m,n)} - V_t^{(n)} = -2 \sum_{u\in\tau_n} \Bigg(\Bigg(\sum_{\substack{u\leq s<u' \\ s\in\tau_n\cup\tau_m}} M_s\left(M_{s'\wedge t} - M_{s\wedge t}\right)\Bigg) - M_u(M_{u'\wedge t} - M_{u\wedge t})\Bigg),$$

where u' is the successor in τ_n and s' the one in $\tau_n \cup \tau_m$. Using the fact that

$$M_{u'\wedge t} - M_{u\wedge t} = \sum_{\substack{u\leq s<u' \\ s\in\tau_n\cup\tau_m}} (M_{s'\wedge t} - M_{s\wedge t})$$

we conclude that

$$V_t^{(m,n)} - V_t^{(n)} = -2 \sum_{u\in\tau_n} \sum_{\substack{u\leq s<u' \\ s\in\tau_n\cup\tau_m}} (M_s - M_u)(M_{s'\wedge t} - M_{s\wedge t}).$$

Hence,

$$E\left[\left(V_t^{(m,n)} - V_t^{(n)}\right)^2\right] = 4E\Bigg[\sum_{u\in\tau_n} \sum_{\substack{u\leq s<u' \\ s\in\tau_n\cup\tau_m}} (M_s - M_u)^2(M_{s'\wedge t} - M_{s\wedge t})^2\Bigg]$$

Note that all occurring mixed terms are zero by the martingale property (exercise!). Furthermore,

$$E\left[\left(V_t^{(m,n)} - V_t^{(n)}\right)^2\right] \leq 4E\left[\Delta_N \sum_{s\in\tau_n\cup\tau_m} (M_{s'\wedge t} - M_{s\wedge t})^2\right],$$

where

$$\Delta_N := \sup_{\tilde{n}\geq N} \sup_{\tilde{m}} \sup \left\{(M_s - M_u)^2 \middle| s \in \tau_{\tilde{n}} \cup \tau_{\tilde{m}}, u \in \tau_{\tilde{n}}, u \leq s \leq u', s, u \leq t\right\} \overset{N\to\infty}{\longrightarrow} 0 \; P\text{-a.s.}$$

is bounded in N and ω, since M is bounded. Then, by Cauchy-Schwarz (since $n, m \geq N$)

$$E\left[\left(V_t^{(m,n)} - V_t^{(n)}\right)^2\right] \leq 4\left(E\left[\Delta_N^2\right]\right)^{1/2} \left\| \sum_{s\in\tau_n\cup\tau_m} (M_{s'\wedge t} - M_{s\wedge t})^2 \right\|_2 .$$

By Lebesgue's dominated convergence theorem,

$$\left(E\left[\Delta_N^2\right]\right)^{1/2} = \|\Delta_N\|_2 \overset{N\to\infty}{\longrightarrow} 0,$$

since $M(\omega)$ is uniformly continuous on $[0,t]$ for all t and $(M_t)_{t\geq 0}$ is bounded in (ω, t). But for $c := \sup_{t,\omega} |M_t(\omega)|$ and for all $n, m \in \mathbb{N}$

$$\begin{aligned}
&E\left[\left(\sum_{s\in\tau_n\cup\tau_m} (M_{s'\wedge t} - M_{s\wedge t})^2\right)^2\right] \\
=&E\left[\sum_{s\in\tau_n\cup\tau_m} (M_{s'\wedge t} - M_{s\wedge t})^4\right] \\
&\quad + 2E\Bigg[\sum_{s\in\tau_n\cup\tau_m} \sum_{\substack{u\in\tau_n\cup\tau_m \\ s<u}} (M_{s'\wedge t} - M_{s\wedge t})^2 (M_{u'\wedge t} - M_{u\wedge t})^2\Bigg] \\
\leq& 4c^2 \underbrace{E\left[\sum_{s\in\tau_n\cup\tau_m} (M_{s'\wedge t} - M_{s\wedge t})^2\right]}_{=:S_1} \\
&\quad + 2 \sum_{s\in\tau_n\cup\tau_m} \underbrace{\sum_{\substack{u\in\tau_n\cup\tau_m \\ s<u}} E\left[(M_{s'\wedge t} - M_{s\wedge t})^2 \cdot E[(M_{u'\wedge t} - M_{u\wedge t})^2 | \mathcal{F}_{u\wedge t}]\right]}_{=:S_2}
\end{aligned}$$

and

$$\begin{aligned}
S_1 &= \sum_{s\in\tau_n\cup\tau_m} E[M_{s'\wedge t}^2 - 2M_{s'\wedge t}M_{s\wedge t} + M_{s\wedge t}^2] \\
&= \sum_{s\in\tau_n\cup\tau_m} E[M_{s'\wedge t}^2 + M_{s\wedge t}^2 - 2E[M_{s'\wedge t}M_{s\wedge t}|\mathcal{F}_s]] \\
&= \sum_{s\in\tau_n\cup\tau_m} E[M_{s'\wedge t}^2 + M_{s\wedge t}^2 - 2M_{s\wedge t}\underbrace{E[M_{s'\wedge t}|\mathcal{F}_s]}_{M_{s\wedge t}}] \\
&= \sum_{s\in\tau_n\cup\tau_m} E[M_{s'\wedge t}^2 - M_{s\wedge t}^2] = E[M_t^2 - M_0^2]
\end{aligned}$$

In addition,

$$\begin{aligned}
S_2 &= \sum_{\substack{u\in\tau_n\cup\tau_m \\ s<u}} E\left[(M_{s'\wedge t} - M_{s\wedge t})^2 \cdot E[M_{u'\wedge t}^2 - M_{u\wedge t}^2 | \mathcal{F}_{u\wedge t}]\right] \\
&= \sum_{\substack{u\in\tau_n\cup\tau_m \\ s<u}} E\left[(M_{s'\wedge t} - M_{s\wedge t})^2 \cdot (M_{u'\wedge t}^2 - M_{u\wedge t}^2)\right] \\
&= E[(M_{s'\wedge t} - M_{s\wedge t})^2 (M_t^2 - M_{s'\wedge t}^2)] \\
&\leq 2c^2 E[E[(M_{s'\wedge t} - M_{s\wedge t})^2 | \mathcal{F}_{s\wedge t}]] = 2c^2 E[M_{s'\wedge t}^2 - M_{s\wedge t}^2].
\end{aligned}$$

Thus,

$$\begin{aligned}
&E\left[\Big(\sum_{s\in\tau_n\cup\tau_m}(M_{s'\wedge t}-M_{s\wedge t})^2\Big)^2\right]\\
\le&4c^2E[M_t^2-M_0^2]+4c^2\sum_{s\in\tau_n\cup\tau_m}E[M_{s'\wedge t}^2-M_{s\wedge t}^2]\\
\le&8c^2E[M_t^2-M_0^2]\le 16c^4.
\end{aligned}$$

Alltogether, we obtain for all $n,m\ge N$

$$\begin{aligned}
\Big(E\left[(V_t^{(n)}-V_t^{(m)})^2\right]\Big)&\le 2E\left[(V_t^{(n)}-V_t^{(n,m)})^2\right]+2E\left[(V_t^{(m,n)}-V_t^{(m)})^2\right]\\
&\le 8\left\|\Delta_N\right\|_2 4c^2+8\left\|\Delta_N\right\|_2 4c^2\overset{N\to\infty}{\longrightarrow}0
\end{aligned}$$

and the claim is proved.

Let $V_t:=\mathcal{L}^2\text{-}\lim_{n\to\infty}V_t^{(n)}$, $t\ge 0$. Define

$$Y_t^{(n)}:=2\sum_{s\in\tau_n}M_s(M_{s'\wedge t}-M_{s\wedge t}).$$

Then each $(Y_t^{(n)})_{t\ge0}$ is an $(\mathcal{F}_t)$-martingale and by Remark 2.2.7 and the above

$$Y_t^{(n)}=M_t^2-M_0^2-V_t^{(n)}\overset{n\to\infty}{\longrightarrow}M_t^2-M_0^2-V_t=:Y_t\quad\text{in }\mathcal{L}^2.\tag{$**$}$$

Hence, by Proposition 2.1.4 Y_t is a martinale and has a P-a.s. continuous version and, therefore, V_t has a continuous version $\langle M\rangle$ and $\langle M\rangle$ is $(\mathcal{F}_t)$-adapted since all P-zero sets in $\mathcal{F}$ are in $\mathcal{F}_0$, hence in every $\mathcal{F}_t$. Furthermore, by Proposition 2.1.4(ii), $Y^{(n)}\to Y$ as $n\to\infty$ in $\mathcal{L}^2(\Omega,\mathcal{F},P;\,C([0,t];\,\mathbb{R}))$, hence $V^{(n)}\to\langle M\rangle$ as $n\to\infty$ in $\mathcal{L}^2(\Omega,\mathcal{F},P;\,C([0,t];\,\mathbb{R}))$, i.e.

$$E\left[\sup_{s\le t}|V_s^{(n)}-\langle M\rangle_s|^2\right]\overset{n\to\infty}{\longrightarrow}0.$$

Case 2: M as in the assertion.

Without loss of generality $M_0\equiv 0$. (Otherwise consider $(M_t-M_0)_{t\ge0}$.). Let R_k be as in 2.1.5, i.e.

$$R_k:=\inf\{t>0\,\big|\,|M_t|>k\}$$

and therefore, R_k is a stopping time. Define

$$M_t^k:=M_{t\wedge R_k},\ t\ge 0.$$

By Remark 2.1.5 M_t^k is a bounded martingale. Hence, by Case 1 there exists $V_t^k:=\langle M^k\rangle_t$, $t\ge0$. Let $(V_t^k)^{(n)}$, $t>0$, be the corresponding approximations along (τ_n). Then there exists a subsequence $(n_l)_{l\in\mathbb{N}}$ and $\Omega_0\in\mathcal{F}_0$, $P(\Omega_0)=1$ such that for all $k\in\mathbb{N}$ (cf. Proposition 2.1.4(ii))

$$(V_s^k)^{(n_l)}(\omega)\overset{l\to\infty}{\longrightarrow}\langle M^k\rangle_s(\omega)$$

locally uniformly for $s\in[0,t]$ and for P-a.e. $\omega\in\Omega_0$. But

$$1_{\{t\le R_k\}}V_t^{(n_l)}=1_{\{t\le R_k\}}V_{t\wedge R_k}^{(n_l)}=1_{\{t\le R_k\}}(V_t^k)^{(n_l)}\overset{l\to\infty}{\longrightarrow}1_{\{t\le R_k\}}\langle M^k\rangle_t\ P-\text{a.s..}\tag{$***$}$$

Hence, we can (well-)define

$$\langle M\rangle_t(\omega) := \begin{cases} \langle M^k\rangle_t(\omega), & \text{with } k \in \mathbb{N} \text{ such that } \omega \in \{t \le R_k\} \cap \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that by the continuity, hence local boundedness of $t \to M_t(\omega)$ we have that $R_k(\omega) \nearrow \infty \ \forall\, \omega \in \Omega$, hence $\Omega_0 = \bigcup_{k\in\mathbb{N}} \{t < R_k\} \cap \Omega_0$, so $\langle M\rangle_t(\omega)$ is defined $\forall\, \omega \in \Omega$ by the above and by $(***)$ this definition is independent of k. Then $\langle M\rangle$ is P-a.s. continuous and $(\mathcal{F}_t)$-adapted.

Furthermore, since $R_k \to \infty$ as $k \to \infty$, for $\delta > 0$ we can fix $k = k(\delta)$ large enough, such that

$$P[R_k < t] < \delta.$$

Then

$$\begin{aligned}
&P\left(\sup_{0\le s\le t} \left|V_s^{(n)} - \langle M\rangle_s\right| \ge \varepsilon\right) \\
=&P\left(\sup_{0\le s\le t} \left|V_s^{(n)} - \langle M\rangle_s\right| \ge \varepsilon,\, R_k < t\right) + P\left(\sup_{0\le s\le t} \left|V_s^{(n)} - \langle M\rangle_s\right| \ge \varepsilon,\, R_k \ge t\right) \\
\le&P\left(R_k < t\right) + P\left(\sup_{0\le s\le t} \left|(V_s^k)^{(n)} - \langle M^k\rangle_s\right| \ge \varepsilon,\, R_k \ge t\right) \\
\le&P\left(R_k < t\right) + P\left(\sup_{0\le s\le t} \left|(V_s^k)^{(n)} - \langle M^k\rangle_s\right| \ge \varepsilon\right).
\end{aligned}$$

But by the first part of the proof and Chebychev's inequality

$$P\left(\sup_{0\le s\le t} \left|(V_s^k)^{(n)} - \langle M^k\rangle_s\right| \ge \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\limsup_{n\to\infty} P\left(\sup_{0\le s\le t} \left|V_s^{(n)} - \langle M\rangle_s\right| \ge \varepsilon\right) \le P[R_k < t] < \delta.$$

It remains to show that $t \mapsto \langle M\rangle_t$ is increasing P-a.s.. But for fixed $t \ge 0$ we have that

$$V_t^{(n)} = \sum_{s\in\tau_n, s\le t} (M_{s'} - M_s)^2 + N_t^{(n)} \overset{n\to\infty}{\longrightarrow} \langle M\rangle_t,$$

where the sum is increasing in t and

$$N_t^{(n)} := (M_t - M_{\delta_t^{(n)}})^2 \overset{n\to\infty}{\longrightarrow} 0$$

and

$$\delta_t^{(n)} := \sup\{s \in \tau_n | s \le t\}.$$

Hence, (i) is completely proved.

(ii) Continuity is clear. By $(**)$ on p. 28 we know, that P-a.s.

$$M_{t\wedge R_k}^2 - \langle M\rangle_{t\wedge R_k} = (M_t^k)^2 - \langle M^k\rangle_t = Y_t^k + M_0^2$$

and the right hand side is a martingale. Hence, $M^2 - \langle M\rangle$ is a local martingale for T up to ∞.

(iii) Without loss of generality $M_0 = 0$. We have to show that $M_T^2 - \langle M \rangle_T \in \mathcal{L}^1(P)$ and that

$$E[\langle M \rangle_T] = E[M_T^2]$$

for all bounded stopping times T. By the monotone convergence theorem, we get that

$$E[\langle M \rangle_T] = \lim_{n\to\infty} E[\langle M \rangle_{T\wedge R_n}] \overset{\substack{\text{proof of}\\(ii)}}{=} \lim_{n\to\infty} E[M_{T\wedge R_n}^2] \overset{\text{Fatou}}{\geq} E[M_T^2].$$

On the other hand, by the submartingale property, $\lim\limits_{n\to\infty} E[M_{T\wedge R_n}^2] \leq E[M_T^2]$, so

$$E[\langle M \rangle_T] = E[M_T^2].$$

Finally, by assumption,

$$E[M_T^2] \leq \left[M^2_{\sup_{\omega\in\Omega} T(\omega)}\right] < \infty,$$

so that

$$E[M_T^2 - \langle M \rangle_T] = 0.$$

□

Remark 2.2.9. *(i) One can drop the assumption that "all P-zero sets in $\mathcal{F}$ are in $\mathcal{F}_0$", but one only gets that the particular version of $\langle M \rangle$ of V is only right-continuous and adapted. But it is continuous only for $\omega \in N^c$, with a P-zero set $N \in \mathcal{F}$.*

(ii) Since convergence in probability implies P-a.s. convergence of a subsequence, it follows by Proposition 2.2.8(i) that for some subsequence $(n_k)_{k\in\mathbb{N}}$

$$P(V_s^{(n_k)} \overset{k\to\infty}{\longrightarrow} \langle M \rangle_s \text{ locally uniformly on } [0,t],\ \forall t \geq 0) = 1.$$

(iii) Since $\langle M \rangle$ is exactly the pathwisely defined continuous quadratic variation process of M in Chapter I, we can apply all results from Chapter I for P-a.e.$\omega \in \Omega$ fixed, i.e.

$$\langle M(\omega) \rangle = \langle M \rangle(\omega).$$

Corollary 2.2.10. *Assume $M_0 \equiv 0$. Then*

$$M_t^2 - \langle M \rangle_t = 2\int_0^t M_s \, dM_s.$$

Hence, the continuous local martingale $M^2 - \langle M \rangle$ is an Itô-integral (cf. Corollary 5.3.10(i)).

Corollary 2.2.11. *P-a.s. the paths of a continuous local martinale are either constant or of unbounded variation.*

Proof. Apply Corollary 5.3.10(ii). □

Corollary 2.2.12. *$\langle M \rangle$ is the unique increasing continuous adapted process such that $\langle M \rangle_0 = 0$ and $M^2 - \langle M \rangle$ is a continuous local martingale. In particular, $\langle M \rangle$ is (up to a P-zero set in $\mathcal{F}$) independent of the chosen sequence of partitions (τ_n) in 2.2.6.*

Proof. Let A, B be two such processes. Then $M^2 - A$, $M^2 - B$ are continuous local martingales. Hence, $B - A$ is a continuous local martingale (with respect to $(\mathcal{F}_t)$ since A, B are $(\mathcal{F}_t)$-adapted) of bounded variation. Therefore, by Corollary 2.2.11

$$B - A = c = B_0 - A_0 = 0 \quad P\text{-a.s.}.$$

□

Definition 2.2.13. *(cf. 5.2.3, 1.3.15) Let M, N be continuous local martingales. Then define the* covariation process *of M, N by*

$$\langle M, N\rangle_t := \frac{1}{2}\left(\langle M+N\rangle_t - \langle M\rangle_t - \langle N\rangle_t\right).$$

Remark 2.2.14. *Since $M+N$ is a continuous local martingale, $\langle M+N\rangle$ exists as a continuous process by Proposition 2.2.8 Therefore, since $\frac{1}{2}[(a+b)^2-a^2-b^2] = ab \quad \forall\, a, b \in \mathbb{R}$ („polarization identity“)*

$$\langle M, N\rangle = \lim_{n\to\infty} \sum_{s\in\tau_n} (M_{s'\wedge t} - M_{s\wedge t})(N_{s'\wedge t} - N_{s\wedge t}).$$

(Exercise!)

Proposition 2.2.15. *Let M, N be continuous local martingales. $\langle M, N\rangle$ is uniquely determined by the following:*

(i) Its paths are (P-a.s.) continuous and of bounded variation and $\langle M, N\rangle_0 = 0$.

(ii) $M\cdot N - \langle M, N\rangle$ is a continuous local martingale. In particular, $\langle M, N\rangle \equiv 0$ if and only if $M \cdot N$ is a local martingale.

Proof. Analogous to the case $M = N$ in Corollary 2.2.12 (or use polarization.). ☐

Lemma 2.2.16. *Let M, N be a continuous local martingale. Let $G(= G_s(\omega), s \geq 0, \omega \in \Omega)$ and H be $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable. Then*

$$\left|\int_0^t H_s(\omega)G_s(\omega)\ d\langle M, N\rangle_s(\omega)\right| \leq \left(\int_0^t H_s^2\ d\langle M\rangle_s(\omega)\right)^{\frac{1}{2}} \left(\int_0^t G_s^2\ d\langle N\rangle_s(\omega)\right)^{\frac{1}{2}}.$$

In particular, we obtain (by Cauchy) the Kunita-Watanabe inequality

$$E\left[\left|\int_0^t H_s(\omega)G_s(\omega)\ d\langle M, N\rangle_s\right|\right] \leq E\left[\int_0^t H_s^2\ d\langle M\rangle_s\right]^{\frac{1}{2}} E\left[\int_0^t G_s^2\ d\langle N\rangle_s(\omega)\right]^{\frac{1}{2}}.$$

Proof. Exercise (cf. [RW87, Vol II, p.50]). ☐

2.3. Construction of stochastic integrals

Fix a probability space $(\Omega, \mathcal{F}, P)$ with right-continuous filtration $(\mathcal{F}_t)$ such that all P-zero sets in $\mathcal{F}$ are in $\mathcal{F}_0$. We want to define

$$H.M := \int_0^{\cdot} H_s \, \mathrm{d}M_s \quad \text{as a martingale}$$

for càdlàg $(\mathcal{F}_t)$-martingales and most general H, where *càdlàg* means right-continuous for all $\omega \in \Omega$ with left limits P-a.s.. (The latter is automatic by [vWW90, p.47, 3.25, 3.26].) M_s is called *integrator process* and H_s *integrand process.*
"Admissible integrators" are given by

$$\mathcal{M}^2 := \mathcal{M}^2(\Omega, \mathcal{F}, P) := \left\{ M \middle| M \text{ is a càdlàg martingale, } M_0 = 0, \ \|M\|^2 := \sup_{t \geq 0} E[M_t^2] < \infty \right\}.$$

We define

$$M_c^2 := \{ M \in \mathcal{M}^2 \mid M \text{ has } P\text{-a.s. continuous sample paths} \}$$

We know that for each $M \in \mathcal{M}^2$ the Doob-Meyer decomposition holds, that is:
For all $M \in \mathcal{M}^2$ there exists a process $\langle M \rangle$ such that it is the unique predictable, right continuous increasing, adapted process and that $M_0 \equiv 0$ and $M^2 - \langle M \rangle$ is a martingale. We only proved that this $\langle M \rangle$ exists and is unique if $M \in \mathcal{M}_c^2$ (see Proposition 2.2.8 and Corollary 2.2.12). For the general case see [Kry] or [vWW90, p.130, Cor. 6.6.3]. However, if $M \in \mathcal{M}^2 \backslash \mathcal{M}_c^2$, then $\langle M \rangle$ is NOT equal to the quadratic variation of M.

Remark 2.3.17. *(i) Define* $\mathcal{F}_\infty := \sigma \left(\bigcup_{t \geq 0} \mathcal{F}_t \right)$*. Clearly,* $M_t \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$ *for all* $M \in \mathcal{M}^2$ *and by the* $\mathcal{L}^2$*-martingale convergence theorem we have for any càdlàg martingale such that* $M_0 = 0$*:*

$$M \in \mathcal{M}^2 \Leftrightarrow \exists M_\infty := \lim_{t \to \infty} M_t \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P).$$

In this case

$$M_t = E[M_\infty | \mathcal{F}_t], \, t \geq 0.$$

So $(M_t)_{t \in [0,\infty]}$ *is a martingale.*

(ii) Starting with the norm $\|\cdot\|$ *define an inner product on* $\mathcal{M}^2$ *by polarization:*

$$(M, N) := \frac{1}{4}(\|M+N\|^2 - \|M-N\|^2), \quad \forall M, N \in \mathcal{M}.$$

It remains to check that it is really an inner product. This could be done by checking that $\|\cdot\|$ *on* $\mathcal{M}^2$ *satisfies*

$$\|M+N\|^2 + \|M-N\|^2 = 2\|M\|^2 + 2\|N\|^2 \quad \textit{(parallelogram identity)}$$

and using J. von Neumann's theorem. But (iii) implies this more easily.

(iii) Let $M \in \mathcal{M}^2$*. Since* M^2 *is a submartingale and because of (i) we have*

$$E[M_\infty^2] \overset{(i)}{=} \lim_{t \to \infty} E[M_t^2] = \sup_{t \geq 0} E[M_t^2] = \|M\|^2.$$

Hence, for $N \in \mathcal{M}^2$*, by polarization*

$$\begin{aligned} E[M_\infty N_\infty] &= \frac{1}{4}\left(E[M_\infty + N_\infty]^2 - E[M_\infty - N_\infty]^2\right) \\ &= \frac{1}{4}(\|M+N\|^2 - \|M-N\|^2) = (M, N). \end{aligned}$$

Therefore, $(\cdot, \cdot)$ *is really an inner product with corresponding norm* $\|\cdot\|$*.*

(iv) *Note that* $\|M\|^2 = \sup_{t\geq 0} E[M_t^2] \leq E[\sup_{t\geq 0} M_t^2] \overset{Doob}{\leq} 4 \sup_{t\geq 0} E[M_t^2] \leq 4\|M\|^2$. *So, on* $\mathcal{M}^2$ *the norms* $\|\cdot\|$ *and* $\|\cdot\|_{L^2(\Omega,\mathcal{F},P;\,C_b([0,\infty);\,\mathbb{R}))}$ *are equivalent.*

Proposition 2.3.18. (i) $\mathcal{M}^2$ *is a Hilbert space and*

$$(M,N) = E[M_\infty N_\infty] = E[\langle M,N\rangle_\infty],$$

where

$$\langle M,N\rangle_\infty := \lim_{t\to\infty} \langle M,N\rangle_t.$$

(ii) $\mathcal{M}_c^2$ *is a closed subspace of* $\mathcal{M}^2$.

Proof. (i) $(\mathcal{M}^2, \|\cdot\|)$ is an inner product space (Pre-Hilbert space) by Remark 2.3.17. Furthermore, it is complete by Proposition 2.1.4(ii) and because

$$\mathcal{L}^2(\Omega,\mathcal{F},P;C_b([0,\infty),\mathbb{R})),$$

is complete.
Finally,

$$\|M\|^2 = E[M_\infty^2] = \lim_{t\to\infty} E[M_t^2] = \lim_{t\to\infty} E[\langle M\rangle_t] \overset{B.Levi}{=} E[\underbrace{\lim_{t\to\infty} \langle M\rangle_t}_{=:\langle M\rangle_\infty}].$$

So, by polarization the last assertion follows.

(ii) Apply Proposition 2.1.4(ii).

□

Now we define stochastic integrals with $M \in \mathcal{M}^2$ as integrators, but first for elementary functions:

Definition 2.3.19. *Define* $\mathcal{E}$ *to be the set of all processes* H *on* $]0,\infty[\times\Omega$ *which are of the following form:*
For $t > 0$, $\omega \in \Omega$

$$H_t(\omega) := \sum_{i=0}^{n-1} h_{t_i}(\omega) 1_{]t_i,t_{i+1}]}(t), \text{ “elementary (predictable) adapted processes”}$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n < \infty$ *and* h_{t_i} *are* $\mathcal{F}_{t_i}$*-measurable bounded. For* $M \in \mathcal{M}^2$ *and* H *as above define*

$$\int_0^t H_s\, dM_s := \sum_{i=0}^{n-1} h_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}),\ t \geq 0.$$

An easy exercise shows that this is independent of the representation of H*! Set*

$$(H.M)_t := \int_0^t H_s\, dM_s, \quad t \geq 0.$$

Then $H.M$ *is called the* stochastic integral of H with respect to M.

Lemma 2.3.20. *Let* $H \in \mathcal{E}$.

(i) $H.M \in \mathcal{M}^2$ *and if* $M \in \mathcal{M}_c^2$*, then* $H.M \in \mathcal{M}_c^2$.

(ii)

$$\begin{aligned}\langle H.M\rangle_t &= \int_0^t H_s^2 \ d\langle M\rangle_s \quad \forall t \geq 0\\ &= \sum_{i=0}^{n-1} h_{t_i}^2(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i\wedge t}).\end{aligned}$$

In particular,

$$\|H.M\|^2 = E\left[\int_0^\infty H_s^2 \ d\langle M\rangle_s\right]. \qquad \text{“Isometry”}$$

Note that here we always set

$$\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s := \int_{]0,t]} H_s^2 \,\mathrm{d}\langle M\rangle_s.$$

Proof. (i) By definition we see that:

- if M is a cadlag (continuous) martingale, then $H.M$ is càdlàg (continuous),
- $H.M$ is adapted,
- $(H.M)_0 = 0$.

Furthermore, since M is a martingale,

$$\begin{aligned}\sup_{t\geq 0} E[(H.M)_t^2] &= \sup_{t\geq 0} E\left[\sum_{i=0}^{n-1} h_{t_i}^2 \left(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}\right)^2\right]\\ &= \sup_{t\geq 0}\sum_{i=0}^{n-1} E\left[h_{t_i}^2 \left(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}\right)^2\right]\\ &\leq \sup_{0\leq i\leq n} \left\|h_{t_i}^2\right\|_\infty \sup_{t\geq 0}\sum_{i=0}^{n-1} E\left[\left(M_{t_{i+1}\wedge t}^2 - M_{t_i\wedge t}^2\right)\right]\\ &\leq \sup_{0\leq i\leq n} \left\|h_{t_i}^2\right\|_\infty \sup_{t\geq 0} E\left[M_{t_n\wedge t}^2\right]\\ &\leq \sup_{0\leq i\leq n} \left\|h_{t_i}^2\right\|_\infty \|M\|^2 < \infty.\end{aligned}$$

It remains to show the martingale property of $H.M$:
Let T be a bounded stopping time. Then

$$\begin{aligned}E[(H.M)_T] &= \sum_{i=0}^{n-1} E\Big[h_{t_i}\left(M_{t_{i+1}\wedge T} - M_{t_i\wedge T}\right)\Big]\\ &= \sum_{i=0}^{n-1} E\Big[h_{t_i} \underbrace{E\left[M_{t_{i+1}\wedge T} - M_{t_i\wedge T}|\mathcal{F}_{t_i}\right]}_{=0, \text{ since } (M_{t\wedge T})_{t\geq 0} \text{ is an } (\mathcal{F}_t)\text{-martingale}}\Big]\\ &= 0.\end{aligned}$$

Thus, $H.M$ is a martingale.

(ii) By uniqueness of the Doob-Meyer decomposition (see Corollary 2.2.12 for $M \in \mathcal{M}_c^2$) it is enough to show that

$$(H.M)_t^2 - \int_0^t H_s^2 \, \mathrm{d}\langle M\rangle_s, \ t \geq 0,$$

is a martingale. (Then $\langle H.M\rangle_t = \int_0^t H_s^2 \, \mathrm{d}\langle M\rangle_s$.) Let T be a bounded stopping time. Then, by defining

$$\Delta_i M := M_{t_{i+1}\wedge T} - M_{t_i \wedge T}$$

and since all mixed terms dissappear, we get that

$$\begin{aligned}
E[(H.M)_T^2] &= \sum_{i,j} E\left[h_{t_i} h_{t_j} (\Delta_i M)(\Delta_j M)\right] \\
&= \sum_i E\left[h_{t_i}^2 E[(\Delta_i M)^2 | \mathcal{F}_{t_i}]\right] \\
&= \sum_i E\left[h_{t_i}^2 E[M_{t_{i+1}\wedge T}^2 - M_{t_i\wedge T}^2 | \mathcal{F}_{t_i}]\right] \\
&= \sum_i E\left[h_{t_i}^2 E[\langle M\rangle_{t_{i+1}\wedge T} - \langle M\rangle_{t_i\wedge T} | \mathcal{F}_{t_i}]\right] \\
&= E\left[\int_0^T H_s^2 \, \mathrm{d}\langle M\rangle_s\right].
\end{aligned}$$

In particular,

$$\|H.M\|^2 = \sup_{t\geq 0} E[(H.M)_t^2] = \sup_{t\geq 0} E\left(\int_0^t H_s^2 \, \mathrm{d}\langle M\rangle_s\right) \overset{\text{B.Levi}}{=} E\left(\int_0^\infty H_s^2 \, \mathrm{d}\langle M\rangle_s\right).$$

□

We now want to consider the above “Isometry Property” more closely:
Let $\bar{\Omega} :=]0,\infty[\times \Omega$, $\bar{\omega} := (t,\omega)$, $\bar{\mathcal{F}} = \mathcal{B}(]0,\infty[) \otimes \mathcal{F}$. $\omega \mapsto \langle M\rangle_t(\omega)$ is $\mathcal{F}$-measurable for fixed t and for all $\omega \in \Omega$ $t \mapsto \langle M\rangle_t(\omega)$ is right-continuous, nonnegative, increasing, with $\langle M\rangle_0(\omega) = 0$. Therefore, for fixed $\omega \in \Omega$ there exists a unique, positive measure

$$\langle M\rangle(\omega, dt) := \mathrm{d}\langle M\rangle_t(\omega).$$

Thus, $\langle M\rangle(\omega, dt)$ defines a transition kernel from $(\Omega, \mathcal{F})$ to $(]0,\infty[, \mathcal{B}(]0,\infty[))$ and it induces a measure

$$P_M(d\bar{\omega}) := P(d\omega) \otimes \langle M\rangle(\omega, dt)$$

on $(\bar{\Omega}, \bar{\mathcal{F}})$. Explicitly,

$$P_M(A) = E\left[\int_0^\infty 1_A(\cdot, t) \, \mathrm{d}\langle M\rangle_t(\cdot)\right], \quad A \in \bar{\mathcal{F}}.$$

Note that this is not a probability measure, but it is finite since $M \in \mathcal{M}_{(c)}^2$. In particular, defining

$$E_M[\,\cdot\,] = \int \cdot \, \mathrm{d}P_M,$$

we have

$$E_M[H^2] = E\left[\int_0^\infty H_s^2 \, \mathrm{d}\langle M\rangle_s\right], \tag{2.3.1}$$

which is the $\mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M)$-norm of $H \in \mathcal{E}$. ($H : \bar{\Omega} \to \mathbb{R}$, $H(\omega, t) = H_t(\omega)$.) Note that the map

$$\begin{aligned} \mathcal{E} &\to \mathcal{M}^2, \\ H &\mapsto H.M \end{aligned}$$

is obviously linear and by Lemmas 2.3.20(ii) and (2.3.1), an isometry from $\mathcal{E} \subset \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M)$ to $(\mathcal{M}^2_{(c)}, \|\cdot\|)$. Therefore, there exists a unique isometric extension to the closure of $\mathcal{E}$ in $\mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M)$ denoted by

$$\bar{\mathcal{E}} := \bar{\mathcal{E}}^M$$

(which depends on M).

Definition 2.3.21. *For $H \in \bar{\mathcal{E}}^M$, let $H.M$ denote the uniquely determined element in $\mathcal{M}^2_{(c)}$ with*

$$\lim_{n\to\infty} \|H^n_{.}M - H.M\| = 0$$

for every sequence $(H^n)_{n\in\mathbb{N}} \subset \mathcal{E}$ which converges in $\mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M)$ to H.

Automatically, we have

$$\|H.M\|^2 = E_{P_M}[H^2].$$

But we also have the analogue of 2.3.20(ii), namely:

Proposition 2.3.22. *Let $H \in \bar{\mathcal{E}}^M$, $M \in \mathcal{M}^2_{(c)}$, and therefore, $H.M \in \mathcal{M}^2_{(c)}$. Then*

$$\langle H.M\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s, \quad t \geq 0,$$

and hence by polarization for $G \in \bar{\mathcal{E}}^M$

$$\langle H.M, G.M\rangle_t = \int_0^t H_s G_s \langle M\rangle_s \ t > 0.$$

Proof. Let T be a bounded stopping time. Then

$$E[(H.M)_T^2] = \lim_{n\to\infty} E[(H^n_{.}M)_T^2],$$

since

$$\begin{aligned} E[((H.M)_T - (H^n_{.}M)_T)^2] &\leq E\left[\sup_t (H.M - H^n_{.}M)_t^2\right] \\ &\overset{\text{Doob}}{\leq} 4\,\|H.M - H^n_{.}M\|^2 \xrightarrow{n\to\infty} 0. \end{aligned}$$

Hence,

$$\begin{aligned} E[(H.M)_T^2] &= \lim_{n\to\infty} E[(H^n_{.}M)_T^2] \\ &\overset{2.3.20(ii)}{=} \lim_{n\to\infty} E\left[\int_0^T (H^n)_s^2 \, \mathrm{d}\langle M\rangle_s\right] \\ &= E\left[\int_0^T H_s^2 \, \mathrm{d}\langle M\rangle_s\right], \end{aligned}$$

since $H^n \xrightarrow{n\to\infty} H$ in $\mathcal{L}^2(P_M)$, so, $1_{]0,T]}H^n \to 1_{]0,T]}H$ in $\mathcal{L}^2(P_M)$. Therefore, $(H.M)_t^2 - \int_0^t H_s^2 \, \mathrm{d}\langle M\rangle_s$ is a martingale and the assertion follows by the uniqueness of the Doob-Meyer decomposition. □

In our next step we want to determine the size of $\bar{\mathcal{E}}^M$. We want to characterize admissible integrands in dependence of M.

Definition 2.3.23.

$$\begin{aligned}\mathcal{P} &:= \sigma\big(H = (H_t)_{t>0} | H \text{ is a (real-valued) left-continuous, adapted process}\big)\\ &\overset{exercise}{=} \sigma\big(H = (H_t)_{t>0} | H \text{ is a (real-valued) continuous, adapted process}\big)\end{aligned}$$

is called σ-algebra (on $]0,\infty[\times\Omega$) of predictable sets. *A process* $H = (H_t)_{t>0}$ *is called* predictable, *if it is* $\mathcal{P}$*-measurable.*
Let $\mathcal{P}_M :=$ *the completion of* $\mathcal{P}$ *with respect to* P_M *in* $\bar{\mathcal{F}}$.

Remark 2.3.24. *Every predictable process* $(H_t)_{t>0}$ *is progressively measurable, i.e.* $H_{|]0,t]\times\Omega}$ *is* $\mathcal{B}(]0,t])\otimes\mathcal{F}_t$*-measurable for all* $t>0$*, hence in particular* $(\mathcal{F}_t)$*-adapted (cf. [vWW90, Lemma 6.1.6]).*

Remark 2.3.25. *Let* τ_n *be a partition of* $[0,\infty)$ *and*

$$\mathcal{P}_n := \sigma(]t,t'[\times A_t \big| t\in\tau_n, t' \text{ is the successor of } t \text{ in } \tau_n \text{ and } A_t \text{ is } \mathcal{F}_t\text{-measurable})$$

$\mathcal{P}_n$ *is called the* σ*-algebra of predictable rectangles (with respect to* τ_n*).*

(i) If $|\tau_n| \overset{n\to\infty}{\longrightarrow} 0$ *and* $t_{N_n} \overset{n\to\infty}{\longrightarrow} \infty$*, then*

$$\mathcal{P} = \sigma\Big(\bigcup_n \mathcal{P}_n\Big).$$

(ii) H is $\mathcal{P}_n$*-measurable and bounded if and only if*

$$H_t(\omega) = \sum_{t_i\in\tau_n} h_{t_i}(\omega) 1_{]t_i,t_{i+1}]}(t),\ t>0,$$

where h_{t_i} *is* $\mathcal{F}_{t_i}$*-measurable and bounded.*

Proof. (i) By definition we have $\mathcal{P}_n\subset\mathcal{P}$. Hence, $\sigma\left(\bigcup_n \mathcal{P}_n\right)\subset\mathcal{P}$. To show that $\mathcal{P}\subset\sigma\left(\bigcup_n\mathcal{P}_n\right)$, let $(H_t)_{t>0}$ be a left-continuous adpated process. It suffices to show that H is $\sigma\left(\bigcup_n\mathcal{P}_n\right)$-mearsurable. H is adapted and left-continuous. Therefore,

$$H_t(\omega) = \lim_{n\to\infty}\sum_{s\in\tau_n}\underbrace{H_s(\omega)1_{]s,s']}(t)}_{\sigma(\mathcal{P}_n)\text{-measurable}},\ t>0,$$

Hence,

$$\mathcal{P} = \sigma\Big(\bigcup_n \mathcal{P}_n\Big).$$

(ii) The assertion follows by a monotone class argument.

□

Proposition 2.3.26.

$$\bar{\mathcal{E}}^M = \mathcal{L}^2(\bar{\Omega},\mathcal{P}_M,P_M).$$

In particular, for all $H\in\mathcal{L}^2(\bar{\Omega},\mathcal{P}_M,P_M)$ *there exists*

$$H.M = \int H\,dM.$$

In particular,

$$\mathcal{M}^2_M := \{H.M | H\in\mathcal{L}^2(\bar{\Omega},\mathcal{P}_M,P_M)\}$$

is a closed subspace of $\mathcal{M}^2$.

Proof. Clearly, $\bar{\mathcal{E}}^M \subset \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$. To prove the dual inclusion, let $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$. Without loss of generality $H \geq 0$. (Otherwise consider H^+ and H^-.) Since

$$H^{(m)} := (m \wedge H) \overset{m\to\infty}{\longrightarrow} H$$

in $\mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$, we may assume $H \geq 0$, bounded. Now let τ_n be a sequence of partitions as in Remark 2.3.25(i). Then by the $\mathcal{L}^1$-martingale convergence theorem (see [Röc11, Korollar 8.5.4 (i)]; ok, since $\mathcal{P}_M$ is a *finite* measure!)

$$H_n := E_{P_M}[H|\mathcal{P}_n] \overset{n\to\infty}{\longrightarrow} E_{P_M}\left[H\middle|\sigma\left(\bigcup_n \mathcal{P}_n\right)\right] \quad \text{in } \mathcal{L}^1(\mathcal{P}_M).$$

Since H is bounded, this convergence also holds in $\mathcal{L}^2(\mathcal{P}_M)$. By Remark 2.3.25(i) this implies that in $\mathcal{L}^2(\mathcal{P}_M)$

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} E_{P_M}[H|\mathcal{P}_n] = E_{P_M}[H|\mathcal{P}] = E_{P_M}[H|\mathcal{P}_M] = H.$$

Since H_n is $\mathcal{P}_n$-measurable, we have by Remark 2.3.25(ii), that

$$H_n \in \mathcal{E},$$

and the assertion follows. □

Proposition 2.3.27. *(i) Let $t \mapsto \langle M\rangle_t$ be continuous (which is e.g. the case if $M \in \mathcal{M}_c^2$). Then*

$$\left(L^2(\bar{\Omega}, \mathcal{P}_M, P_M) \overset{2.3.26}{=}\right) \bar{\mathcal{E}}^M = \mathcal{L}^2(\bar{\Omega}, \mathcal{O}_M, P_M),$$

where the optional σ-algebra $\mathcal{O}$ is defined by

$$\mathcal{O} := \sigma(\{H|\, H \textit{ is an adapted càdlàg process on }]0,\infty[\times\Omega\})$$

and $\mathcal{O}_M$ its completion with respect to P_M in $\bar{\mathcal{F}}$.

(ii) If $M \in \mathcal{M}_c^2$ and $d\langle M\rangle$ is absolutely continuous with respect to Lebesgue measure dt, then

$$\bar{\mathcal{E}}^M = \{H \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_M)|H \textit{ has an adapted version}\}.$$

Proof. (i) We have that $\mathcal{O} \supset \mathcal{P}$. Hence, also $\mathcal{O}_M \supset \mathcal{P}_M$. It remains to prove $\mathcal{O}_M \subset \mathcal{P}_M$. So, let H be càdlàg and adapted. Consider for $\omega \in \Omega$

$$U_\omega := \{t > 0 | s \mapsto H_s(\omega) \text{ is discontinuous in } t\}.$$

Then U_ω is countable, since H is càdlàg (exercise). Define for $\omega \in \Omega$

$$H_t^-(\omega) := \lim_{s\nearrow t} H_s(\omega) =: H_{t-}(\omega) \ (\in \mathbb{R}).$$

Then H^- is left-continuous. Hence, it is $\mathcal{P}$-measurable and

$$U_\omega = \{t > 0 | H_t(\omega) \neq H_t^-(\omega)\}.$$

Note that $\{(t,\omega) \in]0,\infty[\times\Omega|\, H_t(\omega) \neq H_t^-(\omega)\} \in \mathcal{O} \subset \bar{\mathcal{F}}$. Since $\langle M\rangle$ is continuous, we have

$$\int_0^\infty 1_{U_\omega}(t)\ \mathrm{d}\langle M\rangle_t = 0 \quad \text{for} \quad P\text{-a.e. } \omega \in \Omega.$$

But then

$$E_{P_M}[1_{\{H\neq H^-\}}] = \int\int_{]0,\infty[} 1_{U_\omega}(t)\ \mathrm{d}\langle M\rangle_t(\omega)P(d\omega) = 0.$$

Hence, H is P-a.e. equal to a predictable process, therefore, H is $\mathcal{P}_M$-measurable and (i) is proved.

(ii) One has to prove that $\bar{\mathcal{F}} \subset \mathcal{P}_M$ (cf. [CW90, p.60] or [vWW90, p.124]).

□

Remark 2.3.28. *By a modification of the proof of Proposition 2.3.27(i) one can even show that every progressively measurable process* $(H_t)_{t>0}$ *(see Remark 2.3.24 above) has a predictable* $\mathcal{P}_M$*-version (of course still assuming that* $\langle M \rangle$ *is* P*-a.s. continuous).*

Remark 2.3.29 (Extending stochastic integration via localization)**.** *Note that, if* M *is a Brownian motion, then* $M \notin \mathcal{M}^2$*. Therefore, we have to stop. Define*

$$\mathcal{M}^2_{loc} := \{M | \exists \textit{ a sequence } T_n \nearrow \infty \textit{ of stopping times such that } M^{T_n} (:= M_{T_n \wedge \cdot}) \in \mathcal{M}^2 \forall n \in \mathbb{N}\}.$$

Then for all $H \in \mathcal{L}^2_{loc}(\bar{\Omega}, \mathcal{P}_M, P_M)$

$$H.M = \int H \, dM$$

is defined via localization. We have to check that

- *the definition is consistent on* $\{T_n = T_{n+1}\}$,
- $H.M \in \mathcal{M}^2_{loc}$ *(We need Lemma 2.4.33 below).*

Remark 2.3.30 (Semi-martingales as integrals)**.** *If* A *is predictable and of bounded variation and* $M \in \mathcal{M}^2_{loc}$ *we can define for* $H \in \mathcal{L}^2_{loc}(\bar{\Omega}, \mathcal{P}_M, P_M)$

$$\int H \, d(M + A) := \int H \, dM + \int H \, dA,$$

where $M + A$ *is a semi-martingale and* $\int H \, dA$ *is a pathwise defined Lebesgue-Stieltjes-integral. Axiomatic considerations show that, if* $\int H \, d(M + A)$ *is required to have reasonable properties, then this cannot be generalized. As a consequence, reasonable stochastic integrators are semi-martingales (Dellacherie-Mokobodzki-Bichteler) (cf. [vWW90]).*

2.4. Characterization of $H.M$ in $\mathcal{M}^2$

Fix $M \in \mathcal{M}^2$, $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$.

Proposition 2.4.31. *$H.M$ is the unique element $L \in \mathcal{M}^2$ such that*

(i)

$$d\langle L, N\rangle = H\ d\langle M, N\rangle \quad P\text{-a.s. } \forall N \in \mathcal{M}^2,$$

that is

$$\langle L, N\rangle_t = \int_0^t H_s\ d\langle M, N\rangle_s := \int 1_{]0,t]}(s) H_s\ d\langle M, N\rangle_s, \quad \forall t \geq 0,\ P\text{-a.s. } \forall\ N \in \mathcal{M}^2, \forall t > 0,$$

respectively, such that the following weaker property holds:

(ii)

$$E[L_\infty N_\infty] = E\left[\int_0^\infty H_s\ d\langle M, N\rangle_s\right] \left(:= \int 1_{]0,\infty[}(s) H_s\ d\langle M, N\rangle_s\right)$$

for all $N \in \mathcal{M}^2$.

Remark 2.4.32. *(i) Because of $E[L_\infty N_\infty] = E[\langle L, N\rangle_\infty]$ (cf 2.3.18(i)), it follows that in 2.4.31, (i) implies (ii).*

(ii) By 2.4.31(i) with N replaced by $G.N$ for $G \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ we particularly have

$$d\langle H.M, G.N\rangle = H\ d\langle M, G.N\rangle = HG\ d\langle M, N\rangle.$$

Proof of 2.4.31. (a) Uniqueness if (ii) holds:
Let $L, L' \in \mathcal{M}^2$ such that both satisfy 2.4.31(ii). Then

$$E[(L_\infty - L'_\infty)N_\infty] = 0 \quad \forall N_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P).$$

Hence $L_\infty = L'_\infty$ P-a.s.. Therefore,

$$L_t = E[L_\infty | \mathcal{F}_t] = L'_t \quad P\text{-a.s..}$$

Now let us first prove (ii).

(b) $L := H.M$ satisfies (ii):
Step 1: Assume for fixed $s > 0$ that

$$H_t(\omega) := h_s(\omega) 1_{]s,\infty[}(t),$$

where h_s is bounded and $\mathcal{F}_s$ -measurable. Note that $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$. Therefore, for $L := H.M$ we have

$$\begin{aligned} L_\infty = (H.M)_\infty &= \int_0^\infty h_s 1_{]s,\infty[}(t)\ \mathrm{d}M_t \\ &= \lim_{N\to\infty} \int_0^N h_s 1_{]s,\infty]}(t)\ \mathrm{d}M_t \\ &= \lim_{N\to\infty} h_s(M_N - M_s) = h_s(M_\infty - M_s) \end{aligned}$$

Thus, for all $N \in \mathcal{M}^2$

$$\begin{aligned}
E[L_\infty N_\infty] =& E[h_s(M_\infty - M_s)N_\infty] \\
\overset{(N_t)_{t\in[0,\infty]}\text{ mart.}}{=}& E[h_s(M_\infty N_\infty - M_s N_s)] \\
=& E\left[h_s(M_\infty N_\infty - \langle M,N\rangle_\infty - (M_s N_s - \langle M,N\rangle_s))\right] \\
&+ E\left[h_s(\langle M,N\rangle_\infty - \langle M,N\rangle_s)\right] \\
=& E\left[h_s(\langle M,N\rangle_\infty - \langle M,N\rangle_s)\right] \\
=& E\left[h_s \int_{]s,\infty[} \mathrm{d}\langle M, N_t\rangle\right] \\
=& E\left[\int_0^\infty H_t \,\mathrm{d}\langle M,N\rangle_t\right].
\end{aligned}$$

Step 2: Let $H \in \mathcal{E}$.
In this case (ii) is clear by Step 1 and linearity.
Step 3: (ii) holds for all $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$:
Let $H^{(n)} \in \mathcal{E}$, $n \in \mathbb{N}$, $H^{(n)} \to H$ in $\mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)(= \bar{\mathcal{E}})$. Then

$$H^{(n)}.M \to H.M \text{ in } (\mathcal{M}^2, \|\cdot\|).$$

Therefore, by 2.3.17(iii) for all $N \in \mathcal{M}^2$

$$\begin{aligned}
E[(H.M)_\infty N_\infty] &= \lim_{n\to\infty} E[(H^{(n)}.M)_\infty N_\infty] \\
&\overset{\text{Step 2}}{=} \lim_{n\to\infty} E\left[\int_0^\infty H_s^{(n)} \,\mathrm{d}\langle M,N\rangle_s\right] \\
&= E\left[\int_0^\infty H_s \,\mathrm{d}\langle M,N\rangle_s\right],
\end{aligned}$$

because by Kunita-Watanabe-inequality (cf. 2.2.16) we have

$$E\left[\left|\int_0^\infty (H^{(n)} - H)\cdot 1 \,\mathrm{d}\langle M,N\rangle\right|\right] \le \left(E\left[\int_0^\infty (H^{(n)} - H)^2 \,\mathrm{d}\langle M\rangle\right]\right)^{\frac{1}{2}} \cdot (E[\langle N\rangle_\infty])^{\frac{1}{2}} \overset{n\to\infty}{\longrightarrow} 0.$$

This proves (ii).

(c) $L := H.M$ satisfies (i):
By Proposition 2.2.15 we have to show that $L_t N_t - \int_0^t H_s \,\mathrm{d}\langle M,N\rangle_s$, $t \ge 0$, is a martingale. Let T be a bounded stopping time. Then, since $(L_t)_{t\in[0,\infty]}$ is a martingale, for $N_t^T := N_{T\wedge t}$, $t \ge 0$, we have

$$\begin{aligned}
E[L_T N_T] &= E[L_\infty N_T] = E[L_\infty N_\infty^T] \\
&\overset{(ii)}{=} E\left[\int_0^\infty H_s \,\mathrm{d}\langle M, N^T\rangle_s\right] \\
&\overset{2.4.33\text{ below}}{=} E\left[\int_0^\infty H_s \,\mathrm{d}\langle M,N\rangle_{s\wedge T}\right] \\
&= E\left[\int_0^T H_s \,\mathrm{d}\langle M,N\rangle_s\right]
\end{aligned}$$

(where we used that obviously $N^T \in \mathcal{M}^2$, since $N \in \mathcal{M}^2$).

□

Lemma 2.4.33. *Let $M, N \in \mathcal{M}^2$ and T be a stopping time. Then*

$$\langle M, N^T\rangle_t = \langle M, N\rangle_{T\wedge t} \quad \forall t \geq 0.$$

Proof. Exercise. (!) □

Corollary 2.4.34. *Let $M \in \mathcal{M}^2, H_1, H_2 \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ such that $H_1 \cdot H_2 \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$. Then*

$$H_1.(H_2.M) = (H_1 \cdot H_2).M,$$

i.e.

$$\int_0^\infty H_1(s)\ d(H_2.M)_s = \int_0^\infty H_1(s)\ d\left(\int_0^s H_2(r)\ dM_r\right) = \int_0^\infty H_1(r)H_2(r)\ dM_r.$$

Proof. Let $L := H_1.(H_2.M)$. Then by 2.4.31 for all $N \in \mathcal{M}^2$

$$\mathrm{d}\langle L, N\rangle = H_1\ \mathrm{d}\langle H_2.M, N\rangle = H_1H_2\ \mathrm{d}\langle M, N\rangle,$$

and

$$\mathrm{d}\langle (H_1 \cdot H_2).M,\ N\rangle = (H_1 \cdot H_2)\ \mathrm{d}\langle M, N\rangle.$$

Hence (again by 2.4.31)

$$(H_1 \cdot H_2).M = H_1 \cdot (H_2.M).$$

□

Corollary 2.4.35. *Let $M \in \mathcal{M}^2$ and let T be a stopping time.*

(i) Then $(H.M)^T = H1_{]0,T]}.M$, i.e.

$$(H.M)^T_t = \int_0^t H_s 1_{]0,T]}(s)\ dM_s,\ t \geq 0.$$

In particular $(H.M)^T \in \mathcal{M}^2_M$, hence, $\mathcal{M}^2_M$ is "stopping stable", that is, $N \in \mathcal{M}^2_M$ implies $N^T \in \mathcal{M}^2_M$ for all stopping times T.

(ii)

$$(H.M^T)_t = (H.M)^T_t,\ t \geq 0.$$

Proof. (i) Let $N \in \mathcal{M}^2$. Then

$$\begin{aligned} E\left[(H.M)^T_\infty N_\infty\right] &\overset{2.3.18(i)}{=} E\left[\langle (H.M)^T, N\rangle_\infty\right] \\ &\overset{2.4.33}{=} E\left[\langle H.M, N\rangle_T\right] \\ &\overset{2.4.31(i)}{=} E\left[\int_0^\infty H_s 1_{]0,T]}(s)\ \mathrm{d}\langle M, N\rangle_s\right]. \end{aligned}$$

Hence, the assertion follows by 2.4.31(ii).

(ii) Since $M^T \in \mathcal{M}^2$, the proof of (ii) is by 2.4.31(i) analogous to (i)

□

2.4.1. Orthogonality in $\mathcal{M}^2$

Definition 2.4.36. *Let $M, N \in \mathcal{M}^2$.*

(i) M, N are called weakly orthogonal *if $E[M_\infty N_\infty](= (M,N)) = 0$ (i.e. orthogonal in $(\mathcal{M}^2, \|\cdot\|)$).*

(ii) M, N are called strongly orthogonal, *denoted by $M \perp N$, if $\langle M, N\rangle = 0, P$-a.s..*

Remark 2.4.37. *(i) $M \perp N \overset{2.2.15(i)}{\Leftrightarrow} M \cdot N$ is a martingale $\Leftrightarrow E[M_T N_T] = 0$ for all bounded stopping times T. But $M_t \overset{t\to\infty}{\longrightarrow} M_\infty$ and $N_T \overset{t\to\infty}{\longrightarrow} N_\infty$ in $\mathcal{L}^2(P)$. Hence if $M \perp N$, then*

$$E[M_\infty N_\infty] = 0,$$

i.e. M, N are weakly orthogonal.

(ii) $M \perp N \overset{2.4.31(i)}{\Leftrightarrow} \mathcal{M}^2_M \perp \mathcal{M}^2_N$ (since by 2.4.31 $d\langle H.M, \tilde{H}.N\rangle = H\tilde{H}\ d\langle M, N\rangle$).

(iii) Since $(\mathcal{M}^2, \|\cdot\|)$ is a Hilbert space and $\mathcal{M}^2_M$ is a closed linear subspace of $\mathcal{M}^2$ (see 2.3.26), for all $N \in \mathcal{M}^2$ there exists an $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ and $L \in \mathcal{M}^2$ such that

$$N = \underbrace{H.M}_{\in \mathcal{M}^2_M} + \underbrace{L}_{\in \mathcal{M}^2}$$

and L is weakly orthogonal to $\mathcal{M}^2_M$.

But in (iii) we have more because of the following proposition.

Proposition 2.4.38. *Let $M, L \in \mathcal{M}^2$, L weakly orthogonal to $\mathcal{M}^2_M$. Then $L \perp \mathcal{M}^2_M$ (since $\mathcal{M}^2_M$ is stopping stable). Hence, L is weakly orthogonal to $\mathcal{M}^2_M$ if and only if $L \perp \mathcal{M}^2_M (\Leftrightarrow L \perp M)$.*

Proof. Because of 2.4.37(i), it remains to show that $E[L_T M_T] = 0$ for all bounded stopping times T. But by 2.4.35 (stopping stability) we have $M^T \in \mathcal{M}^2_M$. Hence,

$$0 = E[L_\infty M^T_\infty] = E[L_\infty M_T] \overset{L \text{ mart.}}{=} E[L_T M_T].$$

□

Corollary 2.4.39 (Kunita-Watanabe-decomposition)**.** *(i) Let $M, N \in \mathcal{M}^2$. Then there exist a unique $L \in \mathcal{M}^2$ and a unique $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ such that $L \perp \mathcal{M}^2_M$ and $N = H.M + L$.*

(ii) Suppose $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$ separable (e.g. true, if $\mathcal{F}_\infty$ is countably generated). Then there exist $M_i \in \mathcal{M}^2$, $i \in \mathbb{N}$, such that

$$M_i \perp M_j,\ i \neq j,$$

and for all $N \in \mathcal{M}^2$ there exist $H^i \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}, P_M)$, $i \in \mathbb{N}$, (uniquely determined by M_i) such that

$$N = \sum_{i=1}^{\infty} H^i_. M_i, \quad \text{(the series converges in } \|\cdot\|_{\mathcal{M}^2}\text{!)}$$

that is,

$$\mathcal{M}^2 = \bigoplus_{i=1}^{\infty} \mathcal{M}^2_{M_i}$$

and

$$\mathcal{M}^2_{M_i} \perp \mathcal{M}^2_{M_j}, \quad \text{for} \quad i \neq j.$$

Proof. (i) 2.4.37(iv) and 2.4.38.

(ii) By assumption and 2.4.37(iii) $(\mathcal{M}^2, \|\cdot\|)$ is separable. So, we can apply (i) and the Gram-Schmidt orthogonalization procedure (cf. [Wei87],[RS80]).

□

Corollary 2.4.40. *Let* $F \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$*,* $M \in \mathcal{M}^2$*. Then there exist uniquely determined* $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ *and* $L \in \mathcal{M}^2$*,* $L \perp M$*, such that,*

$$F = E[F|\mathcal{F}_0] + \int_0^\infty H_s \, dM_s + L_\infty.$$

Proof. Let $F_t := E[F|\mathcal{F}_t] - E[F|\mathcal{F}_0]$. Since F_t is a martingale, by 2.1.1 there exists a càdlàg version $(\tilde{F}_t)_{t\geq 0}$ of $(F_t)_{t\geq 0}$ (i.e. $P[\{F_t = \tilde{F}_t\}] = 1 \, \forall t \geq 0$). Then $(\tilde{F}_t) \in \mathcal{M}^2$. Hence, by 2.4.39(i) there exist unique $L \in \mathcal{M}^2$, $L \perp M$, and $H \in \mathcal{L}^2(\bar{\Omega}, \mathcal{P}_M, P_M)$ such that P-a.s.

$$\tilde{F}_t = (H.M)_t + L_t, \; t \geq 0.$$

In particular, (since $\tilde{F}_\infty = F - E[F|\mathcal{F}_0]$,) we have P-a.s.

$$F = E[F|\mathcal{F}_0] + \int_0^\infty H_s \, \mathrm{d}M_s + L_\infty.$$

□

2.5. Itô's Representation Theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)$ and let $(W_t)_{t\geq 0}$ be a (real-valued) *Wiener process* on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, that is, (W_t) is a continuous $(\mathcal{F}_t)$-adapted process such that, for all $s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and $N(0, t-s)$ distributed. Let for $t \geq 0$

$$\mathcal{F}_t^W := \sigma(\{W_s | 0 \leq s \leq t\})$$

and $\mathcal{F}_{t+}^W$, $t \geq 0$, the corresponding right-continuous filtration.

Remark 2.5.41. *Compared with our definition of Brownian motion for a Wiener process the increment $W_t - W_s$ is independent of the larger σ-algebra $\mathcal{F}_s (\supseteq \mathcal{F}_s^W)$. In this space of $(\mathcal{F}_t^W)$-martingales belonging to $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ the Kunita-Watanabe decomposition (cf. 4.9(i) with $M = W$) has a particularly simple form:*

Theorem 2.5.42 (Itô's representation theorem)**.** *Let $M = (M_t)_{t\geq 0} \subseteq \mathcal{L}^2(\Omega, \mathcal{F}, P)$ be a right-continuous martingale with respect to $(\mathcal{F}_{t+}^W)$, hence with respect to*

$$(\mathcal{F}_{t+}^W)^P := \sigma(\mathcal{F}_{t+}^W, \sigma(\{N \in \mathcal{F} | P(N) = 0\})), t \geq 0.$$

Then

$$M_t = M_0 + \int_0^t H_s \, dW_s, \; t \geq 0, P\text{-a.s.}, \tag{2.5.2}$$

where H is $(\mathcal{F}_t)$-adapted and $H \cdot 1_{]0,T]} \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_W)$ for all $T > 0$ (and through this it is uniquely determined). In particular, M has P-a.s. continuous sample paths and $\mathcal{M}^2 = \mathcal{M}_W^2$ (where $\mathcal{M}^2$ is defined as in the previous section with respect to $\mathcal{F}_t = (\mathcal{F}_{t^+}^W)^P, t \geq 0$, which by the backward martingale convergence theorem is again right-continuous) and

$$\mathcal{M}_W^2 := \left\{ \left(\int_0^t H_s \, dW_s \right)_{t\geq 0} \middle| H \cdot 1_{]0,t]} \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_W) \quad \forall t \geq 0, \sup_{t\geq 0} \left\| \int_0^t H_s \, dW_s \right\|_{\mathcal{L}^2(P)} < \infty \right\}.$$

Proof. Without loss of generality $M_0 = 0$. Fix $k \in \mathbb{N}$ and set

$$\mathcal{F}_{k+}^W := (\mathcal{F}_{k+}^W)^P, \; M_t^{(k)} := M_{t\wedge k}, \; W_t^{(k+1)} := W_{t\wedge(k+1)}, t \geq 0.$$

Let $(M^n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^2(\Omega, \mathcal{F}_{k+}^W, P)$, such that $\|M^n\|_\infty < \infty$ and

$$\lim_{n\to\infty} M^n = M_k = M_\infty^{(k)} \quad \text{in } \mathcal{L}^2(\Omega, \mathcal{F}_{k+}^W, P).$$

Let $M_t^n := E[M^n | \mathcal{F}_{t+}^W]$, $t \geq 0$. Then by the backward martingale convergence theorem (see [Röc11]) $(M_t^n)_{t\geq 0}$ is right continuous. Then for each $n \in \mathbb{N}$

$$\sup_{t,\omega} |M_t^n(\omega)| < \infty.$$

In particular, $M^n \in \mathcal{M}^2$ for all n. Hence, by the Kunita-Watanabe decomposition (cf. 2.4.39(i)) there exist adapted $H^n \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_W)$ and $N^n \in \mathcal{M}^2$, $N^n \perp \mathcal{M}_{W^{(k+1)}}^2$, such that

$$M_t^n = \left(H_\cdot^n W^{(k+1)} \right)_t + N_t^n, \; t \geq 0, \quad \forall n \in \mathbb{N}.$$

We would like to prove that $N^n = 0$ for all $n \in \mathbb{N}$.

Step 1: Claim: Let $N \in \mathcal{M}^2$, $N \perp W^{(k+1)}$. If N is bounded, then $N_t = 0$ for all $t < k + 1$.

Proof of Claim. Let $c := \sup_{t,\omega} |N_t(\omega)| < \infty$. Define

$$D := 1 + \frac{N_\infty}{2c}.$$

Then $D \geq \frac{1}{2}$ and

$$E[D] = 1 + \frac{1}{2c} E[N_\infty] = 1 + \frac{1}{2c} E[N_0] = 1.$$

Thus, $\tilde{P} := DP$ is a probability measure on $(\Omega, \mathcal{F})$ equivalent to P.
Claim': $(W_t)_{t \leq k+1}$ is a Brownian motion under $\tilde{P}$.
Suppose the claim is true. Then the finite dimensional distributions of $(W_t)_{t \leq k+1}$ under P and $\tilde{P}$ are the same. Hence, by Radon-Nikodym,

$$P = \tilde{P} \quad \text{on } \mathcal{F}^W_{k+1}.$$

Therefore, $P = \tilde{P}$ on $\mathcal{F}^W_{t+}$ for all $t < k+1$. Thus, $E[D|\mathcal{F}^W_{t+}] = 1$ for all $t < k+1$. Hence,

$$N_t = E[N_\infty | \mathcal{F}^W_{t+}] = 0, \quad \forall t < k+1.$$

□

Proof of Claim'.

(i) $(W_t^{(k+1)})$ is a martingale under $\tilde{P}$, because for all bounded stopping times T we have

$$E_{\tilde{P}}[W_T^{(k+1)}] = E_P[W_T^{(k+1)} D] = \underbrace{E_P[W_T^{(k+1)}]}_{=0} + \frac{1}{2c} \underbrace{E_P[W_T^{(k+1)} N_\infty]}_{=:A}.$$

But by 2.4.37 $W_T^{k+1} N_T$ is a martingale, since $W^{k+1} \perp N$. Hence,

$$A = E_P[W_T^{k+1} N_T] = E_P[W_0^{(k+1)} N_0] = 0,$$

and thus

$$E_{\tilde{P}}[W_T^{(k+1)}] = 0.$$

(ii) For P-a.e. $\omega \in \Omega$ we have

$$\langle W \rangle_t(\omega) = t \quad \forall t \geq 0.$$

Hence, for $\tilde{P}$-a.e. $\omega \in \Omega$

$$\langle W \rangle_t(\omega) = t \quad \forall t \geq 0.$$

Thus, $\tilde{P}$-a.s.

$$\langle W_t^{(k+1)} \rangle = t \wedge (k+1) \quad \forall t \geq 0.$$

Therefore, by (i), (ii) and Levy's characterization (cf. Proposition 1.5.34) $(W_t)_{t \leq k+1}$ is a Brownian motion under $\tilde{P} = DP$ (up to $k+1$). □

Step 2: Now fix $n_0 \in \mathbb{N}$ and set $N := N^{n_0}$. Define for $n \in \mathbb{N}$

$$T_n := \inf \left\{ t > 0 \middle| \left| \int_0^t H_s^{n_0} \, \mathrm{d}W_s^{(k+1)} \right| > n \right\} \wedge (k+1).$$

Then T_n are $(\mathcal{F}^W_{t+})$-stopping times such that $T_n \nearrow (k+1)$ as $n \to \infty$. We know that $N^{T_n} (= N_{\cdot \wedge T_n}) \perp \mathcal{M}^2_{W^{(k+1)}}$ (cf. 2.4.33, 2.4.37(ii)).

Furthermore,

$$|N_t^{T_n}| = |N_{t\wedge T_n}| \leq \left|M_{t\wedge T_n}^{n_0}\right| + \left|\int_0^{t\wedge T_n} H_s^{n_0}\, \mathrm{d}W_s^{(k+1)}\right| \leq \|M^{n_0}\|_\infty + n.$$

Hence,

$$\sup_{t,\omega} |N_t^{T_n}(\omega)| \leq \|M^{n_0}\|_\infty + n.$$

So, we can apply Step 1 to conclude that $N_t^{T_n} = 0$, for all $t < k+1$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $N_t = 0$ for all $t < k+1$ P-a.s. (since N is right-continuous and the zero set does not depend on t). Therefore, $N^n = 0$ for all n and $t < k+1$. But for all $n, m \in \mathbb{N}$

$$\begin{aligned} E_{P_{W^{(k+1)}}}\left[(H^n - H^m)^2\right] &= E\left(\int_0^{k+1} (H_s^n - H_s^m)^2\, \mathrm{d}s\right) \\ &= E_P\left[\left((H_\cdot^n W^{(k+1)})_{k+1} - (H_\cdot^m W^{(k+1)})_{k+1}\right)^2\right] \\ &= E\left[\left(M_{k+1}^n - M_{k+1}^m\right)^2\right] \overset{n,m\to\infty}{\longrightarrow} 0. \end{aligned}$$

This is true, since P-a.e.

$$M_{k+1}^n = E\left[M^n | \mathcal{F}_{(k+1)+}^W\right] \overset{n\to\infty}{=\!\!\longrightarrow} E\left[M_k | \mathcal{F}_{(k+1)+}^W\right] = M_k \quad \text{in } \mathcal{L}^2(\Omega, \mathcal{F}, P).$$

Therefore, $(H^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_{W^{(k+1)}})$.
Hence, there exists $H \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_{W^{(k+1)}})$ such that $H^n \to H$ in $\mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_{W^{(k+1)}})$. In particular, we can assume H to be adapted. Furthermore, for $0 \leq t \leq k+1$,

$$M_t = \lim_{n\to\infty} M_t^n = \lim_{n\to\infty} (H^n.W^{(k+1)})_t = (H.W^{(k+1)})_t = \int_0^t H_s\, \mathrm{d}W_s^{(k+1)} \quad P\text{-a.s.}.$$

Since $k \in \mathbb{N}$ was arbitrary, the decomposition (2.5.2) follows, since M is right-continuous and the right hand side of (2.5.2) is continuous, M is P-a.s. continuous. ☐

We now include an independent direct proof that every $M \in \mathcal{M}^2$ (with underlying filtration $(\mathcal{F}_{t+}^W)^P$, $t \geq 0$) is P.a.s. continuous:

Proposition 2.5.43. *Every $(\mathcal{F}_{t+}^W)$-adapted $(M_t) \in \mathcal{M}_{loc}^2$ is P-a.s. continuous.*

Proof. Without loss of generality $M \in \mathcal{M}^2$ (localization). It suffices to consider the case

$$M_t := E[F|\mathcal{F}_{t+}^W],\ t \geq 0,$$

where

$$F := \prod_{i=1}^n f_i(W_{t_i}), \quad f_i \in \mathcal{C}_b(\mathbb{R}) \text{ uniformly continuous, } 0 \leq t_1 < \ldots t_n < \infty \qquad (\star)$$

and to prove that (M_t) has a continuous modification. This is enough because $M_t = E[M_\infty|\mathcal{F}_{t+}^W]$ (since $M \in \mathcal{M}^2$) and F of type $(\star)$ are dense in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty^W, P)$.
(Exercise, by a monotone class argument: By 2.1.4 M has a continuous $(\mathcal{F}_{t+}^W)^P$-adapted version. Define

$$V := \left\{G \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty^W, P) | \exists F_n \text{ of type } (\star) \text{ such that } F_n \to G \text{ in } \mathcal{F}_\infty^W, P)\right\}$$

and prove that $V = \mathcal{L}^\infty(\Omega, \mathcal{F}_\infty^W, P)$.)

Let $t \geq 0$ and $i \in \{1, \dots, n\}$ such that $t \in [t_i, t_{i+1}[$ where $t_{n+1} := +\infty$. Then (cf. section of Markov property of Brownian motion),

$$\begin{aligned}
M_t &= E[F|\mathcal{F}^W_{t+}] \\
&= \prod_{j \leq i} f_j(W_{t_j}) E\Big[\prod_{j>i} f_j(W_{t_j})\Big|\mathcal{F}^W_{t+}\Big] \\
&= \prod_{j \leq i} f_j(W_{t_j}) E_{W_t}\Big[\prod_{j>i} f_j(W_{t_j - t})\Big] \\
&= \prod_{j \leq i} f_j(W_{t_j}) p_{t_{i+1}-t} \underbrace{\left(f_{i+2}(p_{t_{i+1}-t_{i+2}} f_{i+2} \dots p_{t_n - t_{n-1}} f_n) \dots\right)}_{=: g \in C_0^\infty(\mathbb{R})} (\omega_t) \quad P\text{-a.s.},
\end{aligned}$$

where

$$p_s f(x) := \frac{1}{\sqrt{2\pi s}} \int f(y) e^{-\frac{1}{2s}(x-y)^2} \, \mathrm{d}y.$$

So, we have to prove that $p_{t_{i+1}-t} g(\omega_t)$ is P-a.s. continuous in t. Set

$$\mathcal{C}_{b,u}(\mathbb{R}) := \{\varphi \in \mathcal{C}_b(\mathbb{R}) | \varphi \text{ uniformly continuous}\}.$$

(i) Let $f \in \mathcal{C}_{b,u}(\mathbb{R})$. Take $x, y \in \mathbb{R}$. Then

$$|p_t f(x) - p_t f(y)| \leq \frac{1}{\sqrt{2\pi s}} \int |f(x+z) - f(y+z)| e^{-\frac{|z|^2}{2s}} \, \mathrm{d}z.$$

Let $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon) > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. So, $|p_t f(x) - p_t f(y)| \leq \varepsilon$ and $p_s f \in C_{b,u}(\mathbb{R})$.

(ii) Let $f \in \mathcal{C}_{b,u}(\mathbb{R})$. Then

$$\lim_{s \to 0} \|p_s f - f\|_\infty = 0.$$

Proof. Let $\varepsilon > 0$ and $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Since

$$f(x) = \frac{1}{\sqrt{2\pi s}} \int f(x) e^{-\frac{|y|^2}{2s}} dy$$

we have

$$\begin{aligned}
&|p_s f(x) - f(x)| \\
&\leq \frac{1}{\sqrt{2\pi s}} \int |f(x+y) - f(y)| e^{-\frac{|y|^2}{2s}} dy \\
&\leq \frac{1}{\sqrt{2\pi s}} \int_{-\delta}^{\delta} \underbrace{|f(x+y) - f(x)|}_{<\varepsilon} e^{-\frac{|y|^2}{2s}} dy + \frac{1}{\sqrt{2\pi s}} \int_{\{|y|>\delta\}} |f(x+y) - f(x)| e^{-\frac{|y|^2}{2s}} \, \mathrm{d}y \\
&\leq \varepsilon + 2\|f\|_\infty \frac{1}{\sqrt{2\pi s}} \int_{\{|y|>\delta\}} e^{-\frac{|y|^2}{2s}} \, \mathrm{d}y.
\end{aligned}$$

Therefore,

$$\limsup_{s \to 0} \|p_s f - f\|_\infty \leq \varepsilon + 2\|f\|_\infty \limsup_{s \to 0} \frac{1}{\sqrt{2\pi s}} \int_{\{|y|>\delta\}} e^{-\frac{|y|^2}{2s}} \, \mathrm{d}y \overset{y' = \frac{y}{\sqrt{s}}}{=} \varepsilon.$$

□

(iii)

$$p_t(p_s f) = p_{t+s} f,$$

in short,

$$p_t p_s = p_{t+s}.$$

Proof. Exercise (by Fourier transform and by use of

$$p_s f(x) = (\varrho_s * f)(x),$$

where $\varrho_s(z) = \frac{1}{\sqrt{2\pi s}} e^{\frac{-|z|^2}{2s}}$ and recall $\hat{\varrho}_s(\xi) = e^{-\frac{s\xi^2}{2}}$). □

(iv) For $t < t_{i+1}$ we have

$$\lim_{s \searrow t} \left\| p_{t_{i+1}-s} g - p_{t_{i+1}-t} g \right\|_\infty = 0 \quad \text{(right-continuity)}.$$

Proof. Let $h > 0$ such that $s := t + h < t_{i+1}$. Then

$$\begin{aligned} \left\| p_{t_{i+1}-t-h} g - p_{t_{i+1}-t} g \right\|_\infty &\overset{(iii)}{=} \left\| p_{t_{i+1}-t-h}(g - p_h g) \right\|_\infty \\ &\leq \| p_{t_i+1-t-h} \|_\infty \cdot \| g - p_h g \|_\infty \\ &\overset{p_s 1 = 1}{\leq} \| g - p_h g \|_\infty \xrightarrow{s \to t} 0 \quad \text{by (ii)}. \end{aligned}$$

□

(v) Assume $t_i < t$ and let $h > 0$ such that $s := t - h > t_i$. Then

$$\lim_{s \nearrow t} \left\| p_{t_{i+1}-s} g - p_{t_{i+1}-t} g \right\|_\infty = 0.$$

Proof. Let $h > 0$ such that $s := t - h > t_i$. Then

$$\left\| p_{t_{i+1}-t+h} g - p_{t_{i+1}-t} g \right\|_\infty \overset{(iii)}{=} \left\| p_{t_{i+1}-t}(p_h g - g) \right\|_\infty \leq \| p_h g - g \|_\infty \xrightarrow{h \to 0} 0.$$

□

(vi) Consider $t := t_i$:
Let $h > 0$

$$p_{t_i-(t_i-h)}(f_i p_{t_{i+1}-t_i} f_{i+1} \ldots p_{t_n-t_{n-1}} f_n) = p_h \underbrace{(f_i p_{t_{i+1}-t_i} f_{i+1} \ldots p_{t_n-t_{n-1}} f_n)}_{=\tilde{g} \in \mathcal{C}_{b,u}(\mathbb{R})} \xrightarrow{h \to 0} \tilde{g}$$

uniformly in x. Hence, also continuous in $t = t_i$ from the left uniformly in x.
Summary: The function G defined by

$$G_t(x) := \prod_{j \leq i} f_j(x) p_{t_{i+1}-t}(f_{i+1} p_{t_{i+2}-t_{i+1}} f_{i+2} \ldots p_{t_n-t_{n-1}} f_n(x) \ldots) \quad \text{for } t \in [t_i, t_{i+1}[$$

is continuous in $t \geq 0$ uniformly in x.

(vii) $t \mapsto G(t, \omega_t)$ is continuous P-a.s..

Proof. Take $|G(t,\omega_t) - G(s,\omega_s)| \leq |G(t,\omega_t) - G(s,\omega_t)| + |G(s,\omega_t) - G(s,\omega_s)|$. Fix $s > 0$. Then the right hand side is dominated by

$$\underbrace{\lim_{t\to s} \|G(t,\cdot) - G(s,\cdot)\|_\infty}_{=0} + \underbrace{\lim_{t\to s} \|G(s,\omega_t) - G(s,\omega_s)\|_\infty}_{=0},$$

because $t \mapsto \omega_t$ is continuous and G is continuous in x. ☐

☐

Corollary 2.5.44 (cf. 2.5.42)**.** *Let $F \in \mathcal{L}^2(\Omega, \mathcal{F}^W_{t_0+}, P)$, $t_0 > 0$ fixed. Such F are sometimes called "Wiener functional". Then there exists an $H \in \mathcal{L}^2(\bar\Omega, \bar{\mathcal{F}}, P_{W^{(t_0+1)}})$, $(\mathcal{F}^W_{t+})$-adapted, such that*

$$F - E[F] = \int_0^{t_0} H_s \, dW_s.$$

Proof. (cf. 2.4.40) Let

$$F^W_t := E[F|\mathcal{F}^W_{t+}] - E[F|\mathcal{F}^W_{0+}],\ 0 \leq t \leq t_0.$$

Then $(F^W_t)_{1\leq t\leq t_0}$ is a martingale, which by the backward martingale convergence theorem (cf. [Röc11]) is right continous. Hence, by 2.5.42,

$$F - E[F|\mathcal{F}^W_{0+}] = E[F|\mathcal{F}^W_{t_0+}] - E[F|\mathcal{F}^W_{0+}] = F^W_{t_0} \overset{2.5.42}{=} \int_0^{t_0} H_s \, \mathrm{d}W_s.$$

The uniqueness is also clear by martingale property. ☐

Example 2.5.45 (Special case: the canonical Model)**.** *Let $(\Omega, \mathcal{F}, P)$ be the (classical) Wiener space, i.e. $\Omega = \mathcal{C}([0,1], \mathbb{R})$, $\mathcal{F}$ its Borel σ-algebra and P the Wiener measure. Define $X_t(\omega) := \omega(t)$, $t \in [0,1]$, $\omega \in \Omega$, hence X is Brownian motion,*

$$\mathcal{F}_t := \bigcap_{s>t} \sigma(\sigma(X_r | r \leq s \leq 1),\ P\text{-zero sets in } \mathcal{F}),$$

and

$$\mathcal{F} = \bigcup_{0<t\leq 1} \mathcal{F}_t.$$

Then, by 2.5.44

$$F \in \mathcal{L}^2(\Omega, \mathcal{F}, P) \Rightarrow F = E[F] + \int_0^1 H_s \, dX_s.$$

Example 2.5.46. *Consider the situation of 2.5.45 the Wiener functional*

$$F := \int_0^1 X_t \, dt.$$

Because of $E[F] = \int_0^1 \underbrace{E(X_t)}_{=0} \, dt = 0$,

$$F = \int_0^1 H_s \, dX_s$$

for some $H \in \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, P_W)$.
Identification of H*:*
Let

$$M_t := E[F|\mathcal{F}_t] = \int_0^t E[X_s|\mathcal{F}_t]\ ds + \int_t^1 E[X_s|\mathcal{F}_t]\ ds = \underbrace{\int_0^t X_s\ ds + X_t(1-t)}_{continuous!}\ t \geq 0,$$

and the latter is clearly in $\mathcal{M}_c^2$.
Claim: $H_t = 1 - t$.

Proof. In view of 2.4.31(i) it is enough to show

$$\langle M, X\rangle = \int_0^t H_s \,\mathrm{d}\langle X\rangle_s \leq \int_0^t (1-s)\,\mathrm{d}s,$$

since then, by $\mathcal{M}^2 \overset{2.5.42}{=} \mathcal{M}_X^2$, we have $M = H.X$, in particular,

$$\int_0^1 X_t \,\mathrm{d}t = F = M_1 = \int_0^1 (1-s)\,\mathrm{d}X_s.$$

But by Itô's product rule

$$X_t t = \int_0^t X_s \,\mathrm{d}s + \int_0^t s \,\mathrm{d}X_s.$$

Hence

$$d\langle M, X\rangle_t = d\langle X - \int_0^\cdot s\,\mathrm{d}X_s,\ X\rangle_t = \ \mathrm{d}t - t\,\mathrm{d}t,$$

thus $\langle M, X\rangle_t = \int_0^t (1-s)\,\mathrm{d}s$. □

3. Markov Processes

3.1. Markov Processes and Semigroups

Definition 3.1.1. *A family* $(p_t)_{t\geq 0}$ *of kernels on a measurable space* $(S, \mathcal{S})$ *is called a* semigroup of kernels *if*

$$p_t p_s = p_{t+s} \quad \forall t, s \geq 0 \quad \textit{(Chapman-Kolmogorov equation)},$$

i.e.

$$p_t p_s(x, A) = \int p_t(x, dy) p_s(y, A) \left(:= \int p_s(y, A) p_t(x, dy) \right).$$

If for all t, x $p_t(x, S) = 1$, $(p_t)_{t\geq 0}$ *is called* Markovian *and, if* $p_t(x, S) \leq 1$ *for all* x, t, sub-Markovian*, respectively.*

Definition 3.1.2. *Let* $(S, \mathcal{S})$ *be a measurable space. A family of stochastic processes*

$$\left(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in S}\right)$$

with state space $(S, \mathcal{S})$ *is called a* Markov process*, if*

(M1) $x \mapsto P_x(\Gamma)$ *is* $\mathcal{S}$*-measurable for all* $\Gamma \in \mathcal{F}$*,*

(M2) there exists a filtration $(\mathcal{F}_t)_{t\geq 0}$ *such that each* X_t *is* $\mathcal{F}_t$*-measurable and*

$$P_x(X_{s+t} \in B | \mathcal{F}_s) = P_{X_s}(X_t \in B) \ P_x\text{-a.e.} \quad \forall s, t \geq 0, B \in \mathcal{S}, x \in S$$

(Markov property with respect to $(\mathcal{F}_t)_{t\geq 0}$*).*

Remark 3.1.3. *If (M2) is true for* $(\hat{\mathcal{F}}_t)_{t\geq 0}$ *with* $\hat{\mathcal{F}}_t \supset \mathcal{F}_t$ $\forall t \geq 0$*, then (M2) is true for* $(\mathcal{F}_t)_{t\geq 0}$ *if* (X_t) *is* $(\mathcal{F}_t)$*-adapted!*

The following theorem shows, at least if $(S, \mathcal{S})$ is polish, that Markovian semigroups and Markov processes are in one-to-one correspondence to each other.

Theorem 3.1.4. *(i) Let* $(S, \mathcal{S})$ *be a measurable space and let* $\mathbb{M} = \left(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in S}\right)$ *be a family of stochastic processes with state space* $(S, \mathcal{S})$ *and* $\mathcal{F} = \sigma(X_t | \, t \geq 0)$*.*

a) Suppose there exists a Markovian semigroup of kernels $(p_t)_{t\geq 0}$*, such that for all* $0 \leq t_0 < t_1 < \ldots < t_n$*,* $f : S^{n+1} \to \mathbb{R}$*, bounded and* $\mathcal{S}^{n+1}$ *measurable, and all* $x \in S$*,*

$$E_x[f(X_{t_0}, \ldots, X_{t_n})] = \int_S p_{t_0}(x, dx_0) \cdot \ldots \cdot \int p_{t_n - t_{n-1}}(x_{n-1}, dx_n) f(x_0, \ldots, x_n) \quad (3.1.1)$$

Then $\mathbb{M}$ *is a Markov process with respect to* $\mathcal{F}_t := \sigma(X_s | s \leq t)$*,* $t \geq 0$ *(and even* (3.1.3) *below holds).*

b) Suppose $\mathbb{M}$ *is a Markov process and set*

$$p_t f(x) := E_x(f(X_t)),\ x \in S, f : S \to \mathbb{R} \text{ bounded }, \mathcal{S}\text{-measurable}, t \geq 0. \quad (3.1.2)$$

Then $(p_t)_{t\geq 0}$ *is a Markovian semigroup of kernels on* $(S, \mathcal{S})$ *and we have* (3.1.1)*.*

(ii) *If* $(S, \mathcal{S})$ *is polish and* $(p_t)_{t\geq 0}$ *is a Markovian semigroup of kernels on* $(S, \mathcal{S})$*. Then there exists a Markov process* $\mathbb{M}$ *with* (3.1.2) *and* $\Omega = S^{[0,\infty]}$.

Proof. (i): a) (M1) follows by "monotone classes" from (3.1.1), since $\mathcal{F} = \sigma(\{X_t | t \geq 0\})$. For (M2) we show an a-priori stronger fact:

$$E_x[f(X_{t_1+s}, \dots, X_{t_n+s}), \Gamma] = E_x[E_{X_s}[f(X_{t_1}, \dots, X_{t_m})], \Gamma] \tag{3.1.3}$$

for all $x \in S$, bounded and $\mathcal{S}^n$-measurable $f : S^n \to \mathbb{R}$, $0 \leq s_0 < s_1 < \ldots < s_m = s$ and $\Gamma \in \mathcal{F}_s, s \geq 0$. By monotone classes (applied to Γ) this follows from

$$\begin{aligned}&E_x[f(X_{t_1+s}, \dots, X_{t_n+s})g(X_{s_0}, \dots, X_{s_m})]\\ =&E_x[E_{X_s}[f(X_{t_1}, \dots, X_{t_n})]g(X_{s_0}, \dots, X_{s_m})]\end{aligned}$$

for all bounded $\mathcal{S}^{m+1}$ -measurable $g : S^{n+1} \to \mathbb{R}$, $0 \leq s_0 < s_1 < \ldots < s_m = s$. By (3.1.1) the left hand side is equal to

$$\begin{aligned}&\int p_{s_0}(x, dx_0) \cdot \ldots \cdot \int p_{s_m - s_{m-1}}(x_{m-1}, dx_m) \cdot \\ &\underbrace{\int p_{t_1}(x_m, dy_1) \ldots \int p_{t_n - t_{n-1}}(y_{n-1}, dy_n) f(y_1, \dots, y_n)}_{=E_{x_m}[f(X_{t_1},\dots,X_{t_n})]} g(x_0, x_1, \dots, x_m)\\ &= E_x[E_{X_{s_m}}[f(X_{t_1} \ldots X_{t_n})]g(X_{s_0}, X_{s_1}, \dots, X_{s_m})].\end{aligned}$$

b) By (M1) it is clear that (3.1.2) defines a Markovian kernel for all $t \geq 0$. Furthermore,

$$\begin{aligned}p_{t+s}f(x) = E_x[f(X_{t+s})] &= E_x[E_x[f(X_{t+s})|\mathcal{F}_s]]\\ &\overset{\text{M.P.}}{=} E_x[E_{X_s}[f(X_t)]] = E_x[p_t f(X_s)]\\ &= p_s(p_t f)(x) = (p_s p_t)f(x)\end{aligned}$$

To prove (3.1.1) let $f = f_0 \otimes f_1 \otimes \ldots \otimes f_n$, $f_i : S \to \mathbb{R}$, bounded and $\mathcal{S}$-measurable. Then for $0 \leq t_0 < t_1 < \cdots < t_n$

$$\begin{aligned}&E_x[f_0(X_{t_0})f_1(X_{t_1}) \ldots f_n(X_{t_n})]\\ =&E_x[f_0(X_{t_0})f_1(X_{t_1}) \ldots f_{n-1}(X_{t_{n-1}})E_x[f_n(X_{(t_n - t_{n-1}) + t_{n-1}})|\mathcal{F}_{t_{n-1}}]]\\ \overset{\text{M.P.}}{=}&E_x[f_0(X_{t_0})f_1(X_{t_1}) \ldots f_{n-1}(X_{t_{n-1}})E_{X_{t_{n-1}}}[f_n(X_{t_n - t_{n-1}})]]\\ \overset{(3.1.2)}{=}&E_x[f_0(X_{t_0})f_1(X_{t_1}) \ldots f_{n-1}(X_{t_{n-1}})p_{t_n - t_{n-1}}f_n(X_{t_{n-1}})]\\ \overset{\text{Ind.Hyp.}}{=}&\int \ldots \int p_{t_0}(x, dx_0)p_{t_1 - t_0}(x_0, dx_1) \cdots p_{t_{n-1} - t_{n-2}}(x_{n-2}, dx_{n-1})\\ &\quad f_0(x_0)f_1(x_1) \ldots f_{n-1}(x_{n-1})p_{t_n - t_{n-1}}f_n(x_{n-1})\\ =&\int \ldots \int p_{t_0}(x, dx_0)p_{t_1 - t_0}(x_0, dx_1) \cdots p_{t_{n-1} - t_{n-2}}(x_{n-2}, dx_{n-1})p_{t_n - t_{n-1}}(x_{n-1}, dx_n)\\ &\quad f_0(x_0)f_1(x_1) \ldots f_{n-1}(x_{n-1})f_n(x_n).\end{aligned}$$

By monotone classes this equality extends to all $f : S^n \to \mathbb{R}$, bounded, measurable. Hence, (3.1.1) holds.

(ii): see Bauer!

□

Now consider the "canonical model":
Let S be a topological space, $\mathcal{S}$ the Borel-σ-algebra and $\Omega \subset S^{[0,\infty[}$ (e.g. $\Omega = S^{[0,\infty[}$ or $\Omega = \mathcal{C}([0,\infty[, S)$ or Ω are all bounded continuous paths in S). Define

$$X_t(\omega) := \omega(t),\ t \geq 0,\ \omega \in \Omega,$$

$$\mathcal{F} := \sigma(X_t | t \geq 0),$$

$$\mathcal{F}_t^0 := \sigma(X_s | s \leq t) \qquad (\text{“past”})$$

$$\hat{\mathcal{F}}_t^0 := \sigma(X_s | s \geq t) \qquad (\text{“future”}).$$

Let $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t^0)_{t\geq 0}), (X_t)_{t\geq 0}\,, \underbrace{(P_x)_{x\in S}}_{\text{all information here!}})$ be a Markov process. Then $\mathbb{M}$ is called the *canonical model.*

Definition 3.1.5. *In the canonical model define the* shift operator $\vartheta_t : \Omega \to \Omega$ *for* $t \geq 0$ *by*

$$\vartheta_t(\omega)(s) := \omega(s+t),$$

i.e.

$$\vartheta_t(\omega) = \omega(\cdot + t).$$

It is obvious that $\vartheta_t : \Omega \to \Omega$ is $\hat{\mathcal{F}}_t^0/\mathcal{F}$-measurable and moreover, (exercise)

$$\vartheta_t^{-1}(\mathcal{F}) = \hat{\mathcal{F}}_t^0 \quad \forall t \geq 0.$$

Lemma 3.1.6. *(i) ψ is $\hat{\mathcal{F}}_t^0$-measurable if and only if there exists an $\mathcal{F}$-measurable φ such that*

$$\psi = \varphi \circ \vartheta_t.$$

(ii) Suppose P_x, $x \in S$, are given and that

$$\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t^0)_{t\geq 0}, (X_t)_{t\geq 0}\,, (P_x)_{x\in S})$$

satisfies (M1). Then (M2) holds if and only if

$$E_x[\varphi \circ \vartheta_t | \mathcal{F}_t^0] = E_{X_t}[\varphi] \quad P_x\text{-a.s.} \, \forall x \in S,\ \forall \varphi : \Omega \to \mathbb{R} \text{ bounded and } \mathcal{F}\text{-measurable.} \tag{3.1.4}$$

Proof. (i): Exercise (Factorization lemma).

(ii): "$\Leftarrow$" is clear. (Consider $\varphi = 1_{\{X_s \in B\}}, B \in \mathcal{S}$.)
"$\Rightarrow$": By 3.1.4 (i)(b) we know that (3.1.1) holds. But in the proof of 3.1.4(i) we have shown that (3.1.1) implies (3.1.3). Now Lemma 3.1.6 follows by "monotone classes".

□

It is clear that (3.1.4) implies (elementary Markov property)

$$E_x[\varphi \circ \vartheta_t | \mathcal{F}_t^0] = E_x[\varphi \circ \vartheta_t | \sigma(X_t)] \quad \forall t \geq 0. \tag{3.1.5}$$

Interpretation of (3.1.5): The conditional expectation of a future observable, given the past, only depends on the present at time t. The equivalent formulations of (3.1.5) in the following lemma have corresponding interpretations:

Lemma 3.1.7. *Fix $x \in S$. Then the following statements are equivalent to* (3.1.5)*:*

(i)

$$E_x[\varphi_t^0|\hat{\mathcal{F}}_t^0] = E_x[\varphi_t^0|\sigma(X_t)] \quad P\text{-a.s.}, \forall t \geq 0, \forall \varphi_t^0 : \Omega \to \mathbb{R} \text{ bounded and } \mathcal{F}_t^0\text{-measurable},$$

(ii)

$$E_x[\varphi_t^0\hat{\varphi}_t^0|\sigma(X_t)] = E_x[\varphi_t^0|\sigma(X_t)]E_x[\hat{\varphi}_t^0|\sigma(X_t)] \quad P_x\text{-a.s.},$$

for all $t \geq 0$ *and for all* $\varphi_t^0, \hat{\varphi}_t^0 : \Omega \to \mathbb{R}$*, bounded,* φ_t^0 $\mathcal{F}_t^0$*-measurable and* $\hat{\varphi}_t^0$ $\hat{\mathcal{F}}_t^0$*-measurable.*

Proof. (3.1.5) ⇒ (i):

$$\begin{aligned} E_x[\varphi_t^0\hat{\varphi}_t^0] =& E_x[\varphi_t^0 E_x[\hat{\varphi}_t^0|\mathcal{F}_t^0]] \\ \overset{3.1.6(i),(3.1.5)}{=}& E_x[\varphi_t^0 E_x[\hat{\varphi}_t^0|\sigma(X_t)]] \\ =& E_x[E_x[\varphi_t^0|\sigma(X_t)]E_x[\hat{\varphi}_t^0|\sigma(X_t)]] \\ =& E_x[\hat{\varphi}_t^0 E_x[\varphi_t^0|\sigma(X_t)]] \quad \forall \text{ bounded } \hat{\mathcal{F}}_t^0\text{-measurable } \hat{\varphi}_t^0 : \Omega \to \mathbb{R}. \end{aligned}$$

Therefore, $E_x[\varphi_t^0|\sigma(X_t)]$ is a P_x-version of $E_x[\varphi_t^0|\hat{\mathcal{F}}_t^0]$. Hence, (i) holds.
(i) ⇒ (ii):
Let $f : S \to \mathbb{R}$ be bounded and $\mathcal{S}$-measurable and $\hat{\varphi}_t^0 : \Omega \to \mathbb{R}$ be bounded, $\hat{\mathcal{F}}_t^0$-measurable. Then

$$\begin{aligned} E_x[\varphi_t^0\hat{\varphi}_t^0 f(X_t)] &= E_x[E_x[\varphi_t^0|\hat{\mathcal{F}}_t^0]\hat{\varphi}_t^0 f(X_t)] \overset{(i)}{=} E_x[E_x[\varphi_t^0|\sigma(X_t)]\hat{\varphi}_t^0 f(X_t)] \\ &= E_x[E_x[\varphi_t^0|\sigma(X_t)]E_x[\hat{\varphi}_t^0|\sigma(X_t)]f(X_t)]. \end{aligned}$$

(ii) ⇒ (3.1.5):

$$\begin{aligned} E_x[\varphi_t^0\hat{\varphi}_t^0] &\overset{(ii)}{=} E_x[E_x[\varphi_t^0|\sigma(X_t)]E[\hat{\varphi}_t^0|\sigma(X_t)]] \\ &= E_x[\varphi_t^0 E_x[\hat{\varphi}_t^0|\sigma(X_t)]] \quad \forall \text{ bounded } \mathcal{F}_t^0\text{-measurable } \varphi_t^0 : \Omega \to \mathbb{R}. \end{aligned}$$

Hence,

$$E_x[\hat{\varphi}_t^0|\sigma(X_t)] = E_x[\hat{\varphi}_t^0|\mathcal{F}_t^0],$$

and (3.1.5) follows by Lemma 3.1.6(i). □

Remark 3.1.8. *Consider the situation of 3.1.4(i)(b). Then* $P_x(X_0 = x) = 1$ *if and only if* $p_0(x, \cdot) = \delta_x$ *(Dirac-measure in* x*). If this holds for every* $x \in S$*, then* M *is called* normal. *In this case,* $p_t(x, A) = P_x(X_t \in A)$ *is the probability to be at time* t *in* A *starting in* x *(transition probability).*

3.2. The Strong Markov Property

Consider the "canonical model". (← only needed to have ϑ_t, can be done abstractly!)
Recall: Let $(\mathcal{F}_t)$ be some filtration and T an $(\mathcal{F}_t)$-stopping time. Define the *σ-field of the T-past* by

$$\mathcal{F}_T := \{A \in \mathcal{F} | A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t \geq 0\}.$$

Definition 3.2.9. *Let* $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in S}, (\vartheta_t)_{t\geq 0})$ *be a canonical Markov process. Then* $\mathbb{M}$ *satisfies the* strong Markov property *(SMP), if there exists a* right-continuous *filtration* $(\mathcal{F}_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F})$ *(i.e.* $\mathcal{F}_t = \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$*) such that for all* $(\mathcal{F}_t)$*-stopping times* T *we have that* $\vartheta_{T\restriction\{T<\infty\}}$ *is* $\mathcal{F} \cap \{T < \infty\}/\mathcal{F}$*- and* $\mathcal{F}_{T+s} \cap \{T < \infty\}/\mathcal{F}_s$*-measurable for all* $s \leq 0$ *and we have (SMP)*

$$E_x[1_{\{T<\infty\}}\varphi \circ \vartheta_T|\mathcal{F}_T] = 1_{\{T<\infty\}}E_{X_T}[\varphi] \quad P\text{-a.s.} \forall \varphi \text{ bounded, } \mathcal{F}\text{-measurable and } \forall x \in S.$$

Remark 3.2.10. *(i) It is clear that (SMP) implies the Markov property with respect to $(\mathcal{F}_t)$ and therefore, with respect to $(\mathcal{F}_t^0)$ by 3.1.6(ii). In general, the converse is false.*

(ii) (SMP) holds if and only if for all *$(\mathcal{F}_t)$-stopping times T we have*

$$E_x[\varphi \circ \vartheta_T; T < \infty] = E_x[E_{X_T}[\varphi]; T < \infty] \quad P_x\text{-a.s.} \tag{3.2.6}$$

for all $\varphi : \Omega \to \mathbb{R}$, bounded and $\mathcal{F}$-measurable, and for all $x \in S$.

Proof. "$\Rightarrow$" is clear.
"$\Leftarrow$": Let $A \in \mathcal{F}_T$ and $\tilde{T} := T \cdot 1_A + \infty \cdot 1_{A^c}$. Then $\tilde{T}$ is a $(\mathcal{F}_t)$-stopping time because $\forall\, t \geq 0$

$$\{\tilde{T} \leq t\} = A \cap \{T \leq t\} \in \mathcal{F}_t.$$

Hence

$$\begin{aligned} E_x[\varphi \circ \vartheta_T; A \cap \{T < \infty\}] &= E_x[\varphi \circ \vartheta_T; \tilde{T} < \infty] \\ &= E_x[\varphi \circ \vartheta_{\tilde{T}};\ \tilde{T} < \infty] \\ &\overset{(3.2.6)}{=} E_x[E_{X_{\tilde{T}}}(\varphi); \tilde{T} < \infty] \\ &= E_x[E_{X_T}(\varphi); A \cap \{T < \infty\}]. \end{aligned}$$

□

Proposition 3.2.11. *Suppose S is a topological space and $\mathcal{S}$ Borel σ-algebra such that $\mathcal{S} = \sigma(\mathcal{C}_b(S))$. Let $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in S}, (\vartheta_t)_{t\geq 0})$ be a (canonical) Markov process with*

(i) right-continuous paths,

(ii) $p_t(\mathcal{C}_b(S)) \subset C_b(S)$ ("Feller property"),

where p_t is the corresponding semigroup. Then $\mathbb{M}$ fulfills the (SMP) with respect to $\mathcal{F}_t := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^0$, $t \geq 0$ (right-continuous).

Proof. Let T be an $(\mathcal{F}_t)$-stopping time. By [Röc11] X_T is $\mathcal{F}_T$-measurable since X has right-continuous sample paths. Similarly, one can prove that ϑ_T has the required measurability properties. Define

$$T_n := \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{\{\frac{k-1}{2^n} \leq T < \frac{k}{2^n}\}} + \infty \cdot 1_{\{T=\infty\}}.$$

Then $T_n \searrow T$. We have to show that (3.2.6) holds:
Without loss of generality (otherwise use monotone classes)

$$\varphi = f_0(X_{t_0}) f_1(X_{t_1}) \dots f_n(X_{t_n}), \quad f_i \in \mathcal{C}_b(S),\ t_0 \leq t_1 \leq \ldots \leq t_n.$$

Furthermore,

$$\{T < \infty\} = \dot{\bigcup}_{k\in\mathbb{N}} \left\{ \frac{k-1}{2^n} \leq T < \frac{k}{2^n} \right\} \tag{3.2.7}$$

Then

$$\begin{aligned}
E_x[\varphi\circ\vartheta_T, T<\infty] &\overset{(i)}{=} \lim_{n\to\infty} E_x[\varphi\circ\vartheta_{T_n}, T<\infty]\\
&\overset{(3.2.7)}{=} \lim_{n\to\infty}\sum_{k=1}^{\infty} E_x\left[\varphi\circ \underbrace{\vartheta_{T_n}}_{=\vartheta_{k2^{-n}}}, \frac{k-1}{2^n}\le T<\frac{k}{2^n}\right]\\
&\overset{M.P.}{=} \lim_{n\to\infty}\sum_{k=1}^{\infty} E_x\left[E_{X_{k2^{-n}}}[\varphi]; \frac{k-1}{2^n}\le T<\frac{k}{2^n}\right]\\
&\overset{(3.2.7)}{=} \lim_{n\to\infty} E_x[E_{X_{T_n}}[\varphi], T<\infty]\\
&= E_x[E_{X_T}[\varphi]; T<\infty] \quad \text{(Lebesgue)},
\end{aligned}$$

since $X_{T_n}\to X_T$ on $[T<\infty]$ and

$$x\mapsto E_x[\varphi]=E_x[f_0(X_{t_0}),\dots,f_n(X_{t_n})]=p_{t_0}(f_0p_{t_1-t_0}f_1p_{t_2-t_1}f_2\cdots p_{t_n-t_{n-1}}f_n)(x)$$

is continuous on S by (ii). $\square$

Remark 3.2.12 (Blumenthal's $0-1$ law)**.** *Let* $\mathbb{M}$ *be a canonical* normal *Markov process with respect to* $\mathcal{F}_t := \bigcap_{s>t}\mathcal{F}_s^0$, $t\ge 0$ *(on measurable state space* $(S,\mathcal{S})$*). Then* $P_x=0$ *or* $P_x=1$ *on* $\mathcal{F}_0$ *for every* $x\in\mathcal{S}$*.*

Proof. Let φ be $\mathcal{F}_0$-measurable and bounded. Then for all $x\in S$ we have

$$\varphi = E_x[\varphi|\mathcal{F}_0]=E_x[\varphi\circ\vartheta_0|\mathcal{F}_0]\overset{M.P.}{=}E_{X_0}[\varphi]=E_x[\varphi]\quad P_x\text{-a.s..}$$

Hence, φ is constant P_x-a.e.. $\square$

3.3. Application to Brownian Motion

Let $S=\mathbb{R}^d$, $\mathcal{S}=\mathcal{B}(\mathbb{R}^d)$ and

$$p_t(x,A)=\int_A \underbrace{\frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{|y-x|^2}{2t}}}_{=:g_t(y-x)}\,\mathrm{d}y,\quad x\in\mathbb{R}^d,\ t>0, A\in\mathcal{B}(\mathbb{R}^d),$$

$$p_0(x,\cdot)=\delta_x.$$

Lemma 3.3.13. *For all* $t,x\ge 0$*, we have*

$$p_tp_s=p_{t+s}.$$

Proof. Exercise (by Fourier transform). $\square$

Let $\Omega:=\mathcal{C}([0,\infty[,\mathbb{R}^d)$, $X_t(\omega):=\omega(t)$, $P_0:=P$ be the Wiener measure on Ω, $\mathcal{F}:=\sigma(X_t|t\ge 0)$ and P_x the image measure of P_0 under $\omega\mapsto\omega+x$. Then we have (cf. [Röc11])

$$E_x[f(X_{t_0},\dots,X_{t_n})]=\int p_{t_0}(x,dx_0)p_{t_1-t_0}p_{t_n-t_{n-1}}(x_{n-1},dx_n)f(x_0,\dots,x_n),$$

for all $x\in\mathbb{R}^d$, for all $\mathcal{B}((\mathbb{R}^d)^{n+1})$-measurable, bounded functions $f:(\mathbb{R}^d)^{n+1}\to\mathbb{R}$ and for $0\le t_0\le t_1\le\ldots\le t_n$. Hence,

$$\mathbb{M}:=(\Omega,\mathcal{F},(X_t)_{t\ge 0},(P_x)_{x\in\mathbb{R}^d})$$

is by Proposition 3.1.4(i)(a) a Markov Process on $(\mathbb{R}^d, B(\mathbb{R}^d))$. $\mathbb{M}$ has continuous sample paths and is normal. In particular, Remark 3.2.12 is fulfilled. But also, by Proposition 3.2.11 $\mathbb{M}$ has SMP with respect to

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^0,\ t \geq 0,$$

because of the following Proposition.

Proposition 3.3.14. *$(p_t)_{t\geq 0}$ is* strong Feller*, that is,*

$$p_t f \in \mathcal{C}_b(\mathbb{R}^d) \quad \forall f \in \mathcal{B}_b(\mathbb{R}^d).$$

(Moreover, we have $p_t f \in \mathcal{C}^\infty$ for all $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$.)

Proof. Fix $t > 0$. Let $x \in \mathbb{R}^d$ and $x_n \to x$. We have to show that $p_t f(x_n) \to p_t f(x)$. Let n_0 such that $x_n \in B_1(x)$ for all $n \geq n_0$. There exists $h \in \mathcal{L}^1(\mathbb{R}) := \mathcal{L}^1(\mathbb{R}, dx)$ (with dx =Lebesgue measure) such that $e^{-\frac{(x'-y)^2}{2t}} \leq h(y)$ for all $y, x' \in \mathbb{R}$ such that $|x' - x| \leq 1$.
Define $g(\tilde{x}_1, \dots, \tilde{x}_d) = h(\tilde{x}_1) \cdots h(\tilde{x}_d)$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_d) \in \mathbb{R}^d$, then

$$e^{\frac{-|x'-y|^2}{2t}} \leq g(y) \quad \forall x' \in B_1(x)\, \forall y \in \mathbb{R}^d.$$

Hence

$$\lim_{n\to\infty} p_t f(x_n) = \frac{1}{\sqrt{(2\pi t)^d}} \lim_{n\to\infty} \int f(y) e^{-\frac{|y-x_n|^2}{2t}}\,\mathrm{d}y = \frac{1}{\sqrt{(2\pi t)^d}} \int f(y) e^{-\frac{|y-x|^2}{2t}}\,\mathrm{d}y = p_t f(x)$$

by Lebesgue's dominated convergence theorem since f is bounded and $e^{\frac{-|x'-y|^2}{2t}} \leq g(y)$ and $g \in \mathcal{L}^1(\mathbb{R})$. ☐

Corollary 3.3.15. *(i) Let $\mathbb{M}$ be a normal (canonical) Markov-Process with respect to $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0$, $t \geq 0$. Then, by Blumenthal for t not fixed,*

$$P_x[\underbrace{\limsup_{t\searrow 0} 1_{\{X_t = x\}} = 1}_{\in \mathcal{F}_0}] \in \{0, 1\}.$$

For Brownian motion this probability is equal to 1 because of the law of iterated logarithm.

(ii) Let X be Brownian motion starting at $x = 0$ and let

$$N(\omega) := \{0 \leq s \leq 1 |\, X_s(\omega) = 0\}.$$

Then for P_0-a.e. ω we have that

a) $N(\omega)$ is closed,

b) $N(\omega)$ has Lebesgue-measure zero,

c) $N(\omega)$ has no isolated points,

that is $N(\omega)$ is a Cantor set for P_0-a.e. ω, i.e. every point of $N(\omega)$ is a cluster point of $N(\omega)$.

(iii) Let $U \subset \mathbb{R}^d$, U open and bounded and let X be Brownian motion on $\mathbb{R}^d$. Define the first exit time of U

$$T := \inf\{t > 0 | X_t \notin U\} = \sigma_{U^c}.$$

T is an $(\mathcal{F}_t)$-stopping time. Then there exists $\varepsilon > 0$ such that

$$\sup_{x\in U} E_x[e^{\varepsilon T}] < \infty.$$

Proof. (i) Clear.

(ii) a) X is P-a.s. continuous.

b) By Fubini, we get

$$\int\int_0^1 1_{N(\omega)}(s)\,\mathrm{d}sP_0(\,\mathrm{d}\omega) = E_0\left[\int_0^1 1_{\{0\}}\circ X_s\,\mathrm{d}s\right] = \int_0^1 P_0\circ X_s^{-1}(\{0\})\,\mathrm{d}s$$
$$= \int_0^1 p_s(0,\{0\})\,\mathrm{d}s = 0.$$

c) (Intuitively clear: by law of (local) iterated logarithm $t=0$ is not an isolated point of $N(\omega)$. By (SMP) this is true for any $s\in\mathbb{N}(\omega)$). Define

$$\begin{aligned}J :=&\{\omega|N(\omega)\text{ has isolated points}\}\\ =& \bigcup_{r,s\in\mathbb{Q},\,r<s}\underbrace{\{\omega|N(\omega)\cap]r,s[\text{ contains exactly one (isolated) point}\}}_{A_{r,s}}.\end{aligned}$$

Let $T_{\{0\}} := \inf\{t>0|X_t=0\}$. Then $T_{\{0\}}$ is an $(\mathcal{F}_t)$ $\left(=\bigcap_{s>t}\mathcal{F}_s^0\right)$-stopping time (exercise). Therefore, for $T := T_{\{0\}}\circ\vartheta_r + r$ (first hitting time of 0 after time r, $(\mathcal{F}_t)$-stopping time!)

$$\begin{aligned}P_0[A_{r,s}] &\le P_0[r+T_{\{0\}}\circ\vartheta_r<s,\,T_{\{0\}}\circ\vartheta_{r+T_{\{0\}}\circ\vartheta_r}>0]\\ &= E_0[1_{\{T_{\{0\}}>0\}}\circ\vartheta_T;\,T<s]\\ &= E_0[E_0[1_{\{T_{\{0\}}>0\}}\circ\vartheta_T|\mathcal{F}_T];T<s]\\ &\overset{\text{SPM}}{=} E_0[E_{X_T}[1_{\{T_{\{0\}}>0\}}],T<s]\\ &\overset{X_T=0}{=} E_0[\underbrace{P_0[T_{\{0\}}>0]}_{=0};T<s]=0,\end{aligned}$$

by (i).

(iii) Let $R>0$ such that $U\subset B_R$ (closed ball with radius R around 0). Define

$$T_R := \inf\{t>0|\,|X_t|>R\} = \sigma_{B_R^c}.$$

By Ito's formula (or Doob-Meyer), $|X_t|^2 - d\cdot t$ is a martingale under P_x. Hence,

$$\begin{aligned}0\le|x|^2 &= E_x[|X_0|^2-d\cdot 0] = E_x[|X_{T_R\wedge n}|^2-d\cdot T_R\wedge n]\\ &= E_x[|X_{T_R\wedge n}|^2]-d\cdot E_x[T_R\wedge n]\le R^2-d\cdot E_x[T_R\wedge n],\quad\forall x\in U,\end{aligned}$$

Therefore, taking n to ∞, we get

$$E_x[T]\le E_x[T_R]\le\frac{R^2}{d}<\infty,\quad\forall x\in U.$$

Hence, for large t_0

$$\sup_{x\in U}P_x[T>t_0]\le\sup_{x\in U}\frac{1}{t_0}E_x[T]<1.$$

Define

$$\varphi(t) := \sup_{x\in U}P_x[T>t]\quad\text{(decreasing in } t!)$$

Then, we have

$$\begin{aligned}\varphi(t+s) &= \sup_{x\in U} E_x[1_{\{T>t+s\}}] \\ &= \sup_{x\in U} E_x[1_{\{T>s\}} \cdot 1_{\{T>t\}} \circ \vartheta_s] \\ &= \sup_{x\in U} E_x[E_x[1_{\{T>t\}} \circ \vartheta_s | \mathcal{F}_s]; T>s] \quad (\text{since } \{T>s\} \in \mathcal{F}_s) \\ &\overset{\text{MP}}{=} \sup_{x\in U} E_x[P_{X_s}[T>t]; T>s] \\ &\leq \sup_{x\in U} P_x[T>t] \cdot \sup_{x\in U} P_x[T>s] = \varphi(t)\varphi(s).\end{aligned}$$

Claim: These two conditions, i.e.

a) there exists a $t_0>0$ such that $\varphi(t_0)<1$,

b)
$$\varphi(t+s) \leq \varphi(t)\varphi(s), \tag{$\star$}$$

imply that φ is subexponential, i.e. there exist $K>0$ and $\lambda>0$ such that

$$\varphi(t) \leq K e^{-\lambda t} \quad \forall t \geq t_0.$$

Proof. We have

$$\varphi(s) = \varphi\left(\frac{s}{t_0} t_0\right) \overset{\varphi \text{ is decreasing}}{\leq} \varphi\left(\left\lfloor \frac{s}{t_0} \right\rfloor \cdot t_0\right)$$

Hence, inequality ($\star$) implies

$$\varphi(s) \leq \varphi(\underbrace{t_0 + \ldots + t_0}_{\lfloor \frac{s}{t_0} \rfloor -\text{times}}) \leq \underbrace{\varphi(t_0) \cdot \ldots \cdot \varphi(t_0)}_{\lfloor \frac{s}{t_0} \rfloor -\text{times}} = \varphi(t_0)^{\lfloor \frac{s}{t_0} \rfloor}.$$

From now on, without loss of generality we may assume $\varphi(t_0)>0$. Then, since $\left\lfloor \frac{s}{t_0} \right\rfloor \geq \frac{s}{t_0} - 1$, it follows that

$$\varphi(t_0)^{\lfloor \frac{s}{t_0} \rfloor} = e^{\lfloor \frac{s}{t_0} \rfloor \ln \varphi(t_0)} \leq \exp\left(-s \frac{|\ln \varphi(t_0)|}{t_0}\right) \exp(|\ln \varphi(t_0)|) = K e^{-s\lambda},$$

where

$$K := \exp(|\ln \varphi(t_0)|) \text{ and } \lambda := \frac{|\ln \varphi(t_0)|}{t_0}.$$

□

Then for all $x \in U$ and any $\varepsilon > 0$

$$\begin{aligned}E_x[e^{\varepsilon T}] &= E_x\left[\int_0^{e^{\varepsilon T}} 1 \,\mathrm{d}s\right] = E_x\left[\int_0^\infty 1_{[0,e^{\varepsilon T}[}(s) \,\mathrm{d}s\right] \\ &\overset{\text{Fubini}}{=} \int_0^\infty P_x[e^{\varepsilon T} > s] \,\mathrm{d}s \\ &= \int_0^{e^{t_0\varepsilon}} P_x[e^{\varepsilon T} > s] \,\mathrm{d}s + \int_{e^{t_0\varepsilon}}^\infty P_x[e^{\varepsilon T} > s] \,\mathrm{d}s.\end{aligned}$$

For the first term we have easily

$$\int_0^{e^{t_0\varepsilon}} P_x[e^{\varepsilon T} > s]\,\mathrm{d}s \le e^{t_0\varepsilon} < \infty.$$

But, we can estimate the second:

$$\begin{aligned}\int_{e^{t_0\varepsilon}}^{\infty} P_x[e^{\varepsilon T} > s]\,\mathrm{d}s &\le \int_{e^{t_0\varepsilon}}^{\infty} \sup_{x\in U} P_x[e^{\varepsilon T} > s]\,\mathrm{d}s \\ &= \int_{t_0}^{\infty} \varepsilon e^{\varepsilon u} \sup_{x\in U} \underbrace{P_x[e^{\varepsilon T} > e^{\varepsilon u}]}_{=P_x[T>u]}\,du \\ &= \varepsilon \int_{t_0}^{\infty} e^{\varepsilon u}\varphi(u)du \\ &\le \varepsilon\cdot K\int_{t_0}^{\infty} e^{u(\varepsilon-\lambda)}du < \infty,\end{aligned}$$

if $\varepsilon < \lambda$.

□

3.4. Sojourn Time

Let S be a polish space, $\mathcal{S}$ the Borel-σ-algebra and $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\ge 0}, (P_x)_{x\in S})$ a normal Markov process with respect to a right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$ with state space S and continuous paths.

Definition 3.4.16. *Fix $x \in S$. The time*

$$\tau_x := \inf\{t > 0 | X_t \neq x\} = \sigma_{\{x\}^c}$$

is called sojourn time*.*

Clearly, by (right) continuity of the sample paths

$$\tau_x = 0 \text{ on } \{X_0 = y\},\ y \neq x.$$

So, τ_x is only non-trivial under P_x.

Remark 3.4.17. *τ_x is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time because for all $t \ge 0$*

$$\{\tau_x \ge t\} = \{X_s = x | 0 \le s \le t,\ s \in \mathbb{Q}\} \in \mathcal{F}_t,$$

hence $\{\tau_x < t\} \in \mathcal{F}_t$. Since $\mathcal{F}_t = \mathcal{F}_{t+}$, also

$$\{\tau_x \le t\} = \bigcap_{k\in\mathbb{N}} \underbrace{\{\tau_x < t + \frac{1}{k}\}}_{\in\mathcal{F}_{t+\frac{1}{k}}} \in \mathcal{F}_{t+} = \mathcal{F}_t.$$

Proposition 3.4.18.

$$P_x[\tau_x > t] = \exp(-c_x \cdot t),\ t \ge 0,$$

where

$$c_x := -\ln P_x[\tau_x > 1] \in [0, \infty],$$

i.e. τ_x is exponentially distributed with constant c_x (cf. [KL99]).

In particular, if the Markov process started at x does not move for sure before time $t = 1$, it for sure never moves. If it moves for sure before time $t = 1$, it for sure moves immediately.

Proof. (By Markov property.) Define for $t \geq 0$

$$f(t) := P_x[\tau_x > t].$$

Claim: $f(t+s) = f(t)f(s)$ for all $t, s \geq 0$.
Indeed,

$$\begin{aligned} E_x[1_{\{\tau_x > t+s\}} | \mathcal{F}_s] &= E_x[1_{\{\tau_x > t\}} \circ \vartheta_s | \mathcal{F}_s] \cdot 1_{\{\tau_x > s\}} \\ &\overset{\text{M.P.}}{=} E_{X_s}[1_{\{\tau_x > t\}}] 1_{\{\tau_x > s\}} \overset{X_s = x \text{ on } \{\tau_x > s\}}{=} P_x[\tau_x > t] \cdot 1_{\{\tau_x > s\}}. \end{aligned}$$

Hence,

$$\begin{aligned} f(t+s) = P_x[\tau_x > t+s] &= E_x[E_x[1_{\{\tau_x > t+s\}} | \mathcal{F}_s]] \\ &= E_x[P_x[\tau_x > t] 1_{\{\tau_x > s\}}] \\ &= P_x[\tau_x > t] P_x[\tau_x > s] = f(t)f(s). \end{aligned}$$

In particular,

$$f(1) = f\left(2^n \frac{1}{2^n}\right) = f\left(\frac{1}{2^n}\right)^{2^n}$$

and then

$$f\left(\frac{k}{2^n}\right) = f\left(\frac{1}{2^n}\right)^k = f(1)^{\frac{k}{2^n}}.$$

Hence, by the right-continuity of f we have $f(t) = f(1)^t = e^{t \ln f(1)}$. □

Corollary 3.4.19. *Let $T_x = \inf\{t > 0 | X_t = x\} = \sigma_{\{x\}}$. Let $\mathbb{M}$ be as in Proposition 3.4.18, but assume in addition, that $\mathbb{M}$ has (SMP) with respect to $(\mathcal{F}_t)_{t \geq 0}$ (right-continuity!). Let $y \in S$ such that*

$$P_y[T_x < \infty] > 0.$$

Then for all $u \geq 0$, we have

$$P_y[X_s = x, \forall s \in [T_x, T_x + u)\,;\, T_x < \infty] = e^{-c_x u} \cdot P_y[T_x < \infty],$$

where c_x is as in Proposition 3.4.18.

Proof. Let $\varphi := 1_{\{X_s = x, \quad \forall s \in [0,u)\}}$. Then

$$\begin{aligned} P_y[X_s = x, \forall s \in [T_x, T_x + u); T_x < \infty] &= E_y[\varphi \circ \vartheta_{T_x}; T_x < \infty] \\ &= E_y[E_y[\varphi \circ \vartheta_{T_x} | \mathcal{F}_{T_x}]; T_x < \infty] \\ &= E_y[E_{X_{T_x}}[\varphi]; T_x < \infty] \\ &= E_x[\varphi] \cdot P_y[T_x < \infty] \\ &= P_x[X_s = x, \forall s \in [0,u)\,] \cdot P_y[T_x < \infty] \\ &\overset{3.4.18}{=} e^{-c_x u} P_y[T_x < \infty]. \end{aligned}$$

□

4. Girsanov Transformation

4.1. Problem Outline ($d = 1$)

We want to construct a process such that it solves (in a "weak" sense) the following equation ("law of motion for the stochastic dynamics $(X_t)_{t\geq 0}$"):

$$\begin{aligned} \mathrm{d}X_t &= b(X_t, t)\ \mathrm{d}t + \ \mathrm{d}W_t, \\ X_0 &= x_0 \in \mathbb{R}^d, \end{aligned}$$

that is,

$$X_t(\omega) = x_0 + \int_0^t b(X_s(\omega), s)\ \mathrm{d}s + W_t(\omega).$$

Here, $X_t = x_0 + \int_0^t b(X_s, s)\ \mathrm{d}s$ denotes the deterministic part and W_t the stochastic perturbation, i.e. W_t is a Wiener process.

One possible strategy of solving this equation is to find a strong solution, that is, for a *given* Wiener process $(W_t)_{t\geq 0}$ on a given probability space $(\Omega, \mathcal{F}, P)$ construct the paths $(X_t)_{t\geq 0}$ of the solution by classical methods (e.g. Picard-Lindelöf or Euler scheme).

Example: The *Ornstein-Uhlenbeck process* has the "law of motion"

$$\begin{aligned} \mathrm{d}X_t &= -\alpha X_t\ \mathrm{d}t + \ \mathrm{d}W_t, \quad \alpha > 0, \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned} \tag{4.1.1}$$

Heuristics:

$$\begin{aligned} (4.1.1) \overset{\text{Ito's product rule}}{\Rightarrow} d(e^{\alpha t}X_t) &= \alpha e^{\alpha t} X_t\ \mathrm{d}t + e^{\alpha t}\ \mathrm{d}X_t \\ &\underset{(4.1.1)}{=} e^{\alpha t}\ \mathrm{d}W_t \\ \Rightarrow e^{\alpha t} X_t &= x_0 + \int_0^t e^{\alpha s}\ \mathrm{d}W_s \\ \Rightarrow X_t &= e^{-\alpha t} x_0 + \int_0^t e^{-\alpha(t-s)}\ \mathrm{d}W_s. \end{aligned}$$

Claim: (4.1.1) has a strong solution

$$X_t := e^{-\alpha t} x_0 + \underbrace{\int_0^t e^{-\alpha(t-s)}\ \mathrm{d}W_s}_{\text{"stochastic convolution"}} = F(x_0, (W_s)_{s\leq t})(t),$$

hence, X_t is adapted to the Wiener filtration.

Proof. We apply Itô's product rule to $X_t = e^{-\alpha t} \cdot \left(x_0 + \int_0^t e^{\alpha s}\ \mathrm{d}W_s\right)$ to get

$$\begin{aligned} X_t &= x_0 + \int_0^t e^{-\alpha s}\ \mathrm{d}\left(x_0 + \int_0^s e^{\alpha u}\ \mathrm{d}W_u\right) + \int_0^t \underbrace{\left(x_0 + \int_0^s e^{\alpha u}\ \mathrm{d}W_u\right)}_{=X_s \cdot e^{\alpha s} \text{ by definition}} (-\alpha) e^{-\alpha s}\ \mathrm{d}s \\ &= x_0 - \alpha \int_0^t X_s\ \mathrm{d}s + \int_0^t 1\ \mathrm{d}W_s = x_0 - \alpha \int_0^t X_s\ \mathrm{d}s + W_t. \end{aligned}$$

□

Instead of strong solutions one can construct "(probabilistically) weak solutions". We want to construct a Brownian motion $(W_t)_{t\geq 0}$ and a process $(X_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, P)$ such that

$$X_t = x_0 + \int_0^t b(X_s, s)\ \mathrm{d}s + W_t$$

that is, construct $(X_t)_{t\geq 0}$ on a suitable probability space $(\Omega, \mathcal{F}, P)$ such that

$$W_t := X_t - x_0 - \int_0^t b(X_s, s)\ \mathrm{d}s$$

is a Brownian motion, e.g. take $(\Omega, \mathcal{F})$, $(X_t)_{t\in[0,t]}$ canonical, i.e.,

$$\Omega := \mathcal{C}([0,1]),$$

$$X_t(\omega) := \omega(t),$$

$$\mathcal{F} := \sigma(X_t | t \geq 0),$$

such that

$$W_t(\omega) := X_t(\omega) - x_0 - \int_0^t b(X_s(\omega), s)\ \mathrm{d}s (= G((X_s(\omega))\ s \leq t)) \tag{4.1.2}$$

is a Brownian motion under P.
But we have to identify P!
One technique to find P is using the Girsanov transformation. This approach has the following advantages:

- One can do this even if the dependence on the past is very complicated (i.e. $b(X_s(\omega), s)$ replaced by b_s).
- One can do this for very irregular b_s.

From now on for simplicity $x_0 = X_0 = 0$.

Method: Let $\Omega = \mathcal{C}([0,1])$, $(X_t)_{t\geq 0}$ be the coordinate process, i.e. $X_t(\omega) = \omega(t)$ and P_0 is the Wiener measure on $C([0,1]) = \Omega$. Then define

$$P := \exp\left(\int_0^1 b_t\ \mathrm{d}X_t - \frac{1}{2}\int_0^1 b_t^2\ \mathrm{d}t\right) P_0.$$

We can check that

$$W_t := X_t - \int_0^t b_s\ \mathrm{d}s$$

is a Brownian motion under P, where b_s denotes the drift. ($(X_t)_{t\geq 0}$ is a Brownian motion under P_0, hence, a martingale under P_0, but *not* a martingale under P!)
Catch: We have to check, that P is a probability measure, i.e. we have to check, that

$$\int e^{\int_0^1 b_t\ \mathrm{d}X_t - \frac{1}{2}\int_0^1 b_t^2\ \mathrm{d}t}\ \mathrm{d}P_0 = 1.$$

This is the hard work in applications.

Relation with the Transformation Rule for Lebesgue Measure

Define $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by

$$T(\omega) := \underbrace{X(\omega)}_{=\omega} - \int_0^{\cdot} b(X_s(\omega), s)\,\mathrm{d}s.$$

Then by Girsanov (under the conditions specified below)

$$\left(\underbrace{e^{\int_0^1 b_s(X_s,s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^1 b(X_s,s)^2\,\mathrm{d}s}}_{\text{“}=|\det DT|\text{''}} P_0 \right) \circ T^{-1} = P_0.$$

4.2. The General Girsanov Transformation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ be a right-continuous filtration (not necessarily “completed”). Then let $\tilde{P}$ be another probability measure such that

$$\tilde{P} \overset{\text{loc.}}{\ll} P \quad (\text{i.e. } \tilde{P}|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}\ \forall t \geq 0).$$

Then the Radon-Nikodym densities

$$Z_t := \left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} := \frac{d\tilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$$

exist and $(Z_t)_{t\geq 0}$ is a martingale (since for all $t > s$ and $F_s \in \mathcal{F}_s$

$$\int_{F_s} Z_t\,\mathrm{d}P \overset{F_s\in\mathcal{F}_s\subset\mathcal{F}_t}{=} \int_{F_s} \mathrm{d}\tilde{P} = \int_{F_s} Z_s\,\mathrm{d}P,$$

i.e.

$$E[Z_t|\mathcal{F}_s] = Z_s).$$

Assumption (from now on in force): Z has continuous sample paths P-a.s.. (This is enough in many applications. Note that by Ito’s representation theorem any martingale with respect to a filtration “generated” by a Brownian motion has this property!)

Lemma 4.2.1. *For every $(\mathcal{F}_t)$-stopping time T we have*

$$\tilde{P} = Z_T P \quad \text{on } \mathcal{F}_T \cap \{T < \infty\} \text{ (trace } \sigma\text{-field).}$$

In particular, $\tilde{P} \ll P$ on $\mathcal{F}_T$ if T is finite.

Proof. Let $A \in \mathcal{F}_T$. Then, by a lemma in [Röc11] for all $t \geq 0$

$$A \cap \{T \leq t\} \in \mathcal{F}_{T\wedge t} \subset \mathcal{F}_t,$$

Hence,

$$\tilde{P}[\underbrace{A \cap \{T \leq t\}}_{\in\mathcal{F}_t}] = \int_{A\cap\{T\leq t\}} Z_t\,\mathrm{d}P \overset{A\cap\{T\leq t\}\in\mathcal{F}_{T\wedge t}}{=} \int_{A\cap\{T\leq t\}} Z_{T\wedge t}\,\mathrm{d}P = \int_{A\cap\{T\leq t\}} Z_T\,\mathrm{d}P.$$

Letting $t \to \infty$ and applying monotone convergence the assertion follows. □

The following lemma describes how martingales "behave" under change from P to $\tilde{P}$. Define

$$\xi(\omega) := \inf\{t \geq 0 | Z_t(\omega) = 0\} \qquad \text{(Then } Z_\xi = 0\text{)}.$$

$\xi(\omega)$ is a stopping time. Recall that then $Z_t(\omega) = 0$ for all $t \in [\xi(\omega), \infty[$, since $(Z_t)_{t\geq 0}$ is a right continuous positive (super)martingale (cf. [Röc11]).

Lemma 4.2.2. *(i) $\xi = \infty$ $\tilde{P}$-a.s. (not necessarily P-a.s.).*

(ii) For all $s \leq t$, φ_t $\mathcal{F}_t$-measurable and positive we have

$$E_{\tilde{P}}[\varphi_t | \mathcal{F}_s] = 1_{\{Z_s \neq 0\}} Z_s^{-1} E_P[\varphi_t Z_t | \mathcal{F}_s] \quad \tilde{P} - a.s..$$

(iii) Let $\tilde{M} := (\tilde{M}_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted continuous process. Then $\tilde{M}$ is a local $\tilde{P}$-martingale (up to ∞), if $\tilde{M} \cdot Z$ is a local P-martingale (up to ξ).

Proof. (i) By Lemma 4.2.1 with $T = \xi \wedge t$, we have

$$\tilde{P}[\underbrace{\xi \leq t}_{\in \mathcal{F}_{t\wedge\xi}}] = E_P[Z_{\xi\wedge t}, \xi \leq t] = E_P[\underbrace{Z_\xi}_{=0}, \xi \leq t] = 0.$$

Letting $t \to \infty$, the assertion follows.

(ii) For all φ_s $\mathcal{F}_s$-measurable and positive,

$$\begin{aligned} E_{\tilde{P}}[\varphi_s \varphi_t] &= E_{\tilde{P}}[\varphi_s 1_{\{Z_s \neq 0\}} \varphi_t] \\ &= E_P[\varphi_s 1_{\{Z_s \neq 0\}} Z_t \varphi_t] \\ &= E_P[\varphi_s 1_{\{Z_s \neq 0\}} E_P[Z_t \varphi_t | \mathcal{F}_s]] \\ &= E_{\tilde{P}}[\varphi_s 1_{\{Z_s \neq 0\}} Z_s^{-1} E_P[\varphi_t Z_t | \mathcal{F}_s]] \end{aligned}$$

(iii) The assertion directly follows from (ii). But it is also a consequence of Lemma 4.2.1: Suppose $T_1 \leq T_2 \leq \dots$ and that $T_n < \xi$ on $\{\xi > 0\}$ be a localizing sequence for the local P-martingale $\tilde{M} \cdot Z$ (hence in particular $\sup_n T_n = \xi$). Then for all bounded stopping times T

$$E_{\tilde{P}}[\tilde{M}_{T\wedge T_n}] \overset{4.2.1}{=} E_P[\tilde{M}_{T\wedge T_n} Z_{T\wedge T_n}] = E_P[\tilde{M}_0 Z_0] = E_{\tilde{P}}[\tilde{M}_0].$$

Hence, $(\tilde{M}_{T_n\wedge t})_{t\geq 0}$ is a $\tilde{P}$-martingale and the assertion follows, since $\xi = \infty$ $\tilde{P}$-a.s. by (i). □

Proposition 4.2.3 (General Girsanov transform)**.** *Let M be a P-a.s. continuous local P-martingale (up to ∞). Then*

$$\tilde{M} := M - \left\langle M, \int_0^\cdot \frac{1}{Z_s}\, dZ_s \right\rangle$$

is a continuous local $\tilde{P}$-martingale (up to ∞).

Proof. By Lemma 4.2.2(iii) we have to show that $\tilde{M}Z$ is a local P-martingale up to ξ.
We have that $\tilde{M}Z$ is P-a.s. continuous. Let $\xi_n := \inf\{t \geq 0 | Z_t \leq \frac{1}{n}\}$ and $\tilde{T}_n$, $n \in \mathbb{N}$, $\tilde{T}_n \nearrow \infty$, be bounded stopping times forming a localizing sequence for the following three local P-martingales M, $MZ - \langle M, Z\rangle$ and

$$\int_0^\cdot \left(\int_0^r \frac{1}{Z_s}\, \mathrm{d}\langle M, Z\rangle_s \right) \mathrm{d}Z_r.$$

Define $T_n := \xi_n \wedge \tilde{T}_n$. Note that $\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s$ is predictable, since it is P-a.s. continuous in r and adapted. We have that $0 \leq T_1 \leq T_2 \leq \ldots$ and that $T_n < \xi$ on $\{\xi > 0\}$, since $\xi_n < \xi$ on $\{0 < \xi < \infty\}$ because Z is P-a.s. continuous. Furthermore, $\xi_n \nearrow \xi$, because Z is P-a.s. continuous, hence $T_n \nearrow \xi$. Then by Itô's product rule for all $t \geq 0$

$$\begin{aligned}
(\tilde{M}Z)_{t\wedge T_n} &= (MZ)_{t\wedge T_n} - Z_{t\wedge T_n}\int_0^{t\wedge T_n} \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s \\
&= (MZ)_{t\wedge T_n} - \int_0^{t\wedge T_n} Z_r\,\mathrm{d}\left(\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z_s\rangle\right) - \int_0^{t\wedge T_n}\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s\,\mathrm{d}Z_r \\
&= (MZ)_{t\wedge T_n} - \int_0^{t\wedge T_n} Z_r\frac{1}{Z_r}\,\mathrm{d}\langle M, Z\rangle_r - \int_0^{t\wedge T_n}\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s\,\mathrm{d}Z_r \\
&= (MZ)_{t\wedge T_n} - \langle M, Z\rangle_{t\wedge T_n} - \int_0^{t\wedge T_n}\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s\,\mathrm{d}Z_r
\end{aligned}$$

Here, $\int_0^{t\wedge T_n}\int_0^r \frac{1}{Z_s}\,\mathrm{d}\langle M, Z\rangle_s\,\mathrm{d}Z_r$ is a P-martingale, and $(MZ)_{t\wedge T_n} - \langle M, Z\rangle_{t\wedge T_n}$ is a P-martingale. Therefore, $\tilde{M}Z$ is a local P-martingale up to ξ. □

Remark 4.2.4 ("Z as an exponential martingale")**.** *We have by Itô formula for $t < \xi$ P-a.s.*

$$\log Z_t = \log Z_0 + \underbrace{\int_0^t \frac{1}{Z_s}\,dZ_s}_{=:Y_t} - \frac{1}{2}\int_0^t \frac{1}{Z_s^2}\,d\langle Z\rangle_s, \tag{$\star$}$$

where Y_t is a local P-martingale up to ξ, since Z is a P-martingale up to ∞, and for its pathwise quadratic variation (cf. 5.3.9) we have

$$\langle Y\rangle_t = \int_0^t \frac{1}{Z_s^2}\,d\langle Z\rangle_s,\ t < \xi.$$

Exponentiating ($\star$) *yields*

$$Z_t = Z_0 \cdot e^{Y_t - \frac{1}{2}\langle Y\rangle_t} \quad \textit{(exponential P-martingale).}$$

Then Z solves the following SDE for given *Y up to ξ:*

$$dZ = Z\ dY \Leftrightarrow Z_t = Z_0 + \int_0^t Z_s\ dY_s, \quad (Z(0) = Z_0)$$

Proof. By the 2-dimensional Itô's formula we have

$$\begin{aligned}
Z_t &= Z_0 + Z_0\int_0^t e^{Y_s - \frac{1}{2}\langle Y\rangle_s}\,\mathrm{d}Y_s - \frac{1}{2}Z_0\int_0^t e^{Y_s - \frac{1}{2}\langle Y\rangle_s}\,\mathrm{d}\langle Y\rangle_s + \frac{1}{2}Z_0\int_0^t e^{Y_s - \frac{1}{2}\langle Y\rangle_s}\,\mathrm{d}\langle Y\rangle_s \\
&= Z_0 + \int_0^t Z_s\,\mathrm{d}Y_s.
\end{aligned}$$

Note that the remaining terms in Itô's formula vanish, since $\langle Y\rangle$ is of bounded variation. □

Corollary 4.2.5. *For $\tilde{M}$ as in Proposition 4.2.3 we have $\tilde{M} = M - \langle M, Y\rangle$ P-a.s. up to ξ, hence $\tilde{P}$-a.s. up to ∞.*

Proof. The assertion follows by the definition of $\tilde{M}$ and Y. □

4.3. Girsanov Transform with Brownian Motion

Let $(X_t)_{t\in[0,1]}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,1]}$. (Take for example $(\Omega, \mathcal{F}, P)$ to be the canonical Wiener space $(C[0,1])_0, \mathcal{F}, P)$ with classical Wiener measure P.)

Heuristics: Let Y be a local continuous P-martingale and $\tilde{P} \overset{\text{loc}}{\ll} P$ with density

$$Z_t := e^{Y_t - \frac{1}{2}\langle Y\rangle_t} \quad \text{on } \mathcal{F}_t.$$

Then we know that $\tilde{M} = X - \langle X, Y\rangle$ is a local continuous $\tilde{P}$-martingale up to ∞, where X is a martingale with respect to P (as M above). Since $\langle \tilde{M}\rangle_t = \langle X\rangle_t = t$, it follows by Levy's characterization theorem of Brownian motion that $\tilde{M}$ is a Brownian motion under $\tilde{P}$. We want to get that

$$\mathrm{d}\langle X, Y\rangle = b_t \, \mathrm{d}t.$$

We succeed by the following

Ansatz:

$$Y_t := \int_0^t b_s \, \mathrm{d}X_s.$$

Remark 4.3.6. *In order to make this work, we need that*

(i) $(b_t)_{t\in[0,1]}$ *is progressively measurable,*

(ii)

$$P\left[\int_0^1 b_s^2 \, ds < \infty\right] = 1$$

(Then, Y is a continuous local P-martingale!) AND!

(iii)

$$\tilde{P} = e^{\int_0^1 b_s \, dX_s - \frac{1}{2}\int_0^1 b_s^2 \, ds} \cdot P$$

is a probability *measure, i.e.*

$$E_P\left[e^{\int_0^1 b_s \, dX_s - \frac{1}{2}\int_0^1 b_s^2 \, ds}\right] = 1 \tag{4.3.3}$$

Theorem 4.3.7 (Girsanov transform for M as a Brownian motion)**:** Assume (i) - (iii) from Remark 4.3.6. Let

$$Z_t := \exp\left[\int_0^t b_s \, \mathrm{d}X_s - \frac{1}{2}\int_0^t b_s^2 \, \mathrm{d}s\right], \quad t \in [0,1],$$

and

$$\tilde{P} := Z_1 \cdot P.$$

Then

$$W_t := X_t - \int_0^t b_s \, \mathrm{d}s, \quad t \in [0,1],$$

is a Brownian motion under $\tilde{P}$.

Proof. **Claim:** $(Z_t)_{t\in[0,1]}$ is a P-martingale up to 1.

Step 1: $(Z_t)_{t\in[0,1]}$ is a (global) P-supermartingale.

Proof. It is clear, that (Z_t) is a local continuous P-martingale. Let $0 \leq T_1 \leq \ldots \leq T_n \ldots < 1$ be a localizing sequence of stopping times for $(Z_t)_{t\in[0,1]}$. Then for $0 \leq s < t \leq 1$

$$\begin{aligned} E_P[Z_t|\mathcal{F}_s] &= E_P[\lim_{n\to\infty} Z_{t\wedge T_n}|\mathcal{F}_s] \\ &\overset{Z_t\geq 0}{\leq} \liminf_{n\to\infty} E_P[Z_{t\wedge T_n}|\mathcal{F}_s] \\ &= \liminf_{n\to\infty} Z_{s\wedge T_n} = Z_s. \end{aligned}$$

□

Step 2: $(Z_t)_{t\in[0,1]}$ is a P-martingale.

Proof. By (iii) for all $s \in [0,1]$

$$1 = E[Z_1] \leq E[Z_s] \leq E[\underbrace{Z_0}_{=1}] = 1.$$

In addition,

$$0 \leq Z_s - E_P[Z_1|\mathcal{F}_s]$$

and

$$\int (Z_s - E_P[Z_1|\mathcal{F}_s])\, \mathrm{d}P = 0.$$

So, $Z_s = E_P[Z_1|\mathcal{F}_s]$ P-a.s. □

We have

$$\left\langle X, \int_0^{\cdot} b_s\, \mathrm{d}X_s \right\rangle_t = \int_0^t b_s\, \mathrm{d}s.$$

Hence, by Corollary 4.2.5 it follows that W is a continuous local $\tilde{P}$-martingale. But $\langle W\rangle_t = \langle X\rangle_t = t$ P-a.s., hence $\tilde{P}$-a.s.. So, by Lévy (Proposition 1.5.34) the assertion follows. □

Remark 4.3.8. *(i)* ***Special case:***

$$b_t(\omega) := b(X_t(\omega), t),$$

i.e. depending only on the "present" time, where $b : \mathbb{R} \times [0,1] \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0,1])$ measurable. Then 4.3.6(i) is fulfilled.
4.3.6(ii) is fulfilled, if e.g.

$$\begin{aligned} E\left[\int_0^1 b_t^2\, dt\right] &= \int_0^1 E[b^2(X_t,t)]\, dt \\ &\overset{E[f(X_t)]=p_t f(0)}{=} \int_0^1 p_t(b^2(\cdot,t))(0)\, dt \\ &\overset{p_t(0,dx)=N(0,t)}{=} \int_0^1 \int_{\mathbb{R}} b^2(x,t) N(0,t)(dx)\, dt \\ &= \int_0^1 \int_{\mathbb{R}} b^2(x,t) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx\, dt < \infty. \end{aligned}$$

In particular, it is not neccessary that b is bounded.
To satisfy 4.3.6(iii) we have to work harder (see Theorem 4.4.15 and Remark 4.4.16(i), as well as for a very special case Example 4.3.9 below)!

(ii) *In Theorem 4.3.7* $(X_t)_{t\geq 0}$ *solves the following SDE* under $\tilde{P}$

$$dX_t = \, dW_t + b_t \, dt.$$

Here, W_t *is a Brownian motion under* $\tilde{P}$, *not under* P.

Example 4.3.9. *Consider* $b_t = \alpha \in \mathbb{R}$ *fixed for all* t. *Then, clearly, 4.3.6(i),(ii) hold, but also:* ***Claim:*** *(iii) holds.*

Proof. Since X_1 is normal distributed, we have

$$\begin{aligned} E\left[e^{\alpha X_1 - \frac{1}{2}\alpha^2}\right] &= e^{-\frac{1}{2}\alpha^2} \int e^{\alpha x} N(0,1)(dx) \\ &= e^{-\frac{1}{2}\alpha^2} \int_{\mathbb{R}} e^{\alpha x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, \mathrm{d}x \\ &= e^{-\frac{1}{2}\alpha^2} \cdot e^{\frac{1}{2}\alpha^2} = 1. \end{aligned}$$

Here, we have used that for the Laplace-transform $\mathcal{L}$ we have

$$\mathcal{L}(N(0,\sigma^2))(\xi) = \int e^{\xi x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} \, \mathrm{d}x = e^{\frac{1}{2}\xi^2\sigma^2}.$$

□

Hence, by Theorem 4.3.7 under $\tilde{P} := Z_1 \cdot P$ *the process* $W_t = X_t - \alpha t$, $t \geq 0$, *is a Brownian motion.*

Theorem 4.3.10 (Cameron-Martin)**.** *Consider the canonical model for Brownian motion, i.e.* $\Omega = C([0,1])$, P_0 *classical Wiener measure and* $X_t(\omega) := \omega(t)$, $t \geq 0$. *Let*

$$h \in E := C([0,1])_0 := \{h \in C([0,1]) | h(0) = 0\}$$

and

$$X_h : C([0,1])_0 \to C([0,1])_0$$

defined by

$$X_h(\omega) := X(\omega) + h.$$

Let $P_h := P_0 \circ X_h^{-1}$ *be the law of* X_h *on* $C([0,1])$. *Then the following assertions are equivalent:*

(i) $P_h \approx P_0$.

(ii)

$$h \in H := \left\{ h \in C([0,1])_0 \middle| h \textit{ is absolutely continuous and } \dot{h} \in L^2([0,1], ds) \right\}.$$

In this case

$$\frac{dP_h}{dP_0} = \exp\left(\int_0^1 \dot{h}(s) \, \mathrm{d}X(s) - \frac{1}{2} \int_0^1 (\dot{h}(s))^2 \, \mathrm{d}s \right).$$

H is called *Cameron-Martin space.* H is a Hilbert space with inner product

$$\langle h, \tilde{h} \rangle_H := \int_0^1 \dot{h}(s) \dot{\tilde{h}}(s) \, \mathrm{d}s, \quad h, \tilde{h} \in H.$$

Note that $\|h\|_H^2 = \langle h, h \rangle_H = 0$ implies $\dot{h}(s) = 0$ ds-a.s.. Hence,

$$h(t) = h(0) + \int_0^t \dot{h}(s) \, \mathrm{d}s = h(0) = 0 \quad \forall t \geq 0.$$

So, $\|\cdot\|_H$ is not only a semi norm, but a norm. We have that H is dense in $E := C([0,1])_0$ with respect to $\|\cdot\|_H$. Then identifying H with its dual H' by the Riesz map R we get

$$E' \hookrightarrow H' \overset{R}{\to} H \subset E$$

continuously (see below). Before we prove Theorem 4.3.10, we want to characterize E'.

Lemma 4.3.11. *For*

$$M_0 := \left\{ \mu \middle| \mu \text{ is a signed measure of finite total variation on } [0,1], \text{ such that } \mu(1) := \int 1 \, d\mu = 0 \right\},$$

we have

$$E' = M_0.$$

Remark 4.3.12. *A signed measure μ can be written as $\mu = \mu_1 - \mu_2$ with positive measures μ_1 and μ_2 and the total variation of μ is defined as*

$$\sup_{A \in \mathcal{B}([0,1])} \mu(A) = \|\mu\|_{Var} < \infty.$$

Furthermore, for a signed measure μ, there exist positive measures μ^+, μ^- and $S \in \mathcal{B}([0,1])$ such that $\mu = \mu^+ - \mu^-$ ("Hahn-Jordan decomposition") and

$$\mu^+(S^c) = 0 \text{ and } \mu^-(S) = 0.$$

Then

$$\|\mu\|_{Var} = \mu^+([0,1]) + \mu^-([0,1]).$$

Proof of 4.3.11. "$\supset$": Defining $f \mapsto \mu(f) := \int f \, \mathrm{d}\mu$ for $\mu \in M_0$ we see that this way any $\mu \in M_0$ defines an element in E'. Furthermore, if $\mu, \nu \in M_0$ such that $\mu(f) = \nu(f)$ $\forall f \in E$, then for all $f \in C([0,1])$

$$\begin{aligned} \int f \, \mathrm{d}\mu &= \int f \, \mathrm{d}\mu - f(0) \int 1 \, \mathrm{d}\mu = \int \underbrace{(f - f(0))}_{\in E} \, \mathrm{d}\mu \\ &= \int (f - f(0)) \, \mathrm{d}\nu = \int f \, \mathrm{d}\nu - \underbrace{\int f(0) \, \mathrm{d}\nu}_{=0} = \int f \, \mathrm{d}\nu. \end{aligned}$$

Hence, $\mu = \nu$ and, thus $M_0 \hookrightarrow E'$. (See also [Cho69a] and [Cho69b].)
"$\subset$": Let $l \in E'$. Then by the Hahn-Banach theorem there exists an $\bar{l} \in (C[0,1])'$ such that $\bar{l} = l$ on $E (\subset C([0,1]))$. Hence, by Riesz-Markov (cf. [Röc04]) there exists a signed measure ν on $[0,1]$ of finite total variation such that

$$\bar{l}(f) = \int f \, \mathrm{d}\nu \quad \forall f \in C([0,1]).$$

Set $\mu := \nu - \nu(1)\delta_0$, where δ_0 is the Dirac measure at $0 \in [0,1]$. Then $\mu(1) = 0$, so $\mu \in M_0$ and for all $f \in E$

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu - \nu(1) \underbrace{f(0)}_{=0} = \int f \, \mathrm{d}\nu = \bar{l}(f) = l(f).$$

□

Proof of Theorem 4.3.10. (ii) ⇒ (i): (An alternative proof using Fourier transform can be found in [MR92, Chapter II].) We first check the conditions of Remark 4.3.6 (i)-(iii):
(i) and (ii) are obviously satisfied for

$$b_s(\omega) := \dot{h}(s) \quad \forall \omega \in \Omega,\ s \in [0,1].$$

(iii): Define

$$Y_t := \int_0^1 \dot{h}(s)\, \mathrm{d}X_s \quad \text{(Itô-Integral)}.$$

Then Y_1 is centered (i.e. $E[Y_1] = 0$) and normally distributed as an L^2-limit of centered, normally distributed random variables (cf. [Röc11]). Furthermore,

$$E[Y_1^2](= E[\langle Y \rangle_1]) = E\left(\int_0^1 (\dot{h}(s))^2 \, \mathrm{d}s \right) = \|h\|_H^2 ,$$

i.e.

$$Y_1 \sim N(0, \|h\|_H^2).$$

Hence,

$$E\left[e^{Y_1 - \frac{1}{2}\|h\|_H^2} \right] = e^{-\frac{1}{2}\|h\|_H^2} \underbrace{E[e^{Y_1}]}_{= e^{\frac{1}{2}\|h\|_H^2}} = 1$$

So, 4.3.6(iii) holds. Therefore, applying Girsanov (Theorem 4.3.7) we obtain that under

$$\tilde{P} := Z_1 \cdot P := \exp\left[\int_0^1 \dot{h}(t)\, \mathrm{d}X_t - \frac{1}{2}\int_0^1 \dot{h}(t)^2 \, \mathrm{d}t \right] \cdot P$$

$W := X - h$ is a Brownian motion, i.e. $\tilde{P} \circ W^{-1} = P_0$, i.e., since $W : \Omega \to \Omega$ is bijective, $\tilde{P} = P_0 \circ (W^{-1})^{-1} = P_0 \circ X_h^{-1} = P_h$. But, since $E[Y_1^2] < \infty$, we have $Z_1 > 0$ P-a.s., therefore, $P_h \approx P$.

(i) ⇒ (ii): We know that $P_0(E) = 1$ with $E := C([0,1])_0$. Clearly, we have that $H \subset E$ dense with respect to the norm $\|\cdot\|_\infty$ (cf. functional analysis).Let $h \in H$. Then (by definition of absolute continuity)

$$h(t) = \underbrace{h(0)}_{=0} + \int_0^t \dot{h}(s)\, \mathrm{d}s \quad \forall t \in [0,1],$$

thus,

$$|h(t)| \leq \int_0^t |\dot{h}(s)|\, \mathrm{d}s \leq \int_0^1 |\dot{h}(s)|\, \mathrm{d}s \leq \left(\int_0^1 |\dot{h}(s)|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} = \|h\|_H , \quad \forall t \in [0,1],$$

hence,

$$\|h\|_\infty \leq \|h\|_H .$$

Hence $H \subset E$ continuously w.r.t. the norms $\|\cdot\|_H$ and $\|\cdot\|_\infty$. Let $R : H' \to H$ be the Riesz isomorphism. Then

$$E' \subset H' \xrightarrow{R} H \subset E \quad \text{(continuously)}.$$

(Cf. Lemma 4.5.20 below: $R(\mu) = -\int_0^\cdot \mu([0,s[)\, \mathrm{d}s, \quad \mu \in E'$.)

Claim 1: Let $\mu \in E'(= M_0)$. The map $C([0,1])_0 \ni \omega \mapsto \mu(\omega) \in \mathbb{R}$ is Gaussian distributed under P_0, more precisely

$$P_0 \circ \mu^{-1} = N(0, \underbrace{E[\mu^2]}_{\int \mu(\omega)^2 \, P_0(\mathrm{d}\omega)}).$$

Proof. Recall that $\mu = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{N_n} \alpha_i^{(n)} \delta_{t_i^{(n)}}}_{=:\mu_n}$, $\alpha_i^{(n)} \in \mathbb{R}$, $t_i^{(n)} \in [0,1]$ weakly (cf. [Bau78]). But $\mu_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, are jointly (centred) Gaussian under P_0, because

$$\sum_{l=1}^{N} (\gamma_l \mu_{n_l})(\omega) = \sum_{l=1}^{N} \gamma_l \sum_{i=1}^{N_{n_l}} \alpha_i^{(n_l)} \delta_{t_i^{(n_l)}}(\omega) = \sum_{l=1}^{N} \gamma_l \sum_{i=1}^{N_{n_l}} \alpha_i^{(n_l)} \underbrace{\omega(t_i^{(n_l)})}_{=X_{t_i^{(n_l)}}(\omega)} \qquad \forall f \in E$$

i.e.

$$\sum_{l=1}^{N} \gamma_l \mu_{n_l} = \underbrace{\sum_{l=1}^{N} \gamma_l \sum_{i=1}^{N_{n_l}} \alpha_i^{(n_l)} X_{t_i}^{(n_l)}}_{(\mathbb{R}-\text{valued) centred Gaussian}} \sim N(0, E(\sum_{i=1}^{N} \gamma_l \mu_{n_l})^2)$$

Hence, since $\mu_n \overset{n\to\infty}{\longrightarrow} \mu$ weakly, i.e. pointwise on E, we have that $\hat{\mu}_n \overset{n\to\infty}{\longrightarrow} \hat{\mu}$ and μ is centered Gaussian with variance $E[\mu^2]$. $\square$

Claim 2: Let $\mu \in M_0$ of the form $\mu = \varrho \cdot \mathrm{d}t$ and ϱ bounded and

$$\mu(1) = \int_0^1 \varrho \,\mathrm{d}t = 0. \qquad (\star)$$

Then

$$E[\mu^2] = \int_0^1 \left(\int_0^t \varrho(s) \,\mathrm{d}s \right)^2 \mathrm{d}t = \|R(\mu)\|_H^2 . \qquad (4.3.4)$$

Proof.

$$\begin{aligned}
E[\mu^2] &\overset{X_t(\omega):=\omega(t)}{=} E\left(\int_0^1 X_t \varrho(t) \,\mathrm{d}t \int_0^1 X_{t'} \varrho(t') \,\mathrm{d}t' \right) \\
&\overset{\text{Fubini}}{=} \int_0^1 \int_0^1 \varrho(t)\varrho(t') E[X_t X_{t'}] \,\mathrm{d}t' \,\mathrm{d}t \\
&= \int_0^1 \int_0^1 \varrho(t)\varrho(t')(t \wedge t') \,\mathrm{d}t' \,\mathrm{d}t = 2 \int_0^1 \int_0^1 t \mathbb{1}_{\{t<t'\}} \varrho(t)\varrho(t') \,\mathrm{d}t \,\mathrm{d}t' \\
&= 2 \int_0^1 t\varrho(t) \int_t^1 \varrho(t') \,\mathrm{d}t' \,\mathrm{d}t \\
&= -\int_0^1 t \frac{d}{\mathrm{d}t} \left(\left(\int_t^1 \varrho(t') \,\mathrm{d}t' \right)^2 \right) \mathrm{d}t \qquad (4.3.5) \\
&\overset{\text{I.b.p.}}{=} \int_0^1 \left(\int_t^1 \varrho(t') \,\mathrm{d}t' \right)^2 \mathrm{d}t \\
&= \int_0^1 \left(\int_0^t -\varrho(s) \,\mathrm{d}s \right)^2 \mathrm{d}t \\
&= \int_0^1 \left(\frac{d}{dt} \underbrace{\int_0^t \left(\int_0^{t'} -\varrho(s) \,\mathrm{d}s \right) \mathrm{d}t'}_{=:\tilde{R}(\mu)(t)} \right)^2 \mathrm{d}t = \int_0^1 \left(\frac{d}{dt} \tilde{R}(\mu)(t) \right)^2 \mathrm{d}t = \left\| \tilde{R}(\mu) \right\|_H^2 .
\end{aligned}$$

For Claim 2 it remains to show that $\tilde{R}(\mu) = R(\mu)$, i.e.

$$R(\mu) = -\int_0^{\cdot} \left(\int_0^{t'} \varrho(s) \,\mathrm{d}s \right) \mathrm{d}t'. \qquad (4.3.6)$$

To this end let $\tilde{h} \in H$. Then

$$\mu(\tilde{h}) = \int_0^1 \tilde{h}(t)\varrho(t)\,\mathrm{d}t \overset{\text{I.b.p.}}{=} -\int_0^1 \frac{d}{dt}\tilde{h}(t)\int_0^t \varrho(t)\,\mathrm{d}s\,\mathrm{d}t = \left\langle \tilde{h}, -\int_0^{\cdot} \left(\int_0^{t'} \varrho(s)\,\mathrm{d}s\right)\,\mathrm{d}t' \right\rangle_H \tag{4.3.7}$$

and Claim 2 is proved. □

Since $C_0^1(]0,1[)$ is dense in $L^2([0,1],dt)$, for all $\tilde{h} \in H$ there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $C_0^1(]0,1[)$ such that $v_n \to \dot{\tilde{h}}$ as $n \to \infty$ in $L^2([0,1],dt)$. Hence

$$u_n := \int_0^{\cdot} v_n\,\mathrm{d}t \overset{n\to\infty}{\longrightarrow} \tilde{h} \quad \text{in } H.$$

Since by (4.3.6)

$$u_n = R(\mu_n) \quad \text{for} \quad \mu_n := -\dot{v}_n\,\mathrm{d}t \in E'$$

it follows that for

$$\tilde{M}_0 := \left\{ \varrho\,\mathrm{d}t \,\middle|\, \varrho \in C([0,1]), \int_0^1 \varrho\,\mathrm{d}t = 0 \right\}$$

that $R(\tilde{M}_0)$ is dense in H with respect to $\|\cdot\|_H$. $R(\tilde{M}_0)$ is also a linear subspace of H.
Let $h \in E(= C([0,1])_0)$ such that $P_h \ll P$.
Claim: Let $\mu_n \in \tilde{M}_0$ such that $R(\mu_n) \overset{n\to\infty}{\longrightarrow} 0$ in H. Then $\mu_n(h) \overset{n\to\infty}{\longrightarrow} 0$.

Suppose the claim is true. Then $h \in H$.

Proof. By the claim the map $R(\tilde{M}_0) \ni R(\mu) \mapsto \mu(h) \in \mathbb{R}$ is a linear continuous functional on $(R(\tilde{M}_0), \|\cdot\|_H)$. Hence, by Riesz (note $R(\tilde{M}_0)$ is dense in H, hence Riesz is applicable) there exists an unique $h_0 \in H$ such that

$$\mu(h) = \langle R(\mu), h_0\rangle_H \quad \forall \mu \in \tilde{M}_0.$$

But by 4.3.11 we also have $\langle R(\mu), h_0\rangle_H = \mu(h_0)$. Therefore,

$$\mu(h) = \mu(h_0), \forall\, \mu \in \tilde{M}_0 \tag{4.3.8}$$

Hence, for all $\varrho \in C([0,1])$ and for

$$\tilde{h} := h - \int_0^1 h(t)\,\mathrm{d}t, \quad \tilde{h}_0 := h_0 - \int_0^1 h_0(t)\,\mathrm{d}t$$

we have

$$\int \tilde{h}\varrho\,\mathrm{d}t = \int \tilde{h}\underbrace{\left(\varrho - \int_0^1 \varrho\,\mathrm{d}s\right)\,\mathrm{d}t}_{\in \tilde{M}_0} = \int h\left(\varrho - \int_0^1 \varrho\,\mathrm{d}s\right)\,\mathrm{d}t$$

$$\overset{(4.3.8)}{=} \int h_0\left(\varrho - \int_0^1 \varrho\,\mathrm{d}s\right)\,\mathrm{d}t = \int \tilde{h}_0\left(\varrho - \int_0^1 \varrho\,\mathrm{d}s\right)\,\mathrm{d}t = \int h_0\varrho\,\tilde{\mathrm{d}}t.$$

Hence, $\tilde{h} = \tilde{h}_0$, therefore, $h = h_0$, because $h(0) = h_0(0) = 0$. Therefore, the assertion is true as long as the claim holds. □

Proof of Claim. Since $R(\mu_n) \to 0$ in H, it follows by (4.3.4) that

$$E_{P_0}[\mu_n^2] = \|R(\mu_n)\|_H^2 \overset{n\to\infty}{\longrightarrow} 0,$$

i.e. $\mu_n \to 0$ in $L^2(P)$, hence, also in P_0-measure, therefore, because of $P_h \ll P_0$, also in P_h-measure. Since $\{\mu_n, n \in \mathbb{N}\}$ is a Gaussian family under P_h, it follows by [Röc11] that

$$\mu_n \overset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^p(P_h) \quad \forall p \geq 1. \tag{4.3.9}$$

Since we have

$$E_{P_h}[\mu_n] = \int \underbrace{\mu_n(\omega + h)}_{\mu_n(\omega)+\mu_n(h)} P(d\omega) = \underbrace{\int \mu_n(\omega) P(d\omega)}_{=0} + \mu_n(h), \tag{4.3.10}$$

applying (4.3.9) for $p = 1$, it follows that

$$\limsup_{n\to\infty} |\mu_n(h)| \overset{(4.3.10)}{\leq} \limsup_{n\to\infty} E_{P_h}[|\mu_n|] = 0.$$

So, the claim is proved. □

4.4. Novikov condition

Let $(\Omega, \mathcal{F}, P)$ be a probability space together with a right-continuous complete filtration $(\mathcal{F}_t)$ and let $(Y_t)_{t\geq 0}$ be a P-a.s. continuous local martingale such that $Y_0 = 0$. Then by 1.4.33

$$Z_t := \exp(Y_t - \frac{1}{2}\langle Y\rangle_t), \quad t \geq 0,$$

is a (P-a.s.) continuous local martingale. We want to derive a condition for (4.3.3).

Lemma 4.4.13. *Let $t \geq 0$ and $\langle Y\rangle_t$ be bounded. Then $Z_t \in \mathcal{L}^p$ for all $p > 1$ and*

$$E(Z_t^p) \leq \exp(\frac{1}{2}p(p-1)\,\|\langle Y\rangle_t\|_\infty).$$

Proof.

$$\begin{aligned} E(Z_t^p) &= E\left[\exp\left(pY_t - \frac{1}{2}p^2\langle Y\rangle_t + \left(\frac{1}{2}p^2 - \frac{1}{2}p\right)\langle Y\rangle_t\right)\right] \\ &\leq \exp\left(\frac{1}{2}p(p-1)\,\|\langle Y\rangle_t\|_\infty\right) E\left[\exp\left(pY_t - \frac{1}{2}p^2\langle Y\rangle_t\right)\right]. \end{aligned}$$

Set $\tilde{Y}_t := pY_t$. Then $\langle \tilde{Y}\rangle_t = p^2\langle Y\rangle_t$ and, therefore,

$$E(\exp(pY_t - \frac{1}{2}p^2\langle Y\rangle_t)) = E(\exp(\tilde{Y}_t - \frac{1}{2}\langle\tilde{Y}\rangle_t)) \leq E(\exp(\tilde{Y}_0 - \frac{1}{2}\langle\tilde{Y}\rangle_0)) = 1,$$

since Z is a supermartingale (by Fatou). □

Now we can prove a condition implying (4.3.3).

Theorem 4.4.14 (Novikov)**.** *If $E(\exp(\frac{1}{2}\langle Y\rangle_t)) < \infty$ for all t, then $E(Z_t) = 1$ for all t.*

Remark 4.4.15. *(i) In the examples above (see 4.3.8 and 4.3.9) we had $Y_t = \int_0^t b_s\, dX_s$ and*

$$E\left(\exp\left(\frac{1}{2}\int_0^t b_s^2\, ds\right)\right) < \infty$$

was always satisfied.

(ii) *For the proof of Theorem 4.4.14 we need that one can construct a Wiener process* W*, such that*

$$Y_t = W_{\langle Y \rangle_t}$$

and (for every fixed t*)* $\langle Y \rangle_t$ *is a stopping time with respect to a suitable filtration for which* W *is adapted. This means that* Y_t *has the form* W_T*, with*

$$\langle Y \rangle_t = T.$$

In other words, any P*-a.s. continuous local martingale is the time change of a Brownian motion. The details are presented in Appendix A.*

Now Theorem 4.4.14 follows from

Theorem 4.4.16. *Let* $(W_t)_{t \geq 0}$ *be a Wiener process on* $(\Omega, \mathcal{F}, P)$ *and* T *be a stopping time. If*

$$E(\exp(\frac{1}{2}T)) < \infty,$$

then the following "Wald identity" holds:

$$E(\exp(W_T - \frac{1}{2}T)) = 1.$$

Remark 4.4.17. *Set* $M_t := \exp(W_t - \frac{1}{2}t)$*. Then* (M_t) *is a continuous martingale, since it is a continuous positive supermartingale and* $E[M_t] = 1$ *(cf. Example 4.3.9). By the optional sampling theorem for unbounded stopping times (cf. [Röc11]), we have*

$$E(M_T) \leq E(M_0) = 1.$$

Thus, it is clear that $E(\exp(W_T - \frac{1}{2}T)) \leq 1$ *in 4.4.16. For the proof of "*$\geq$*" in 4.4.16 we will need two Lemmas.*

Lemma 4.4.18. *Let* $\tilde{P}$ *be a probability measure on* $(\Omega, \mathcal{F}, P)$ *with*

$$\tilde{P}|_{\mathcal{F}_t} = \exp(W_t - \frac{1}{2}t) \cdot P|_{\mathcal{F}_t}, \quad \forall t \geq 0,$$

and T *be a stopping time with* $P(T < \infty) = 1$*. Then*

$$E(\exp(W_T - \frac{1}{2}T)) = \tilde{P}(T < \infty). \tag{4.4.11}$$

In particular, the above Wald identity (cf. 4.4.16) holds if and only if

$$\tilde{P}(T < \infty) = 1.$$

Proof. Since $M_t := \exp(W_t - \frac{1}{2}t)$ is a martingale and $\{T \leq t\} \in \mathcal{F}_{t \wedge T}$ we have

$$\tilde{P}[T \leq t] = E[1_{\{T \leq t\}} M_t] = E[1_{\{T \leq t\}} M_{t \wedge T}] = E[1_{\{T \leq t\}} M_T].$$

Letting $t \to \infty$ we get (since $P[T < \infty] = 1$)

$$\tilde{P}[T < \infty] = E[M_T].$$

□

Lemma 4.4.19. *Let $c > 0$ and define the "passage time of $(W_t - t)$" by*

$$T_c := \inf\{t > 0 | W_t = t - c\}. \quad \big(\Rightarrow W_{T_c} = T_c - c\big)$$

Then T_c is a stopping time such that

$$\tilde{P}(T_c < \infty) = 1$$

and, thus, the above Wald identity holds for T_c. Furthermore,

$$E\left(\exp\left(\frac{1}{2}T_c\right)\right) = e^c.$$

Proof. By the law of iterated logarithm we have

$$P(T_c < \infty) = 1.$$

$\tilde{W}_t = W_t - t$ is a Brownian motion with respect to $\tilde{P}$ (cf. Example 4.3.9 with $\alpha = 1$), which means that T_c is a passage time of $(\tilde{W}_t)$ with respect to $\tilde{P}$. Therefore, again by the law of iterated logarithm

$$\tilde{P}(T_c < \infty) = 1.$$

Thus, by 4.4.18

$$1 = E\left(\exp\left(W_{T_c} - \frac{1}{2}T_c\right)\right) = e^{-c}E\left(\exp\left(\frac{1}{2}T_c\right)\right).$$

□

Proof of Theorem 4.4.16. It remains to show that "$\geq$" holds:
$M_t := \exp(W_t - \frac{1}{2}t)$, $t \geq 0$, is a positive continuous supermartingale. Hence,

$$1 \geq E(M_{T_c \wedge T}) \geq E(M_{T_c}) \overset{4.4.19}{=} 1.$$

But then for all $c > 0$

$$\begin{aligned}
1 = E(M_{T_c \wedge T}) &\overset{W_{T_c} = T_c - c}{=} E\left(\exp\left(\frac{1}{2}T_c\right)\exp(-c), T_c \leq T\right) + E\left(\exp\left(W_T - \frac{1}{2}T\right), T_c > T\right) \\
&\leq e^{-c} \underbrace{E(e^{\frac{1}{2}T})}_{<\infty (\text{Novikov})} + E\left(\exp\left(W_T - \frac{1}{2}T\right)\right) \\
&\overset{c\to\infty}{\longrightarrow} E\left(\exp\left(W_T - \frac{1}{2}T\right)\right)
\end{aligned}$$

□

4.5. Integration by Parts on Wiener Space: A First Introduction to the Malliavin Calculus Following J.M. Bismut

Fix the following framework:
Let $P := P_0$ be the Wiener measure on $\Omega = C([0,1])_0 =: E$, (X_t) the coordinate process,

$$H := \left\{ h \in C([0,1])_0 \middle| h \text{ is absolutely continuous and } \int_0^1 \dot{h}(s)^2 \, \mathrm{d}s < \infty \right\}$$

the Cameron-Martin space. H is a Hilbert space with inner product

$$\langle h, g \rangle_H = \int_0^1 \dot{h}(s)\dot{g}(s) \, \mathrm{d}s = \langle \dot{h}, \dot{g} \rangle_{L^2([0,1],dt)},$$

and $F : \Omega \to \mathbb{R}$ be the Wiener functional. We already know (by Lemma 4.3.11) that

$$E' = \{\mu | \mu \text{ is a signed measure on } [0,1] \text{ of bounded variation such that } \mu(1) = 0\}$$

and

$$E' \subset H' \overset{R}{\to} H \subset E$$

continuous and densely, where R denotes the Riesz map.

Lemma 4.5.20. *Let $\mu \in E'$. Then*

$$R(\mu) = \int_0^{\cdot} \mu(]s,1]) \, ds \left(= - \int_0^{\cdot} \mu([0,s[) \, ds \right).$$

(Cf. (4.3.6) *as a special case.)*

Proof. Let $h \in H$. Then

$$\begin{aligned} \langle R(\mu),\ h \rangle_H = \mu(h) = \int_0^1 h(t)\mu(dt) &= \int_0^1 \int_0^t \dot{h}(s) \, \mathrm{d}s \, \mu(dt) = \int_0^1 \int_0^1 1_{[0,t[}(s)\dot{h}(s) \ \mathrm{d}s\mu(dt) \\ &\overset{\text{Fubini}}{=} \int_0^1 \int_0^1 1_{]s,1]}(t)\mu(dt)\dot{h}(s) \, \mathrm{d}s = \int_0^1 \mu(]s,1])\dot{h}(s) \, \mathrm{d}s \\ &= \int_0^1 \frac{d}{ds} \left(\int_0^s \mu(]r,1]) \, \mathrm{d}r \right)^{\cdot} \dot{h}(s) \, \mathrm{d}s = \left\langle \int_0^{\cdot} \mu(]s,1]) \, \mathrm{d}s, h \right\rangle_H \end{aligned}$$

Hence $R(\mu) = \int_0^{\cdot} \mu(]s,1]) \, \mathrm{d}s$. □

We recall

Definition 4.5.21. *$F : C([0,1])_0 \to \mathbb{R}$ is called* Frêchet-differentiable *in $\omega \in C([0,1])_0$, if there exists $F'(\omega) \in E'$ such that*

$$F(\omega + \eta) = F(\omega) + F'(\omega)(\eta) + o(\|\eta\|), \quad \forall \eta \in C([0,1])_0.$$

In this case

$$\nabla F(\omega) := R(F'(\omega)) \in H$$

is called gradient *of F at ω. Note that $\nabla F(\omega)$ is in the "tangent space" of H in ω.*

Remark 4.5.22. *Because of Lemma 4.5.20 we have for the measure*

$$F'(\omega)(dt) =: F'(\omega, dt)$$

that

$$\nabla F(\omega)(\cdot) = R(F'(\omega, dt)) = \int_0^{\cdot} F'(\omega,]s,1])\ ds. \tag{4.5.12}$$

Then by definition of the derivative

$$\lim_{\lambda\to 0} \frac{F(\omega+\lambda\eta)-F(\omega)}{\lambda} = \langle \nabla F(\omega), \eta\rangle_H \overset{(4.5.12)}{=} \int_0^1 F'(\omega,]s,1])\, \dot{\eta}(s)\ ds \quad \forall \eta \in H(\subset E := C([0,1])_0).$$

Definition 4.5.23. *$F \in \mathcal{L}^2(P)$ is called* H-differentiable, *if for all $(\mathcal{F}_t)$-adapted real processes $(u_s)_{s\in[0,1]}$, product-measurable, bounded and*

$$U_t(\omega) := \int_0^t u_s(\omega)\ ds, \quad t \in [0,1] \quad (H-\textit{vector field on } E)$$

(i.e. $U(\omega) \in H$) there exists a $\mathcal{F}/\mathcal{B}(H)$-measurable map $\nabla F : E \to H$ such that

$$E(\|\nabla F\|_H^2) < \infty \quad \big(\nabla F \in L^2(\Omega \to H, P)\big)$$

and

$$\frac{F(\omega+\lambda U(\omega))-F(\omega)}{\lambda} \xrightarrow{\lambda\to 0} \langle \nabla F(\omega), U(\omega)\rangle_H \quad \textit{in } \mathcal{L}^2(P)$$

or equivalently

$$\nabla_U F := \lim_{\lambda\to 0} \frac{F(X+\lambda U)-F(X)}{\lambda} = \langle \nabla F, U\rangle_H \quad \textit{in } \mathcal{L}^2(P).$$

Here, ∇F is called the Malliavin gradient *(cf. [Wat84]).*
Define the Malliavin derivative

$$D_t F := (\nabla F(\omega))^{\bullet}(t).$$

In particular, $(D_tF)_{0\le t\le 1}$ is a process. (This process is product-measurable in (ω, t)!)

Geometric interpretation of ∇F : H-vector field on E.
Geometric interpretation of DF : $L^2([0,1], dt)$-vector field on E.

Remark 4.5.24. *Let u and U as in Definition 4.5.23.*

(i) We have

$$\langle \nabla F(\omega), U(\omega)\rangle_H = \langle D.F(\omega), u\rangle_{L^2([0,1],dt)}$$

(ii) Let

$$Z_t^\lambda := \exp\left(\lambda \int_0^t u_s\ dX_s - \frac{1}{2}\lambda^2 \int_0^t u_s^2\ ds\right).$$

Then Novikov's condition is fulfilled, since u is bounded. So, Girsanov's theorem implies that $X^\lambda := X - \lambda U$ is a Wiener process under $P^\lambda := Z_1^\lambda P$. Hence,

$$E_P[F(X^\lambda)Z_1^\lambda] = \int F(X^\lambda)\ dP^\lambda \overset{P^\lambda\circ(X^\lambda)^{-1}=P}{=} \int F(X)\ dP = E_P[F(X)] \tag{4.5.13}$$

Lemma 4.5.25.

$$\lim_{\lambda\to 0} \frac{Z_1^\lambda - 1}{\lambda} = \int_0^1 u_s\ dX_s \textit{ in } \mathcal{L}^2(P).$$

In fact this limit exists even in $\mathcal{L}^p(P)$ for all $p \geq 1$ (exercise).

Proof. By Itô

$$Z_t^\lambda = 1 + \lambda \int_0^t Z_s^\lambda u_s \,\mathrm{d}X_s. \tag{$\star$}$$

Hence, for $t = 1$

$$\frac{Z_1^\lambda - 1}{\lambda} - \int_0^1 u_s \,\mathrm{d}X_s = \int_0^1 (Z_s^\lambda - 1) u_s \,\mathrm{d}X_s. \tag{4.5.14}$$

But by Ito-isometry

$$E\left[\left(\int_0^1 (Z_s^\lambda - 1) u_s \,\mathrm{d}X_s\right)^2\right] = E\left[\int_0^1 (Z_s^\lambda - 1)^2 u_s^2 \,\mathrm{d}s\right] \leq \|u\|_\infty^2 \, E\left[\int_0^1 (Z_s^\lambda - 1)^2 \,\mathrm{d}s\right].$$

$Z_s^\lambda - 1$ is a martingale, since Z_s^λ is in $\mathcal{L}^2(P)$ (see below), thus, $(Z_s^\lambda - 1)^2$ is a submartingale. Therefore,

$$E\left[\left(\int_0^1 (Z_s^\lambda - 1) u_s \,\mathrm{d}X_s\right)^2\right] \leq \|u\|_\infty^2 \, E\left[(Z_1^\lambda - 1)^2\right]$$

It remains to prove that $(Z_1^\lambda - 1) \to 0 \in \mathcal{L}^2(P)$ as $\lambda \to 0$. Clearly, $(Z_1^\lambda - 1) \to 0$ P-a.s. But the set $\{(Z_1^\lambda - 1)^2 | 0 \leq \lambda \leq 1\}$ is uniformally P-integrable, because

$$E[(Z^\lambda)^p] \overset{4.4.13}{\leq} \exp\left[\frac{1}{2}\lambda^2 p(p-1) \left\| \int_0^1 u_s^2 \,\mathrm{d}s \right\|_\infty\right] \leq \exp\left[\frac{1}{2} p(p-1)\lambda^2 \|u\|_\infty^2\right],$$

i.e. $\{Z_1^\lambda | 0 \leq \lambda \leq 1\}$ is $\mathcal{L}^p$ bounded for all $p \geq 2$. Hence, the assertion follows by Lebesgue's dominated convergence theorem. □

Proposition 4.5.26 (Bismut's integration by parts formula on Wiener space)**.** *Let u, U be as in Proposition 4.5.23 and let $F : E \to \mathbb{R}$ be H-differentiable. Then*

$$\left(E\left[\langle D.F, u\rangle_{L^2([0,1],dt)}\right]\right) = E[\langle \nabla F, U\rangle_H] = E\Big[F \underbrace{\int_0^1 u_s \, dX_s}_{=-div\, U = D^* u}\Big]. \quad (\Rightarrow \dot{u} \in Dom\ D^*)$$

Remark 4.5.27. *Proposition 4.5.26 identifies a duality between D and $\int \cdot \, dX$ (i.e. between the Malliavin derivative and the Ito-integral). This is the starting point for defining an extension of the Itô-integral, namely the* Skorohod integral. *This extension is simply defined as the adjoint D^* of D. Note that* $\operatorname{dom} D^*$ *contains also non-$(\mathcal{F}_t)$-adapted processes (cf. Lectures on Malliavin calculus!).* $\operatorname{dom} D$ *is the set of all H-differentiable functions $F :\in L^2(E, P_0)$,*

$$D : \operatorname{dom} D \subset L^2(E, P) \to L^2(E \to L^2([0,1], dt), P).$$

Hence,

$$D^* : \operatorname{dom} D^* \subset L^2(E \to L^2([0,1], dt), P) \to L^2(E, P).$$

Proof of Proposition 4.5.26. (4.5.13) implies

$$E_{(P)}\left[\frac{F(X^\lambda) Z_1^\lambda}{\lambda}\right] = E_{(P)}\left[\frac{F(X)}{\lambda}\right].$$

Therefore, (recall $X(\omega) = \omega$)

$$E\left[\frac{F(X^\lambda) - F(X)}{\lambda} Z_1^\lambda\right] = E\left[-F(X)\frac{Z_1^\lambda - 1}{\lambda}\right] \overset{\lambda \to 0}{\longrightarrow} E\left[-F \int_0^1 u_s \,\mathrm{d}X_s\right]$$

by Lemma 4.5.25. But the left hand side is equal to

$$E\left[\left(\frac{1}{\lambda}(F(X^{\lambda})-F(X))+\langle\nabla F,U\rangle_H\right)Z_1^{\lambda}\right]-E\left[\langle\nabla F,U\rangle_H Z_1^{\lambda}\right].$$

Since $\langle\nabla F,U\rangle_H\in\mathcal{L}^2(E,P)$ and $Z_1^{\lambda}\xrightarrow{\lambda\to 0}1$ in $\mathcal{L}^2(P)$ (see Proof of 4.5.25), the second summand converges to $-E[\langle\nabla F,U\rangle_H]$ as $\lambda\to 0$. The first summand converges to 0 by Cauchy-Schwarz as $\lambda\to 0$, and the assertion follows. □

First application:
Identification of the integrand in Ito's representation theorem.

Corollary 4.5.28 (Clark-Formula)**.** *Let $F\in\mathcal{L}^2(P)$ be H-differentiable. Then for $\mathcal{F}_t:=\mathcal{F}_{t+}^0$ with $\mathcal{F}_t^0:=\sigma(X_s|s\le t)$, $t\in[0,1]$,*

$$F=E[F]+\int_0^1 E[D_tF|\mathcal{F}_t]\;dX_t\quad P\text{-a.s.}.$$

Exercise: Show that the process $(D_tF)_{0\le t\le 1}$ has a version which is a $\mathcal{B}([0,1])\otimes\mathcal{F}$-measurable function. Hint: First consider $N_t(\omega)=1_{[a,b[}(t)1_A(\omega)$ for $A\in\mathcal{F}$ instead of D_tF. (Recall $\bar{P}_x(dt,d\omega)=d\langle X\rangle_t(\omega)P(d\omega)=dt\,P(d\omega)$).

Proof of 4.5.28. Without loss of generality $E[F]=0$. Let $G\in\mathcal{L}^2(P)$. Then by Corollary 2.5.44

$$G=E[G]+\int_0^1 u_t\,\mathrm{d}X_t,$$

where $u\in\mathcal{L}^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P}_X)$ and u is $(\mathcal{F}_t)$-adapted. Then by Bismut's Integration by Parts-formula we have for

$$u^{(n)}:=(u\wedge n)\vee(-n),\quad n\in\mathbb{N},$$

that

$$\begin{aligned}E[FG]&=\lim_{n\to\infty}E\left[F\int_0^1 u_t^{(n)}\,\mathrm{d}X_t\right]\overset{4.5.26}{=}\lim_{n\to\infty}E\left[\int_0^1 D_tF\,u_t^{(n)}\,\mathrm{d}t\right]\\&\overset{\text{Fubini}}{=}\lim_{n\to\infty}\int_0^1 E[D_tF\,u_t^{(n)}]\,\mathrm{d}t=\int_0^1 E[D_tF\,u_t]\,\mathrm{d}t\\&\overset{u\text{ adapted}}{=}E\left[\int_0^1 E[D_tF|\mathcal{F}_t]u_t\,\mathrm{d}t\right]\\&\overset{2.3.22}{=}E\left[\left(\int_0^1 E[D_tF|\mathcal{F}_t]\,\mathrm{d}X_t\right)G\right].\end{aligned}$$

and the assertion follows, since $G\in\mathcal{L}^2(P)$ was arbitrary. □

Example 4.5.29. *(i) Define*

$$F:=\int_0^1 X_t\,dt.$$

Note that $E\ni\omega\mapsto F(\omega)=\int_0^1 X_t\,dt$ is linear and continuous on E, hence $F\in E'$. Then $E'\ni F'(\omega,dt)=dt-\delta_0$, hence by (4.5.12) for all $t\in[0,1]$

$$D_tF(\omega)=F'(\omega,]t,1])=1-t.$$

So, by 4.5.28

$$\int_0^1 X_t\,dt=F=\int_0^1(1-t)\;dX_t.$$

Note that this can be also (in fact much more easily) proved by Itô's product rule (see 2.5.46).

(ii) *$F := f(X_1)$, where $f \in \mathcal{C}^2(\mathbb{R})$. Then by the chain rule (which also holds for Frechet-differentiable functions on Banach spaces) F is Frechet-differentiable and*

$$F'(\omega, dt) = f'(X_1(\omega))\delta_1 - f'(X_1(\omega))\delta_0 \quad (\in E').$$

Thus,

$$D_t F(\omega) = F'(\omega,]t, 1]) = f'(X_1(\omega)).$$

Hence, by Corollary 4.5.28

$$f(X_1) = F = E[f(X_1)] + \int_0^1 E[f'(X_1)|\mathcal{F}_t]\; dX_t. \qquad (\star)$$

F depends only on time $t = 1$, but is represented as an integral along paths. The corresponding integrands can be interpreted in terms of solutions $h(X_t(\omega), t)$ to the following "final value problem")

$$h(\cdot, 1) = f(\cdot),$$
$$\frac{1}{2} h_{xx} + h_t = 0.$$

The solution has the form $h(x,t) = p_{1-t} f(x)$ (as easy to see by elementary calculation, cf. Chapter III). Hence by Itô's formula (cf. 1.3.1(iii))

$$h(X_1, 1) = h(0,0) + \int_0^1 h_x(X_s, s)\; dX_s, \qquad (\star\star)$$

where $h(X_1, 1) = f(X_1)$ and $h(0,0) = p_1 f(0) = E[f(X_1)]$. By $(\star)$ and $(\star\star)$ and uniqueness in Itô's representation theorem we can conclude that

$$E[f'(X_1)|\mathcal{F}_s] = h_x(X_s, s) = (\partial_x\, p_{1-s}\, f)(X_s) = (\partial_x (E[f(X_{1-s} + x)]))_{x = X_s}.$$

To show that $F \in \mathcal{L}^2(P)$ is H-differentiable is rather difficult in general. The following sufficient condition might be useful to check H-differentiability.

Proposition 4.5.30. *The following is a sufficient condition for the H-differentiability of F in $\mathcal{L}^2(P)$:*
There exists a kernel $F'(\omega, dt)$ from Ω to $\mathcal{B}([0,1])$ such that for all U as in 4.5.23

(i)

$$\frac{F(X + \lambda U) - F(X)}{\lambda}(\omega) \xrightarrow{\lambda \to 0} \int_0^1 F'(\omega, dt) U_t(\omega) \quad \text{for } P\text{-a.e. } \omega \in E.$$

(ii) *For all $c > 0$*

$$|F(X + U) - F(X)| \leq c\, \|U\|_\infty \quad P\text{-a.s.}.$$

In this case

$$(H \ni) \nabla F(\omega) = R(\underbrace{F'(\omega, dt) - F'(\omega, [0,1]) \cdot \delta_0}_{=:\mu_\omega(dt) \in E'}) = \int_0^\cdot F'(\omega,]s, 1])\; ds.$$

Proof. By 4.5.20 we have

$$\int F'(\omega, dt) U_t(\omega) \overset{U_0 = 0}{=} \int \mu_\omega(dt) U_t(\omega) = \langle R(\mu_\omega), U(\omega) \rangle_H = \langle \nabla F(\omega), U(\omega) \rangle_H.$$

Hence, the assertion follows by Lebesgue's dominated convergence theorem. □

Example 4.5.31. *$F(\omega) := \max_{0\leq t\leq 1} \omega(t)$ is* not *Frechet-differentiable, but it is H-differentiable. To see this, define*

$$T := \inf\{t > 0 | X_t = F\}.$$

Then it follows by Proposition 4.5.30 that F is H-differentiable and

$$D_t F(\omega) = \delta_{T(\omega)}(]t, 1]) = 1_{\{T>t\}}(\omega)$$

(cf. Exercises).
Next step: *Identify $E[D_t F|\mathcal{F}_t]$ in order to* use *the Clark formula.*
Define

$$M_t := \max_{0\leq s\leq t} X_s.$$

Then we have for P-a.e. $\omega \in E$

$$\begin{aligned} E[D_t F|\mathcal{F}_t](\omega) &= P[T > t|\mathcal{F}_t](\omega) \\ &= P\left(\max_{t\leq s\leq 1} X_s > M_t|\mathcal{F}_t\right)(\omega) \\ &= P\left(\max_{0\leq s\leq 1-t} X_{s+t} > M_t|\mathcal{F}_t\right)(\omega) \end{aligned}$$

Now, we use the superstrong Markov property, i.e. for any $\mathcal{F}\otimes\mathcal{F}_t$-measurable positive or bounded function $G : E \times E \to \mathbb{R}$ we have

$$E_x[G(\vartheta_t, \cdot)|\mathcal{F}_t](\omega) = E_{X_t(\omega)}[H(\cdot, \omega)] \quad \text{for } P_x\text{-a.e. } \omega \in E,$$

where P_x is the law of Brownian motion started at $x \in \mathbb{R}$ and $\vartheta_t(\omega) := \omega(\cdot + t)$.
Hence,

$$\begin{aligned} E[D_t F|\mathcal{F}_t](\omega) &= P_{X_t}(\omega)\left(\max_{0\leq s\leq 1-t} X_s > M_t(\omega)\right) \\ &= P_{X_t(\omega)}\left(\max_{0\leq s\leq 1-t} X_s - X_t(\omega) > M_t(\omega) - X_t(\omega)\right) \quad = P_0\left(\max_{0\leq s\leq 1-t} X_s > M_t(\omega) - X_t(\omega)\right) \end{aligned}$$

By the reflection principle, this is equal to

$$\begin{aligned} &2 \cdot P_0(X_{1-t} > M_t(\omega) - X_t(\omega)) \\ =&2 \cdot N(0, (1-t))(]M_t(\omega) - X_t(\omega), \infty[) \\ =&2 \cdot N(0, 1)\left(\frac{M_t(\omega) - X_t(\omega)}{\sqrt{1-t}}, \infty[\right) \\ =&2 \int_{\frac{M_t(\omega)-X_t(\omega)}{\sqrt{1-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ =&2 \cdot \left(1 - \Phi\left(\frac{M_t(\omega) - X_t(\omega)}{\sqrt{1-t}}\right)\right). \end{aligned}$$

(Vgl. hierzu [Röc11, Beispiel 10.2.8].)

5. Brownian motion and potential theory

Attention: This chapter has not been proofread yet!

5.1. Brownian motion and Laplace Operator

Let $(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in R^d})$ be the canonical Brownian Motion on $\mathbb{R}^d$, i.e.:

$$\Omega := C([0,\infty[, \mathbb{R}^d), \quad X_t(\omega) = \omega(t),\ t \geq 0, \quad \mathcal{F} := \sigma(X_t | t \geq 0), \quad \mathcal{F}_t := \bigcap_{\varepsilon > 0} \sigma\{X_s | s \leq t + \varepsilon\},$$

$$P_x := \text{ Pushforward measure of the Wiener measure } P_0 \text{ under } \omega \mapsto x + \omega$$

Let $(p_t)_{t>0}$ be the corresponding semi-group of kernels, i.e. for $f : \mathbb{R}^d \to \mathbb{R}_+$, $\mathcal{B}(\mathbb{R}^d)$-measurable, $t > 0$,

$$\int f(X_t) dP_x = E_x[f(X_t)] = \int f(y) p_t(x, dy) = \int f(y) \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{\frac{-||x-y||^2}{2t}} dy, \quad x \in \mathbb{R}^d.$$

Let $D \subset \mathbb{R}^d$, open.

$$T_D := \text{exit time from } D \ (:= inf\{t > 0 | X_t \notin D\} = \sigma_{D^c}).$$

$\forall x \in D$ the following holds for P_x-a.e. ω and all $t \in [0, T_D(\omega)[$ (cf. I.2.72):

$$f(X_t(\omega)) - \underbrace{f(X_0(\omega))}_{=f(x)} = \int_0^t \nabla f(X_s) dX_s(\omega) + \frac{1}{2} \int_0^t \Delta f(X_s(\omega)) ds, \quad \text{if } f \in C^2(D)$$

In particular

$$M_t^f(\omega) := f(X_t(\omega) - f(X_0(\omega)) - \frac{1}{2} \int_0^t \Delta f(X_s(\omega)) ds$$

is Itô-Integral with respect to Brownian Motion with

$$\langle M^f \rangle(\omega) = \int_0^t ||\nabla f(X_s)||^2 ds. \tag{$*$}$$

Corollary 5.1.1. *(i) $\forall f \in C^2(D)$ (M_t^f) is local martingale (until T_D) under P_x $\forall x \in D$ (cf. I.3.4.), i.e. Brownian Motion is solution for the (so called)* martingale problem *for $(\mathcal{L} := \frac{1}{2}\Delta, C^2(D))$ with initial condition $x \in D$ (cf. [SV06]).*

(ii) ***Preliminary remark:*** *D_0 open rel. kp with $\bar{D}_0 \subset D \Rightarrow E_x[T_{D_0}] < \infty \quad \forall x \in D_0$ (cf. III.3.3.3). Hence, because of $(*)$, $E_X[\langle M^f \rangle_{T_{D_0}}] < \infty \quad \forall x \in D_0$*

$\overset{I.3.5.}{\Longrightarrow}$ Dynkin formula: Let T be stopping time, $\leq T_{D_0} (< T_D$ as $\bar{D}_0 \subset D)$, then $\forall x \in D_0$

$$E_x[f(X_t)] - f(x) = \frac{1}{2} E_x \left[\int_0^T \Delta f(X_s) ds \right]$$

Application 5.1.2: (i) $T = T_\varepsilon$ = exit time from open ε-ball $K_\varepsilon(x)$ around x. Then there exists for $f \in C^2(D)$

$$\mathfrak{a}f(x) := \lim_{\varepsilon \searrow 0} \frac{E_x[f(X_{T_\varepsilon})] - f(x)}{E_x[T_\varepsilon]} \quad \text{“characteristic (Dynkin) operator” (local),}$$

and we have

$$\mathfrak{a}f(x) = \frac{1}{2}\Delta f(x), \quad \mathfrak{a} = \frac{1}{2}\Delta \text{ on } C^2(D) \text{ for Brownian Motion (cf. “special script”)}$$

(ii) $D = \mathbb{R}^d$, $f \in C_0^2(\mathbb{R}^d)$:

$$Af(x) := \lim_{t \downarrow 0} \frac{p_t f(x) - f(x)}{t} = \lim_{t \downarrow 0} \frac{E_x[f(X_t)] - f(x)}{t} = \frac{1}{2}\Delta f(x)$$

the “infinitesimale operator or generator” of the semi-group.

$$A = \frac{1}{2}\Delta \text{ on } C_0^\infty(\mathbb{R}^d) \text{ for Brownian Motion}$$

Attention: In what way $\frac{1}{2}\Delta$ really generates $(p_t)_{t>0}$ and with it the Brownian Motion, hence in particular is uniquely determined, can not be seen from above (cf. “special script” or [MR92]).

(iii) It follows from the Dynkin formula with T_r = exit time from $K_r(x_0)$ and $f(x) = ||x - x_0||^2$, since $\frac{1}{2}\Delta f = d$, $\forall x \in K_r(x_0)$ that

$$E_x[\underbrace{(X_{T_r} - x_0)^2}_{=r^2}] - ||x - x_0||^2 = dE_x[T_R]$$
$$\implies E_x[T_r] = \frac{r^2 - ||x - x_0||^2}{d}.$$

5.2. Stochastic solution of the Dirichlet- resp. Poisson-Problem

$D \subset \mathbb{R}^d$, open, rel. *kp.*
T = exit time from D, then $E_x[T] < \infty \quad \forall x \in D$.
Poisson problem
Have: $f \in C(\partial D)$, $g \in C_b(D)$
Want: $h \in C^2(D) \bigcap C(\bar{D})$ with

$$\frac{1}{2}\Delta h = g \text{ on } D$$
$$h = f \text{ on } \partial D$$

If $g \equiv 0$:
Dirichlet problem

Theorem 5.2.3 (“Representation Theorem”). *Let h be a solution of the Poisson problem, then*

$$\forall x \in D \quad h(x) = E_x[f(X_{T_D})] - E_x\left[\int_0^{T_D} g(X_s) ds\right]$$

Proof. Let $D_n \nearrow D$, $\bar{D}_n \subsetneqq D$, i.e. in particular $\bar{D}_n$ kp.

$$\underset{\text{Dynkin formula}}{\Longrightarrow} \forall x \in D_n \; \forall n \quad E_x \underbrace{\left[h(X_{T_{D_n}})\right]}_{\nearrow X_{T_D} \in \partial D} - h(x) = E_x \left[\underbrace{\int_0^{T_{D_n}} g(X_s) ds}_{T_{D_n} \nearrow T_D} \right]$$

$$\underset{\substack{n\to\infty \\ \text{Lebesgue}}}{\longrightarrow} E_x \left[f(X_{T_D}) \right] - h(x) = E_x \left[\int_0^{T_D} g(X_s) ds \right]$$

□

We consider now the case $g \equiv 0$, i.e. a Dirichlet problem. For D_0 open, rel. kp in D we we define for $x \in D_0$.

$$\mu_x^{D_0}(A) := P_x \left[X_{T_{D_0}} \in A \right], \quad A \in \mathcal{B}(\partial D_0)$$

Then we get

Corollary 5.2.4 ("generalized mean value property")**.** *Let h be harmonic in D. Then $\forall D_0$ open, rel. kp in D*

$$h(x) = E_x \left[h(X_{T_{D_0}}) \right] = \int_{\partial D_0} h \; d\mu_x^{D_0}, \quad x \in D_0.$$

Definition 5.2.5. *$\mu_x^{D_0}$ is called the* harmonic measure *for D_0 in x ($\in D_0$). (=exit distrubition $P_x \circ X_{T_{D_0}}^{-1}$ of the Brownian motion from D_0 with start in x)*

Remark 5.2.6. *It follows from the definiton of the Brownian motion, that it is translation- and rotation invariant. Thus, $\mu_x^{K_r(x)}$ is also rotation invariant.*

$$\Longrightarrow \mu_x^{K_r(x)} = const \cdot \underbrace{\sigma_{x,r}}_{\substack{\text{norm. surface} \\ \text{measure}}} \underset{\mu_x^{K_r(x)}(\partial K_r(x))=1}{\Longrightarrow} \mu_x^{K_r(x)} = \sigma_{x,r}.$$

More generally, we have $\forall y \in K_r(x)$

$$\mu_y^{K_r(x)}(dz) = \underbrace{r^{n-2} \frac{r^2 - ||y-x||^2}{||y-z||^n}}_{\text{Poisson kernel for } K_r(x)} d\sigma_{x,r}$$

Theorem 5.2.7 ("Existence of a generalized solution of the Dirichlet problem")**.** *$D \subset \mathbb{R}^d$, open, f bounded, measurable on ∂D. Then we have for*

$$h(x) := E_x \left[f(X_{T_D}) \right], \quad x \in D:$$

(i) h harmonic on D

(ii) $\lim_{t \nearrow T_D} h(X_t) = f(X_{T_D}) \quad P_x$-a.s $\forall x \in D$.

Proof. Set $T := T_D$ and let $D_n \nearrow D$, D_n open, $\bar{D}_n \subset D_{n+1} \subset D$. Set $T_n := T_{D_n}$. $x \in D$ fixed. Without loss of generality : $x \in D_1$.

(i) Harmonicity: $\forall n$

$$h(X_{T_n}) = E_{X_{T_n}}[f(X_T)] \underset{SMP}{=} E_x[f\underbrace{(X_T\circ\theta_{T_n})}_{=X_{\underbrace{T\circ\theta_n+T_n}_{=T,\text{ since }T\geq T_n}}}|\mathcal{F}_{T_n}]$$

$$= E_x[f(X_T)|\mathcal{F}_{T_n}]$$

$\implies (h(X_{T_n}))_{h\in\mathbb{N}}$ P_x-martingale and

$$h(x) = E_x[f(X_T)] = E_x[E_x[f(X_T)|\mathcal{F}_{T_n}]] \underset{\text{see above}}{=} E_x[h(X_{T_n})]$$

$\implies$ h harmonic. Since $\forall K_r(x)\subset\overline{K_r(x)}\subset D$, Without loss of generality : $D_1 = K_r(x)$, hence $h(x) = E_x[h(X_{T_1})] \underset{\text{see above}}{=} \int h(y)\sigma_{x,r}(dy)$, i.e. h satisfies classic mean value property $\underset{\text{Analysis}}{\implies}$ h harmonic.

(ii) $(h(X_{T_n}))_{n\in\mathbb{N}}$ bounded martingale under P_x

$$\underset{\substack{\text{mart. conv.}\\ \text{theorem + above}}}{\implies} \quad h(X_{T_n}) \longrightarrow E_x\left[f(X_T)|\sigma(\bigcup_n \mathcal{F}_{T_n})\right]$$

$$= f(X_T) \quad P_x-\text{a.s., since } \underset{\substack{\sigma(\cup_n\mathcal{F}_{T_n})-\\ \text{measurable}}}{X_T = \lim X_{T_n}} \text{ (as } T_n\to T \text{ and continuity)}$$

But according to the upcrossing lemma $\exists \lim h(X_t)$ P_x-a.s., (more detailled: Cor. 5.2, ch. II, W-Theorie II-Vorlesung), hence it must be the above one.

(Existence: Set $h(X_t) = f(X_T)$ for $t\geq T$, then $(h(X_t))_{t\geq 0}$ P_x-martingale, since

$$1_{\{t<T\}}h(X_t) = 1_{\{t<T\}}E_{X_t}[f(X_T)] \underset{MP}{=} E_x[\underbrace{1_{\{t<T\}}f(X_{T\circ\theta_t+t})}_{=1_{\{t<T\}}f(X_T)}|\mathcal{F}_t]$$

$$= E_x[f(X_T)|\mathcal{F}_t] - \underbrace{1_{\{t\geq T\}}f(X_T)}_{\mathcal{F}_{t\wedge T}\subset\mathcal{F}_t\text{ measurable}} \quad \text{(cf. lemma 4.1 in ch. II, W-Th.II).)}$$

□

5.3. Application: Reccurence/transience of the Brownian motion

We know (because of the iterated Log) that $\forall d$: $\overline{\lim_{t\uparrow\infty}}||X_t||_{\mathbb{R}^d} = +\infty$ P_x-a.s. $\forall x\in\mathbb{R}^d$. We want to show:

Theorem 5.3.8. *(i) $d\geq 3 \implies$ Brownian motion* transient, *i.e.:* $\lim_{t\nearrow\infty}||X_t|| = +\infty$ *P_x-a.s. $\forall x\in\mathbb{R}^d$.*

(ii) $d=2 \implies$ Brownian motion recurrent, *i.e.: Brownian motion hits $K_\varepsilon(x_0)$ P_x-a.s. $\forall\varepsilon>0$ $\forall x_0\in\mathbb{R}^d$; i.e. P_x-a.e. path is dense in $\mathbb{R}^2$.*

Preparation for the proof

Let $A_{r,R} := \{x \in \mathbb{R}^d | r < ||x|| < R\}$ and $f = \begin{cases} 1 & \text{in } \{x \mid ||x|| = r\} \\ 0 & \text{in } \{x \mid ||x|| = R\} \end{cases}$ be a solution to the corresponding Dirichlet problem

$$h_{r,R}(x) := E_x\left[f \circ X_{T_{A_{r,R}}}\right] = P_x\left[||X_{T_{A_{r,R}}}|| = r\right].$$

One can easily show that analytically (write Δ in spherical coordinates)

$$h_{r,R}(x) = \frac{\varphi_d(||x||) - \varphi_d(R)}{\varphi_d(r) - \varphi_d(R)} \quad \text{with } \varphi_d(\rho) = \begin{cases} \rho & , d = 1 \\ -log\rho & , d = 2 \\ \rho^{2-d} & , d \geq 3 \end{cases} \tag{5.3.1}$$

(cf. Ruinproblem W.theorie II)

Theorem 5.3.9. *$\forall d \geq 2$ every point in $\mathbb{R}^d$ is* polar*, i.e.:*

$$P_y[\sigma_{\{x\}} < \infty] = 0 \quad \forall y \in \mathbb{R}^d.$$

(with $\sigma_{\{x\}} := inf\{t > 0 | X_t = x\}$.)

Proof. Without loss of generality : $x = 0$ (because auf translation invariance)

(i) $y \neq x = 0$:

$$\begin{aligned} P_y[\sigma_{\{0\}} < \infty] &= \lim_{n\to\infty} P_y[\sigma_{\{0\}} < \sigma_{K_n(0)^c}] \\ &\quad \sigma_{K_n(0)^c} \underset{n\to\infty}{\nearrow} P_y - \text{a.s.} \\ &\quad (\text{ since for} P_y - \text{a.a. } \omega \{X_s(\omega) | 0 \leq s \leq t\} \text{ bounded } \forall t) \\ &\leq \overline{\lim_{n\to\infty}}\ \overline{\lim_{\substack{r\to 0 \\ r<||y||}}}\ P_y\left[||X_{T_{A_{r,n}}}|| = r\right] \\ &= \overline{\lim_{n\to\infty}}\ \underline{\lim_{\substack{r\to 0 \\ r<||y||}}}\ h_{r,n}(y) = 0 \quad \text{because of (5.3.1), since } d \geq 2. \end{aligned}$$

(ii) $y = x$:

$$\begin{aligned} P_x[\sigma_{\{x\}} < \infty] &\leq \lim_{t\downarrow 0} P_x[\sigma_{\{x\}} \circ \theta_t < \infty] \\ &\underset{\text{(MP)}}{=} \lim_{t\downarrow 0} E_x[\underbrace{P_{X_t}[\sigma_{\{x\}} < \infty]}_{\substack{\overset{1)}{=} 0 \ P_x-\text{a.s.,} \\ \text{since } X_t \neq x \ P_x-\text{a.s.}}}] \end{aligned}$$

$\square$

Remark 5.3.10. *For $x \notin K_r(0)$*

$$\begin{aligned} P_x[\sigma_{K_r(0)} < \infty] &= \lim_{n\to\infty} P_x[\overbrace{||X_{T_{A_{r,n}}}|| = r}^{\nearrow n \nearrow P_x-a.s.}] \\ &\underset{(5.3.1)}{=} \begin{cases} 1, & d = 2 \\ \frac{||x||^{2-d}}{r^{2-d}}, & d \geq 3. \end{cases} \end{aligned}$$

Now:

Proof of Theorem 5.3.8. Let $x_o \in \mathbb{R}^d$, Without loss of generality $x_0 = 0$. $T_n :=$ exit time from $K_n(0)$, then $\forall x \in \mathbb{R}^d$

$$P_x \underbrace{\left[\underline{\lim}_{t\uparrow\infty} ||X_t|| < r\right]}_{=\bigcap_n \underbrace{\left[\sigma_{K_r(0)} \circ \theta_{T_n} < \infty\right]}_{\searrow n \nearrow}} = \lim_{n\to\infty} P_x\left[\sigma_{K_r(0)} \circ \theta_{T_n} < \infty\right]$$

$$\underset{\text{(SMP)}}{=} \lim_{n\to\infty} E_x\left[P_{X_{T_n}}\left[\sigma_{K_r(0)} < \infty\right]\right]$$

$$\underset{5.3.10}{=} \lim_{n\to\infty} \begin{cases} 1 & , d = 2 \\ \frac{n^{2-d}}{r^{2-d}} \underset{n\to\infty}{\longrightarrow} & , d = 3. \end{cases}$$

□

5.4. Regularity of boundary points

Let $D \subset \mathbb{R}^d$, open, rel. kp., $\partial D := \bar{D} \cap \overline{(\mathbb{R}^d \setminus D)}$

Definition 5.4.11. *$z \in \partial D$ is called* regular, *if $P_z[\sigma_{D^c} = 0] = 1$.*

Remark 5.4.12. *Since $\{\sigma_{D^c} = 0\} \in \mathcal{F}_0 := \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$, we have acoordingly to Blumenthal's zero-one law a priori $P_x[\sigma_{D^c} = 0] \in \{0, 1\}$.*

Example 5.4.13. *Let $D = U \setminus \{x\}$ with U rel. kp. open, $z \in U$. Then $z \in \partial D$ and z is irregular, if $d \geq 2$, since z polar. We consider the behaviour of the stochastic solution of the Dirichlet problem for D with boundary condition $f = 1_{\{z\}} + 0 \cdot 1_{(\partial D)\setminus\{z\}}$*

$$\begin{aligned} h(x) &= E_x\left[f(X_{T_D})\right] \\ &= 0 \cdot P_x\left[X_{T_D} \in \partial D \setminus \{z\}\right] + 1 \cdot \underbrace{P_x\left[X_{T_D} \in \{z\}\right]}_{\substack{=0, \\ \text{if } d\geq 2, \text{ since then } z \text{ polar.}}} \\ &\equiv 0 \end{aligned}$$

If we understand the above as an "inteference" of the solution of the Dirichlet problem for $D \cup \{z\}$, we then obtain "stability".

Theorem 5.4.14. *$z \in \partial D$ regular $\Leftrightarrow \forall f$ bounded, measurable on ∂D, continous in z we have:*

$$h(x) := E_x\left[f(X_{T_D})\right] \underset{\substack{x\to z \\ x\in D}}{\longrightarrow} f(z)$$

Proof. We only proof "$\Rightarrow$": Let in the following always $x \in D$.
Claim: $\lim_{x\to z} P_x\left[X_{T_D} \notin K_\delta(z)\right] = 0 \quad \forall \delta > 0$. Here $K_\delta(z)$ open ball.
If this claim is true, then select $\delta = \delta(\varepsilon)$, such that $|f(x) - f(z)| < \varepsilon \quad \forall x \in K_\delta(z) \cap \partial D$. It follows that

$$|h(x) - f(z)| \leq E_x\left[|f(X_{T_D}) - f(z)|\right] \leq \varepsilon + 2||f||_\infty P_x\left[X_{T_D} \notin K_\delta(z)\right]$$

$\Longrightarrow \overline{\lim}_{x\to z} |h(x) - f(z)| \leq \varepsilon \quad \forall \varepsilon > 0$, hence "$\Rightarrow$" proved. □

Proof of the claim. Let $\delta > 0$ fixed. If $X_0(\omega) = x \in K_{\frac{\delta}{2}}(z)$, then $\forall s > 0$:

$$X_{T_D(\omega)} \notin K_\delta(z) \implies T_D(\omega) > s \text{ or } T_{K_{\frac{\delta}{2}}(x)} \leq s.$$

So we have $\forall x \in K_{\frac{\delta}{2}}(z) \quad \forall s > 0$

$$P_x\left[X_{T_D} \notin K_\delta(z)\right] \leq P_x[T_D > s] + P_x\left[T_{K_{\frac{\delta}{2}}(x)} \leq s\right]$$

Because of the translation invariance we get

$$\underbrace{P_x\left[T_{K_{\frac{\delta}{2}}(x)} \leq s\right] = P_0\left[T_{K_{\frac{\delta}{2}}(0)} \leq s\right]}_{<\varepsilon \quad \text{for } s \text{ small}} \quad \left(\underset{s\downarrow 0}{\downarrow}\ P_0\left[T_{K_{\frac{\delta}{2}}(0)} = 0\right] \underset{\substack{\text{because of right}\\ \text{continous pathes}}}{=} 0\right)$$

Let $s > 0$. To show:

$$y \mapsto \varphi_s(y) := P_y[T_D > s],\ y \in \mathbb{R}^d,\ \text{is not semi-continuous in } y := z$$

Because then $0 \leq \overline{\lim_{x\to z}}\, \varphi_s(x) \leq \varphi_s(z) \overset{z\text{ regular}}{=} 0$.
But $\forall y \in \mathbb{R}^d$

$$\begin{aligned}\varphi_s(y) = P_y[T_D > s] = \lim_{t\downarrow 0}\ & \underbrace{P_y[\overbrace{\underbrace{t + T_D \circ \theta_t}_{\text{first exit time after } t} \quad > s}^{\searrow\text{, if } t\searrow\text{, hence lim=inf}}]} \\ & = P_y[T_D \circ \theta_t > s - t] \\ & \underset{MP}{=} E_y[\underbrace{P_{X_t}[T_D > s - t]}_{\varphi_{s-t}(X_t)}] \\ = \lim_{t\downarrow 0}\ & \left.\underbrace{p_t\varphi_{s-t}(y)}_{\substack{\text{cont. as } p_t\\ \text{strong Feller}}}\ \right\}\text{n.o.h.cont.}\end{aligned}$$

□

Now an analytic criterion for regularity:

Theorem 5.4.15 (“Zaremba's cone condition”). *Let $z \in \partial D$ such that there exixts a cone C with apex in 0 and $h > 0$ with $z + C_h \subseteq D^c$, where $C_h := C \cap K_n(0)$. Then z is regular.*

Proof.

$$\begin{aligned}P_z[T_D \leq t] &\geq P_z[X_t \in z + C_h] \quad \forall t \in]0,1] \\ &= P_0\left[X_t \in C_h\right] \\ &= P_0\left[\sqrt{t}X_{\frac{t}{(\sqrt{t})^2}} \in C_h\right] \text{ , since } \left(cX_{\frac{t}{c^2}}\right)_{t\geq 0} \text{ again B.M.; cf. W.th.II, ch. III, Satz 5.1} \\ &= P_0\left[X_1 \in t^{\frac{-1}{2}}C_h\right] \\ &\underset{(t\leq 1)}{\geq} P_0[X_1 \in C_h] > 0\end{aligned}$$

$$\implies P_z[T_D = 0] = \lim_{t_n \searrow 0} P_z[T_D \le t_n] \ge P_0[X_1 \in C_h] > 0 \ \forall n$$

$$\underset{\substack{\text{Blumentahl's}\\ \text{0-1 law}}}{\implies} \quad P_z[T_D = 0] = 1.$$

□

Idea: D_c shouldn't be to "small" in z.
Exercice: for $d = 2$ C-line is enough.
In general $(z + C_h) \cap H \subseteq D^c$ is enough, where H hypersurface (cf. [?], p. 276 or 350, analytical with Wiener theorem).

Example 5.4.16 ("Lebesgue-sting"). *Let*

$$\begin{aligned} I_n &:= [2^{-n}, 2^{-(n-1)}] \\ k_n &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \in I_n,\ x_2 = x_3 = 0\} \\ F_n &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \in I_n,\ x_2^2 + x_3^2 \le \varepsilon_n^2\} \\ D &= \underset{(height\ z)}{cylinder} \setminus \bigcup_n F_n \end{aligned}$$

We want to choose ε_n such that $0 \in \partial D$ irregular.
$d = 3 : \implies P_0[X_0$ *never hit* $K_n] = 1$*, since*

$$\begin{aligned} &P_0[X_0 \textit{ never goes back to } \{x_2 = x_3 = 0\}] \quad (\leftarrow \textit{ no condition for } X_0^{(1)}) \\ &= P_0^{d=2}[\textit{ B.M. in } \mathbb{R}^2 \textit{ never goes back to } 0] = 1 \quad , 0 \textit{ polar!} \end{aligned}$$

Since Brownian Motion in $\mathbb{R}^3$ is transient $\exists$ for P_0-a.a. ω an $s(\omega) \ge 0$ with $||X_t(\omega)|| \ge 2$ $\forall t \ge s(\omega)$. It is obvious that

$$a_n(\omega) := dist\ (\{X_t(\omega) | 0 \le t \le s(\omega)\}, K_n) > 0$$

(as both sets kp. and disjoint). Hence $\{a_n \ge \frac{1}{m}\} \underset{m \uparrow \infty}{\nearrow} \Omega$ P-a.s., thus $\exists \varepsilon_n$ with $P_0[a_n \le \varepsilon_n] \le 3^{-n}$. Then

$$P_0[\underbrace{X_t \in F_n \textit{ for a } t}_{\{\sigma_{F_n} < \infty\}}] \le P_0[a_n \le \varepsilon_n] \le 3^{-n}$$

$$\implies \quad P_0[T_D = 0] \le \sum_n P_0[\underbrace{X_t \in F_n \textit{ for a } t}_{\{\sigma_{F_n} < \infty\}}] \le \sum 3^{-n} < 1$$

$$\implies \quad 0 \textit{ irregular.}$$

Excursus to Brownian motion in $\mathbb{R}^1$. Without loss of generality $a \ge 0$
Define *passage time* $T_a := \inf\{t > 0 | X_t > a\}$.
Because $X_{T_a} = a$ and continuity of paths, we have:

$$\begin{aligned} T_a &= \inf\{t > 0 | X_t \ge a\} = \inf\{t > 0 | X_t = a\} \\ P &:= P_0 = \text{Brownian Motion with start in } 0. \end{aligned}$$

Theorem 5.4.17 (Reflection principle).

$$\frac{1}{2} P[T_a \le t] = P[X_t \ge a] \quad \left(= \frac{1}{2} P[|X_t| \ge a]\right)$$

Proof.

$$P[X_t \geq a] = P[T_a \leq t, \quad \underbrace{X_{t-T_a}(\theta_{T_a})}_{\substack{=X_t \text{ since} \\ \text{canonical model}}} \geq a]$$

$$= E[\Phi(\cdot, \theta_{T_a(\cdot)}(\cdot));\ T_a \leq t],$$

where $\Phi(\omega, \eta) = 1_{[a,\infty[} \circ X_{t-T_a(\omega)}(\eta), \quad (\omega, \eta) \in \Omega \times \Omega = C([0,\infty[) \times C([0,\infty[)$.
Φ is $\mathcal{F}_{T_a} \otimes \mathcal{F}$-measurable. We need:

Lemma 5.4.18 ("(very) SMP")**.** *Let T be stopping time, $\Phi : \Omega \times \Omega \to \mathbb{R}^+ \quad \mathcal{F}_T \otimes \mathcal{F}$-measurable. Then it follows for $H(\omega) := \Phi(\omega, \theta_{T(\omega)}(\omega))$ and all $x \in \mathbb{R}^d$*

$$E_x[H|\mathcal{F}_T](\omega) = E_{X_{T(\omega)}}[\Phi(\omega, \cdot)] \quad P_x\text{-a.a. } \omega \in \Omega.$$

Proof. Let F $\mathcal{F}_T$-measurable, G $\mathcal{F}$-measurable and $\Phi(\omega, \eta) = F(\omega)G(\eta)$. Then because of SMP for P_x-a.a. $\omega \in \Omega$

$$E_x[F\ G \circ \theta_T|\mathcal{F}_T](\omega) = F(\omega)E_x[G \circ \theta_T|\mathcal{F}_t](\omega) = F(\omega)E_{X_{T(\omega)}}[G] = E_{X_{T(\omega)}}[F(\omega)G].$$

The rest follows with monoton classes. ☐

Continuation of proof 5.4.17.
According to the lemma: for P-a.a. $\omega \in \Omega$

$$\begin{aligned} &E[\Phi(\cdot, \theta_{T_a(\cdot)}(\cdot));\ T_a \leq t](\omega) \\ &= \int_{\{T_a \leq t\}} \underbrace{P_{\underbrace{X_{T_a(\omega)}}_{a}}[X_{\underbrace{t - T_a(\omega)}_{=:s}}(\cdot) \geq a]}_{=P_s(a,[a,\infty[)=\frac{1}{2}} P(d\omega) = \frac{1}{2} P[T_a \leq t]. \end{aligned}$$

☐

Corollary 5.4.19 ("Distribution of the passage time")**.** *$P \circ T_a^{-1} = \varphi_a \cdot ds$, where*

$$\varphi_a(s) = \begin{cases} \frac{|a|}{\sqrt{2\pi s^3}}\ e^{-\frac{a^2}{2s}} & , s > 0 \\ 0 & , s \leq 0. \end{cases}$$

("stable with exponent a", since then $E[e^{ixT_a}] = e^{-|x|^a}$)

Proof.

$$\begin{aligned} P[T_a \leq t] &\underset{5.4.17}{=} 2P[X_t \geq a] \\ &= 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\ &= \int_0^t \underbrace{s^{\frac{-3}{2}} \frac{a}{\sqrt{2\pi}} e^{-\frac{a^2}{2s}}}_{\varphi_a(s)}\, ds. \end{aligned}$$

since $\frac{a^2}{2s} = \frac{x^2}{st} \implies x = a\sqrt{\frac{t}{s}}$ and $\frac{dx}{ds} = a\sqrt{t}(-\frac{1}{2})s^{-\frac{3}{2}}$ ☐

Corollary 5.4.20. *(i)* $P[T_a > t]\sqrt{t} \underset{t\nearrow\infty}{\longrightarrow} \sqrt{\frac{2}{\pi}}a$. $(a \geq 0)$

(ii) $E[T_a] = +\infty$

Proof. (i) :

$$P[T_a > t]\sqrt{t} \underset{5.4.17}{=} \sqrt{t}P\left[|X_t| \leq a\right] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-\frac{x^2}{2t}} dx \underset{t\nearrow\infty}{\longrightarrow} \frac{2a}{\sqrt{2\pi}}.$$

(ii) :

$$\begin{aligned} E[T_a] &= \int_0^\infty P[T_a > t] dt \\ &\underset{\substack{(i)\ \varepsilon \text{ small} \\ t_0 = t_0(\varepsilon)\text{big}}}{\geq} \int_{t_0}^\infty \left(\sqrt{\frac{2}{\pi}}a - \varepsilon\right) t^{-\frac{1}{2}} dt = \infty \end{aligned}$$

□

Example 5.4.21 (Solution of the Dirichlet problem on the half-space.).
$D := \{x = (x_1, ..., x_d) \in \mathbb{R}^d | x_d > 0\}$. *Consider D* not *rel. kp. as always assumed before! But an essential assumption for the existence of a solution is only* $P_x[T_D < \infty] = 1\ \forall x \in D$. *(cf. 5.2.7 and 5.4.15). Since* $\partial D := \{z = (y, 0) | y \in \mathbb{R}^{d-1}\}$ *no longer kp, we have to assume* $f \subset C_b(\partial D)$.

Stochastic:
Cone condition fulfilled $\forall z \in \partial D$!
Calculation of the harmonic measure
Under P_x the exit time $T := T_D = \inf\{t > 0 | X_t^d = 0\}$ (=passage time) has density φ_a from 5.4.19 with $a = x_d$. With notation $\omega = (\omega_1, ..., \omega_d) \in (C[0, \infty))^d$, $X_T := (Y_T, 0)$ it follows $\forall x \in D$

$$h(x) \underset{2.5!}{=} E_x[f(X_T)] = E_{(x_1,...,x_{d-1})} \underbrace{\left[E_{x_d}[f(Y_{T(\omega_d)}(\omega_1, ..., \omega_{d-1}), 0)]\right]}_{\underset{5.4.19}{=} \int_0^\infty f(Y_t(\omega_1,...,\omega_{d-1}),0) \underbrace{\frac{x_d}{\sqrt{2\pi t^3}} e^{-\frac{x_d^2}{2t}}}_{\varphi_{x_d}(t)} dt}$$

$$\begin{aligned} &\underset{\text{Fubini}}{=} \int_0^\infty \frac{x_d}{\sqrt{2\pi t^3}} e^{-\frac{x_d^2}{2t}} \left(\int_{\mathbb{R}^{d-1}} f(y, 0) \frac{1}{\sqrt{2\pi t}^{d-1}} \exp\left[\frac{-||y - (x_1, ..., x_{d-1})||^2}{2t}\right] dy\right) dt \\ &\underset{\text{Fubini}}{=} \int_{\mathbb{R}^{d-1}} f(y, 0) \frac{x_d}{\sqrt{2\pi}^d} \underbrace{\int_0^\infty t^{-(\frac{d}{2}+1)} \exp\left[-\frac{||(y, 0) - x||^2}{2t}\right] dt}_{=\Gamma(\frac{d}{2})||(y,0)-x||^{-d} 2^{\frac{d}{2}}} dy \end{aligned}$$

because: $\int_0^\infty t^{-\beta} e^{-\frac{\alpha}{t}} dt \underset{s=\frac{\alpha}{t}}{=} \alpha^{1-\beta} \int_0^\infty s^{\beta-2} e^{-s} ds = \alpha^{1-\beta}\Gamma(\beta - 1)$

$$\Longrightarrow \mu_x^D(d_y) = \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}}} \frac{x_d}{||x - (y, 0)||^d} \underbrace{\lambda_{d-1}}_{\substack{\text{Leb.meas.} \\ \text{on } \mathbb{R}^{d-1}}} (dy)$$

Remark 5.4.22. *The stochastic solution of the Dirichlet problem leads us to the harmonic measure* μ_x^D *= exit distribution of the Brownian Motion. For the Poisson problem we get* Green function = residence density of the BM in *D. We want to show this in the next section.*

5.5. Poisson equality and Green function

Assumption: $P_x[T_D < \infty] = 1\ \forall x \in D$. (D not necessarily bounded)
Have $f \in C_b(D)$. Want $u \in C^2(D) \cap C(\bar{D})$ with

$$\left.\begin{array}{rll} \frac{1}{2}\Delta u & = -f & \text{on } D \\ u & = 0 & \text{on } \partial D \end{array}\right\Vert \quad \text{Poisson problem with homog. boundary condition}$$

Remark 5.5.23. *The general case $u = g$ on ∂D can be traced back to the above by adding the solution of the Dirichlet problem with boundary condition g. We then know, because of 5.2.3, that*

$$u(x) = E_x\left[\int_0^T f(X_s)ds\right], \quad x \in D. \tag{5.5.2}$$

with $T := T_D$ $((\mathcal{F}_t)-$ stopping time with $\mathcal{F}_t := \bigcap_{\varepsilon>0} \{\sigma(X_s) | s \leq t+\varepsilon\})$.

Remark 5.5.24. *$p_t^D(x,A) := P_x[X_t \in A, t < T]$, $A \subseteq D$, measurable, defines a measure on D. It is obvious that*

$$p_t^D(x,\cdot) \leq \overbrace{p_t(x,\cdot)}^{B.sgr.} \ll dy \quad (= \textit{Lebesgue-measure on } \mathbb{R}^d)$$

$\implies \exists p_t^D(x,y) : p_t^D(x,dy) = p_t^D(x,y)dy.$

Definition 5.5.25.

$$G^D(x,y) := \int_0^\infty p_t^D(x,y)dt \quad , \ x,y \in D$$

is called the Green function *of D.*
Attention: only if $d \geq 3$, $G^D(x,y) < \infty\ \forall x \neq y$. If $d = 1,2$ and $D = \mathbb{R}^d$, $G^D(x,y) = \infty\ \forall x,y$.

Aim: Representation of $p_t^D(x,y)$ and thus of $G^D(x,y)$. Because then for u as in (5.5.2)

$$\begin{aligned} u(x) = E_x\left[\int_0^\infty f(X_s)1_{\{T>s\}}ds\right] &\underset{\text{Fubini}}{=} \int_0^\infty \underbrace{E_x[f(X_s); T > s]}_{\int f(y)p_s^D(x,y)dy} ds \\ &= \int f(y) \int_0^\infty p_s^D(x,y)ds\ dy \end{aligned}$$

Thus

Theorem 5.5.26. *$u(x) = \int G^D(x,y)f(y)dy$ solves Poission problem for D with boundary condition 0.*
If G^D "explicit", then u explicit.

Theorem 5.5.27. *(i)*

$$p_t^D(x,y) = \overbrace{p_t(x,y)}^{\textit{Density of the Br. sgr.}} - \underbrace{E_x[\overbrace{p_{t-T}(X_T,y)}^{\textit{cont. in } t};\ \overbrace{T<t}^{\textit{left cont. in } t}\ ]}_{\textit{left cont. (Lebesgue)}}, \quad x,y \in D$$

Note $|y - X_T| \geq d(y,\partial D) > 0$ since $y \in D$.

(ii) $p_t^D(x,y)$ is symmetric in x,y.

Proof. (i) $A \in \mathcal{B}(D)$

$$\begin{aligned}
P_x[X_t \in A,\ t < T] &= P_x[X_t \in A] - \underbrace{P_x[X_t \in A,\ T \le t]}_{\substack{=P_x[X_{t-T}\circ\theta_T\in A,\ \underbrace{T\le t}_{\in\mathcal{F}_T}] \\ \underset{5.4.18}{=} \int_{\{T<t\}} P_{X_{T(\omega)}}[X_{t-T(\omega)}\in A]dP_x(\omega)}} \\
&= \int\limits_A p_t(x,y)dy - \int\limits_{\{T<t\}} \int\limits_A p_{t-T(\omega)}(X_{T(\omega)}, y)dy\ dP_x(\omega) \\
&= \int\limits_A \left[p_t(x,y) - E_x[p_{t-T}(X_T,y),\ T<t]\right] dy
\end{aligned}$$

$\Longrightarrow$ (i)

(ii) By approximation: Let $\overline{D_n} \subset D$, $D_n \nearrow D$, $\overline{D_n}$ kp., $T_n := T_{D_n}$. Then $T_n \nearrow T$ and hence $\{T > t\} = \bigcup\limits_n \{T_n > t\}$. Let $\tau_m = m$-th dyadic decomposition of $[0, \infty[$.

$$\begin{aligned}
\Longrightarrow\ & P_x[X_t \in A,\ T > t] \\
&= \lim_n(\nearrow) P_x[X_t \in A,\ T_n > t] \\
&= \lim_n(\nearrow)\lim_m(\searrow) \underbrace{P_x[X_t \in A,\ X_{t_i}\in D_n\ \forall t_i \in \tau_m,\ t_i \le t]}_{=\int_A \underbrace{\int\limits_{D_n} \dots \int\limits_{D_n} p_{\underbrace{t_1}_{=\frac{1}{2}m}}(x,x_1)p_{\underbrace{t_2-t_1}_{=\frac{1}{2}m}}(x_1,x_2)\dots p_{\underbrace{t\_t_M}_{=\frac{1}{2}m}}(x_M,y)dx_1\dots dx_M}_{=:q_{n,m}(x,y)\text{ is symm., since } p_s(x',y')=p_s(y',x').}\, dy} \\
&\underset{\text{Levi}}{=} \int_A \underbrace{\lim_n(\nearrow)\lim_m(\searrow) q_{n,m}(x,y)}_{=p_t^D(x,y)\text{ symm. as limit of symm. ones}} \quad dy
\end{aligned}$$

□

Corollary 5.5.28. *Let $d \ge 3$.*

(i) $G^D(x,y) = G(x,y) - E_x[G(X_T,y)]$,
where

$$G(x,y) := 2^{1-\frac{d}{2}} \frac{\Gamma(\frac{d}{2}-1)}{(2\pi)^{\frac{d}{2}}} \cdot ||x-y||^{2-d} = \int_0^\infty p_t(x,y)dt. \qquad \textbf{\textit{Green function on }} \mathbb{R}^d$$

(ii) G^D *is symm. and* $G^D(\cdot, y)$ *is harmonic on* $D \setminus \{y\}$.

Proof. (i) Clear, by 5.5.27 and calculation of $\int_0^\infty p_t(x,y)dt$ (Excercise!)

(ii) Symm. clear. Also clear that $x \mapsto E_x[G(X_T,y)]$ harmonic on D and $\Delta G(\cdot,y) = 0$ on $\mathbb{R}^d \setminus \{y\}$ (Excercise!)

□

5.6. Self intersection of the paths

$d \geq 2$. $t \geq 0$ fix.

$$\begin{aligned} P_x[\text{Returning to } X_t] &= P_x[X_s = S_t \text{ for one } s > t] \\ &= \int 1_{\{\sigma_{X_t(\omega)}<\infty\}} \cdot \theta_t(\omega) dP_x(\omega) \\ &\underset{\substack{5.4.18 \\ X_t\ \mathcal{F}-\text{meas.}}}{=} \int \underbrace{P_{X_t(\omega)}[\sigma_{X_t(\omega)} < \infty]}_{\substack{=0\text{ , since pt. polar,} \\ \text{as } d\geq 2}} dP_x(\omega) = 0 \end{aligned} \tag{5.6.3}$$

Nevertheless we have P_x-a.s.

$d \leq 2$: $\forall n$ $\exists$ n-times self intersection, i.e.
$\exists\ t_1(\omega) < ... < t_n(\omega):\ X_{t_1(\omega)}(\omega) = ... = X_{t_n(\omega)}(\omega)$.

$d \leq 3$: the double points lie dense on path

$d = 3$: $\nexists$ triple points

$d \geq 4$: $\nexists$ double points.

We want to show now:

Theorem 5.6.29. $d = 3 \implies \forall x \in \mathbb{R}^3$

$$P_x[\textit{ double points lie dense on path }] = 1$$

Therefor we need some preparations.

Excursus to the capacity

$$d \geq 3: \quad G(x,y) := \frac{1}{||x-y||^{d-2}} \quad \text{(i.e. as before up to constants)}$$

Definition 5.6.30. *(i) Let μ be positive measure on $\mathbb{R}^d$.*

$$G\mu(x) := \int G(x,y)\mu(dy) \quad , \ x \in \mathbb{R}^d, \quad (\in [0,\infty])$$

is called the potential generated by μ *and*

$$e(\mu) := \int G\mu d\mu$$

the energy of μ.

(ii) For $K \subset \mathbb{R}^d$ kp.,

$$C(K) := (\inf\{e(\mu) | \mu \textit{ p.measure on } K\})^{-1}$$

is the capacity of K.

Remark 5.6.31. *(i) $C(\{x\}) = 0$ $\forall x \in \mathbb{R}^d$, since $e(\varepsilon_x) = \infty$.*

(ii) $C(K) > 0 \Leftrightarrow \exists \mu$ p.measure on $K:\ e(\mu) < \infty$

(iii) $C(K) = 0 \implies \lambda(K) = 0$ *($\lambda :=$ Leb. measure),*
($\not\Leftarrow$) *because:*

$$\lambda(K) > 0 \implies \underbrace{\int G\mu d\mu}_{=\frac{1}{\lambda(K)}\int \frac{1}{|x|^{d-2}} \underbrace{1_K \times 1_K}_{kp.\ support}(x)\lambda(dx)} < \infty \textit{ with } \mu := \lambda(K)^{-1}1_K \cdot \lambda$$

Theorem 5.6.32 ("equilibrium principle")**.** $C(K) > 0 \implies \exists$ *finite measure μ_K on K with:*

$$G_{\mu_K} \leq 1 \ , \quad = 1\mu_K \textit{ a.e. on } K$$

without proof.
In particular: $e(\mu_K) = ||\mu_K|| := \mu_K(K)$ and for $\tilde{\mu}_K := \frac{\mu_K}{||\mu_K||}$ (p.measure):

$$e(\tilde{\mu}_K)\frac{1}{||\mu_K||^2}e(\mu_K) = \frac{1}{||\mu_K||}$$

$\implies C(K) \geq ||\mu_K||$.

Theorem 5.6.33. $C(K) > 0 \implies h_K(x) := P_x[\overbrace{T_{K^c}}^{=\sigma_K} < \infty] > 0$ *on K^c (and in fact harmonic on K^c).*

Proof. $u := G\mu_K$ harmonic on K^c, since for $x \in B_r(x_0) \subset K^c$

$$\int G\mu_K(z)\mu_x^{B_r(x_0)}(dz) = \int_K \int \underbrace{G(z,y)}_{\substack{\text{harm. on} \\ \mathbb{R}^d\setminus\{y\}}} \mu_x^{B_r(x_0)}(dz)\mu_K(dy) = \int G(x,y)\mu_K(dy).$$

Clear: $0 < u \leq 1$ and $u(x) \leq ||\mu||_K \sup_{y\in K} G(x,y) = ||\mu||_K d(x,K)^{2-d} \longrightarrow 0$ for $||x|| \to \infty$.
Choose D_n rel. kp. $\bar{D}_n \subseteq K^c$, $D_n \nearrow K^c$, thus $T_n := T_{D_n} \nearrow T := T_{K^c}$. Hence $\forall x \in K^c$

$$\begin{aligned}
0 < u(x) &= \lim_{n\to\infty} E_x\left[u(X_{T_n})\right] \quad (u \text{ harmonice on } K^c) \\
&\leq E_x\left[\overline{\lim_n}\, u(X_{T_n})\right] \\
&= \underbrace{E_x\left[\overline{\lim_n}\, u(X_{T_n}); \lim_n T_n = \infty\right]}_{\substack{=0, \text{ since on } \{\lim T_n = \infty\}: \\ ||X_{T_n}|| \to \infty \text{ because of transience}}} + \underbrace{E_x\left[\overline{\lim_n}\, u(X_{T_n}); \sup_n T_n < \infty\right]}_{\substack{\leq P_x[\sup T_n < \infty] \\ = P_x[T_{K^c} < \infty]}}
\end{aligned}$$

$\implies P_x\left[T_{K^c} < \infty\right] \geq u(x) > 0.$ $\square$

End of excursus.

Theorem 5.6.34. *Let $0 \leq a < b$, $d = 3$. $\forall x \in \mathbb{R}^3$, we have for P_x-a.a. ω:*

$$C(\underbrace{\{X_T(\omega) | a \leq t \leq b\}}_{=:K_\omega}) > 0$$

Proof. $\mu_\omega :=$ Pushforward measure of $\underbrace{\lambda_{|[a,b]}}_{\substack{1-\text{ dim.}\\ \text{Leb. meas.}}}$ under $s \mapsto X_s(\omega)$.

("occupation measuere" for $[a,b]$)
μ_ω measure on K_ω and

$$e(\mu_\omega) = \int\int G(x,y)\mu_\omega(dx)\mu_\omega(dy) = \text{const.} \underbrace{\int_a^b\int_a^b ||X_s(\omega)-X_t(\omega)||^{2-d}dsdt}_{(*)}$$

$$< \infty \; P_x-\text{a.s., if } d=3, \text{ since:}$$

$$\begin{aligned}E_x[(*)] &= \int_a^b\int_a^b E_0\left[||X_s-X_t||^{2-d}\right]dsdt\\ &= \underbrace{E_0\left[||X_1||^{2-d}\right]}_{<\infty}\underbrace{\int_a^b\int_a^b \frac{1}{|t-s|^{\frac{d-2}{2}}}dsdt}_{(**)=\begin{cases}<\infty & , d=3\\ =\infty & , d>3\end{cases}}\\ &< \infty\end{aligned}$$

since

$$(**) = 2\int_a^b\int_t^b \frac{1}{(s-t)^{\frac{d-2}{2}}}dsdt = 2\int_a^b \underbrace{\int_0^{b-t} s^{1-\frac{d}{2}}ds}_{\substack{<\infty\\ \Leftrightarrow 1-\frac{d}{2}>-1\\ \Leftrightarrow 4>d}}\; dt$$

and then bounded in t. ☐

Proof of 5.6.29. Let $K_\omega := \{X_s(\omega)|0\le s\le 1\}$

$$(\omega,\eta)\mapsto T(\omega,\eta) := \sigma_{K_\omega}(\eta) \quad (:= \inf\{t>0|X_t(\eta)\in K_\omega\})$$

is $\mathcal{F}_1\otimes\mathcal{F}$-measurable,
since $\{T\le t\} \overset{!}{=} \bigcap_n \bigcup_{z\in(2^{-n}\mathbb{Z})^3}\{\sigma_{K_{2^{-n-1}}(z)}\le 1\}\times\{\sigma_{K_{2^{-n-1}}(z)}\le t\}\in\mathcal{F}_1\otimes\mathcal{F}_t$
where $K_{2^{-n-1}}(z)$ is a bounded ball around z with radius 2^{-n-1} in max. norm.

$$\begin{aligned}\Longrightarrow P_x&\left[X_t = X_s \text{ for an } s\in[0,1] \text{ and a } t\in]2,\infty[\;\right]\\ &= P_x[\sigma_{K_0}\circ\theta_2<\infty]\\ &= \int E_x[1_{\{\sigma_{K_0}<\infty\}}\circ\theta_2|\mathcal{F}_2](\omega)P_x(d\omega)\\ &\overset{5.4.18}{=} \int \underbrace{P_{X_2(\omega)}[\sigma_{K_\omega}<\infty]}_{P_x-\text{ a.s. }>0\text{, because:}} P_x(d\omega) > 0\end{aligned}$$

$$\begin{aligned}\int 1_{K_\omega}\circ X_2(\omega)P_x(d\omega) &= E_x[E_x[1_{K_0}\circ X_2(\cdot)|\mathcal{F}_1]]\\ &= \int E_{X_1(\omega)}[1_{K_\omega}\circ X_1]dP_x = 0\\ \text{as } E_y[1_{K_\omega}\circ X_1] = N(y,1)(K_\omega) &\underset{\substack{5.6.34,\\ 5.6.31(iii)}}{=} 0\end{aligned}$$

and thus, because of 5.6.34 and 5.6.31(iii), 5.6.33 can be used.
$\Longrightarrow$ $\exists N \geq 2$, (large) such that for

$$T := (2+\sigma_{K_0}\circ\theta_2)\wedge N\ ,$$

$(\mathcal{F}_t)$-stopping time (with $\mathcal{F}_t := \bigcap_{\varepsilon>0}\sigma\{X_s|s\leq t+\varepsilon\}$).

$$\begin{aligned}\alpha_T :&= P_x\left[X_t = X_s \text{ for an } s\in[0,1] \text{ and a } t\in]2,T]\right]\\ &= P_0[2+\underbrace{\sigma_{K_0}\circ\theta_2\leq N]}_{=:A_o\in\mathcal{F}_T} > 0\end{aligned}$$

(independent of x due to translation invariance)
Let $A_n := \{X_t = X_s \text{ for an s } \in [nT, nT+1] \text{ and a } t\ \in]nT+2,(n+1)T]\}$, $n\in\mathbb{N}$. Then $A_n = \theta_{nT}^{-1}(A_0)$, $A_n\in\mathcal{F}_{(n+1)T}$, thus

$$\begin{aligned}P_x[A_0\cap\ldots\cap A_n] &= E_x[\underbrace{P_x[A_n|\mathcal{F}_{nT}]}_{\substack{=P_{X_{nT}}[A_0]\\ =P_x[A_0]=\alpha_T}},\ A_0\cap\ldots\cap A_{n-1}] \underset{\text{Induction}}{=} \alpha_T^{n+1}\\ &= P_x[A_0]\ldots P_x[A_n] \implies \text{ independent}\end{aligned}$$

(as $P_x[A_n] = E_x[E_x[1_{A_0}\circ\theta_{nT}|\mathcal{F}_{nT}]] \underset{\substack{\text{see}\\ \text{above}}}{=} \alpha_T > 0$)

$\underset{\substack{\text{Borel-}\\ \text{Cantelli}}}{\Longrightarrow}$ ∞-many A_n occur

$\Longrightarrow$ P_x-a.e. path has ∞-many double points $\underset{\text{excercise}}{\Longrightarrow}$ density. ☐

5.7. Feyman-Kac-Formula

Let $V\in C(D)$. Poisson-problem for $\frac{1}{2}\Delta + V$:
Have: $g\in C_b(D),\ f\in C(\partial D)$
Want: $u\in C^2(D)\cap C(\bar{D})$ with

$$\begin{cases}\frac{1}{2}\Delta u + V\cdot u = -g & \text{on } D\\ u = f & \text{on } \partial D\end{cases} \tag{5.7.4}$$

Theorem 5.7.35. *If* $E_x[\exp((1+T)\int_0^T V^+(X_s)ds)] < \infty\ \forall x\in D$ *and* $T := T_D$*, then we have for* u *with (5.7.4)*

$$u(x) = E_x\left[f(X_T)\exp\left(\int_0^T V(X_s)ds\right)\right] + E_x\left[\ \underbrace{\int_0^T g(X_t)\exp\left(\int_0^T V(X_s)ds\right)dt}_{\textit{not obvious if } \in\mathcal{L}^1,\ \textit{but as } u(x)<\infty,\ \textit{needs to be}}\ \right],\ x\in D$$

Proof. Define $Z_t := u(X_t) \cdot A_t$ with $A_t := \exp(\int_0^t V(X_s)ds)$. ($C^1$!)

$$\underset{\substack{\text{Ito's}\\ \text{product rule}}}{\Longrightarrow} \qquad Z_t = Z_0 + \int_0^t u(X_s) \underbrace{dA_s}_{=V(X_s)A_s ds} + \int_0^t A_s du(X_s)\ , \quad t < T$$

with

$$\int_0^t A_s du(X_s) = \lim_n \sum_{\substack{t_i \in \tau_n\\ t_i \le t}} A_{t_i}(u(X_{t_{i+1}}) - u(X_{t_i}))$$

$$\underset{\text{Ito}}{=} \lim_n \underbrace{\sum_{t_i \in \tau_m} A_{t_i} \left(\int_{t_i}^{t_{i+1}} (\nabla u, dX_s) \right)}_{=:M_t^n} + \frac{1}{2} \int_0^t A_s \Delta u(X_s) ds$$

Let $\bar{D}_m \subset D_{m+1}$, D_m open, $D_m \nearrow D$. Then (M_t^n) is local martingale until T with local sequence $T_m := T_{D_m} \wedge m$. Since for $s \le T_{D_m} \wedge m,\ A_s \le \exp(m||V||_{D_m})$, it follows that

$$M_t := \lim_{n \to \infty} M_t^n\ ,$$

is local martingale with local sequence (T_m) (cf. poof of I.3.2, II.1.4).
Furthermore

$$Z_t = Z_0 + M_t + \int_0^t \underbrace{\left(\frac{1}{2} \Delta u + V u \right) (X_s)}_{-g(X_s)} A_s\ ds\ , \quad t < T$$

$$\Longrightarrow \underbrace{E_x[Z_{T_m \wedge t}]}_{\substack{m \to \infty\\ \text{then } t \to \infty\\ \longrightarrow E_x[Z_T]}} = \underbrace{E_x[Z_0]}_{u(x)} + \underbrace{E_x\left[- \int_0^{t \wedge T_m} g(X_s) A_s\ ds \right]}_{\substack{m \to \infty\\ \text{then } t \to \infty\\ \longrightarrow E_x[- \int_0^T g(X_s) A_s ds]}}$$

since $|Z_{T_m \wedge t}| \le ||u||_\infty \exp(\int_0^T v^+(X_s)ds) \in \mathcal{L}^1$. ☐

Example 5.7.36.

$$\frac{1}{2} \Delta u + \alpha u = -g \quad \text{on } D$$
$$u \equiv 0 \quad \text{on } \partial D$$

At least if $\alpha < \varepsilon = \varepsilon(D)$, ε *so small (cf. III 3.3 3)), such that* $E_x\left[e^{\varepsilon T}\right] < \infty$,$x \in D$. *Then*

$$u(x) = E_x \left[\int_0^{T_D} e^{\alpha T} g(X_t) dt \right]$$

5.8. The heat equation

$D = \mathbb{R}^d \times]0, T]$.
Have: $f \in C(\mathbb{R}^d)$ with $\exists b, c > 0 :\ |f(x)| \le b e^{c||x||^2}\ \forall x \in \mathbb{R}^d$ $(*)$
Want: $v \in C^2(D)$, continuous on $\mathbb{R}^d \times [0, T]$

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2} \Delta v \text{ on } D \quad (\Delta \text{ on } \mathbb{R}^d) \\ v(\cdot, 0) = f \text{ on } \mathbb{R}^d \end{cases} \tag{5.8.5}$$

Set $v(x,t) := \underset{\substack{\uparrow \\ \text{Br.sgr.}}}{p_t} \; f(x) = E_x[f(X_t)] = \frac{1}{\sqrt{2\pi t}^d} \underbrace{\int f(y) e^{\frac{-\|y-x\|^2}{2t}} dy}_{\text{Exists only if } (*) \text{ is fulfilled}}$, $(x,t) \in D$.

Then v is a solution of (5.8.5) if $T < \frac{1}{2c}$ (excercise). $\implies$ existence

Dual heat equation on $\mathbb{R}^d \times [0,T]$:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\Delta u & = 0 \text{ on } \mathbb{R}^d \times [0,T[\\ u(\cdot,T) & = f \text{ on } \mathbb{R}^d \end{cases} \qquad \left(\begin{matrix} u \in C^2(D),\ u \text{ cont. on } \mathbb{R}^d \times [0,T]; \\ f \in C(\mathbb{R}^d) \end{matrix} \right) \tag{5.8.6}$$

Solution via time reversal on T, i.e.: Let v be solution of (5.8.5), then

$$\begin{aligned} u(x,t) &:= v(x, T-t) \quad \text{solution of (5.8.6)} \\ &\underset{\text{if } T<\frac{1}{2c}}{=} E_x[f(X_{T-t})] \ , \quad (x,t) \in \mathbb{R}^d \times [0,T] \end{aligned}$$

$\implies$ existence for 5.8.6

Theorem 5.8.37 (Uniqueness)**.** *Let $f \in C(\mathbb{R}^d)$ with $(*)$ and $T < \frac{1}{2c}$. Then $\exists!$ solution $u \in C^2(D)$ of $\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0$ on $\mathbb{R}^d \times [0,T[$ with*

(i) $\lim\limits_{\substack{t \nearrow T \\ y \to x}} u(y,t) = f(x)$

(ii) $\sup 0 \le t < T |U(x,t)| \le K e^{a\|x\|^2}$; $K,\ a > 0$.

, and $u(x,t) = E_x[f(X_{T-t})]$ is this solution.

Proof. $u(x,t) := E_x[f(X_{T-t}] = p_{T-t}f(x)$ fulfilles (i), (ii) (Exercise!)
(with $K = b$, $a = \frac{c}{1-2cT}$, $|p_{T-t}f(x)| \le$ const $e^{\frac{c\|x\|^2}{1-2c(T-t)}}$)
Uniqueness: Let u be solution of (5.8.6) with (i), (ii). Without loss of generality : $f = 0$.
Need to show: $u(x,t) = 0$

$$u(X_x, t+s) = u(X_0,t) + \underbrace{\int_0^s \nabla u(X_r,r) dX_r}_{\substack{=:M_s \text{ (Def.)} \\ \text{but: local martingale (cf. proof of 5.3.9)}}} + \int_0^s \underbrace{\left(\frac{1}{2}\Delta + \frac{\partial}{\partial r}\right) u(X_r,r)\, dr}_{=0, \text{ since } r<s<T-t<T}$$

Define

$$R_n := \inf\{s > 0 |\ |X_s| > n\} \ , \ n \in \mathbb{N},$$

and let $t_k \nearrow T - t$, $t_k < T - t$.

$$\begin{aligned} \underset{\substack{\text{stopping theorem} \\ \text{+ remark II.1.5}}}{\implies} \quad u(x,t) &= E_x[u(X_{R_n \wedge t_k},\ t + R_n \wedge t_k)] \\ &= \underbrace{E_x[u(X_{t_k},\ t+t_k);\ R_n > t_k]}_{=:I_{k,n}} + \underbrace{E_x[u(X_{R_n},\ t+R_n);\ R_n \le t_k]} \\ & \qquad \qquad \underset{(ii)}{\le} K e^{a\|x_{R_n}\|^2} P_x[R_n \le t_k] \\ & \qquad \qquad \underset{t_k \nearrow T-t}{\longrightarrow} K e^{an^2} P_x[R_n < T-t] \\ & \qquad \qquad \text{since } 1_{\{R_n \le t_k\}} \nearrow 1_{|R_n < T-t|} \end{aligned}$$

since $I_{k,n} \underset{t_k \nearrow T-t}{\longrightarrow} E_x[f(X_{T-t})\ ;\ R_n \geq T-t] = 0\ \forall n$ because of (i), (ii) and because of $1_{\{R_n > t_k\}} \searrow 1_{\{R_n \geq T-t\}}$

Let $Q_n := \left\{ x \in \mathbb{R}^d | \max\limits_{1 \leq i \leq d} |x_i| \leq \frac{n}{\sqrt{d}} \right\} \subset K_n(0)$

$$\begin{aligned}\implies P_x[R_n < T-t] &\leq \sum_{j=1}^{d} P_x\left[\max_{0 \leq s \leq T-t} X_s^d \geq \frac{n}{\sqrt{d}}\right] + P_x\left[\max_{0 \leq s \leq T-t} (-X_s^j) \geq \frac{n}{\sqrt{d}}\right] \\ &\underset{5.4.17}{=} \sum_{j=1}^{d} 2P_x\left[X_{T-t}^j \geq \frac{n}{\sqrt{d}}\right] + 2P_x\left[-X_{T-t}^j \geq \frac{n}{\sqrt{d}}\right] \\ &\leq \text{const.} \sum_{j=1}^{d}\left[e^{-\left(\frac{n}{\sqrt{d}} - x_j\right)^2/2(T-t)} + e^{-\left(\frac{n}{\sqrt{d}} - x_j\right)^2/2(T-t)}\right]\end{aligned}$$

where the last inequality holds because of $\int_a^\infty e^{-\frac{x^2}{2}} dx \leq a^{-1} e^{-\frac{a^2}{2}}$

$\implies Ke^{an^2} P_x[R_n < T-t] \underset{n\to\infty}{\longrightarrow} 0$, if $\frac{1}{2d(T-t)} > a$.

$\implies u(x,t) = 0$, if $t > T - \frac{1}{2ad}$.

If we apply the above to $T_1 := T - \frac{1}{3ad}$, we get

$$u(x,t) = 0\ , \quad \text{if } t > T_1 - \frac{1}{2ad} = T - \frac{1}{3ad} - \frac{1}{2ad}$$

It follows with iteration

$$u(x,t) = 0 \quad \forall t \in [0,T]\ , \quad x \in \mathbb{R}^d.$$

□

More general:

$D \subset \mathbb{R}^d \times [0,\infty[$. We want solution of

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{1}{2}\Delta) u &= 0 \quad \text{on } D \\ u &= f \quad \text{on } \partial D \end{cases} \tag{5.8.7}$$

Idea: Interpretation as *Dirichlet problem for the "space-time process".*

We define for $r \geq 0$, $\sigma_r : [0,\infty[\to [0,\infty[$ by $\sigma_r(s) := r + s$.

$$\begin{aligned}\text{Let } \bar{\Omega} :&= \Omega \times \{\sigma_r | r \in [0,\infty[\} \\ \bar{\omega} &= (\omega, \sigma_r) \quad \text{(sarting point } t) \\ \bar{X}_s(\bar{\omega}) &= (X_s(\omega), r+s) \in \mathbb{R}^d \times [0,\infty[\\ P_{(x,t)} :&= P_x \otimes \delta_{\sigma_r}.\end{aligned}$$

If we have again everything "enough" bounded, we get

$$u(x,t) = E_{(x,t)}[f(\bar{X}_{\bar{T}})]$$

with $\bar{T}(\bar{\omega}) := \inf\{t > 0 | \bar{X}_T(\bar{\omega}) \notin D\}$.

In the special case above: $D := \mathbb{R}^d \times [0,T[$ with $f : \partial D = \mathbb{R}^d \times \{T\} \to \mathbb{R}$ we have

$$\bar{T} = T - t\ P_{(x,t)} - \text{ a.s., i.e.}$$

$$\begin{aligned}u(x,t) &= E_{(x,t)}[f(X_{T-t}, \sigma_0(T-t))] \\ &= E_x[f(X_{T-t};\ T)]\end{aligned}$$

Hence as before.

6. Stochastic Differential Equations

6.1. Solution definitons, examples

We consider the following SDE:

$$\begin{cases} X_0 = \xi_0 \quad \text{(initial condition)} \\ dX_t = \sigma_t(X)dW_t + b_t(X)dt \end{cases} \tag{6.1.1}$$

For the sake of simplicity we will mostly consider only the Markov case:

$$\begin{cases} \sigma_t(x) = \sigma(X_t, t) \\ b_t(X) = b(X_t, t) \end{cases} \tag{6.1.2}$$

and only in $\mathbb{R}^1$ (thus $X_t \in \mathbb{R}$), where we assume that

$$\sigma = \sigma(x,t) \ , \quad b = b(x,t) \ \text{ Borel measurable on } \mathbb{R} \times [0, \infty[. \tag{6.1.3}$$

Let the right continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ be as in the following definitions:

$$\mathcal{F}_t \supset \ \text{completion of } \bigcap_{\varepsilon > 0} \sigma(\xi_0,\ W_s;\ s \leq t + \varepsilon) \quad \text{ with respect to } P \text{ in } \mathcal{F},$$

where $\xi_0 : \Omega \to \mathbb{R}$, $\mathcal{F}$-measurable.

Definition 6.1.1. *(i) A* weak solution *of (6.1.1) is a triple (X, W), $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \geq 0}$ with: W $(\mathcal{F}_t)$-BM on $(\Omega, \mathcal{F}, P)$, X $(\mathcal{F}_t)$-adapted, such that P-a.s.*

$$X_t = \xi_0 + \underbrace{\int_0^t \sigma_s(X) dW_s}_{\textit{in part. } \exists} + \underbrace{\int_0^t b_s(X) ds}_{\textit{in part. } \exists} \quad \forall t \geq 0 \ \textit{(resp. } \forall t < \zeta : \ \textit{stopping time).} \tag{6.1.4}$$

(ii) This solution is called weakly unique*, if the distribution of ξ_0 (on $\mathbb{R}$) determines uniquely the distribution of X.*

Definition 6.1.2. *(i) (6.1.1) has a* strong solution*, if $\exists\ F : \mathbb{R} \times C([0,\infty[)_0 \to C([0,\infty[)_0$, such that $F(x,\cdot)$ is $\mathcal{F}_t^0/\mathcal{F}_t^0$-measurable $\forall x, t$ and $\forall$ $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)$, and all $(\mathcal{F}_t)$-BM's W,*

$$X := F(\xi_0, W)$$

fulfills (6.1.4) ("F is the solution"). Here $\mathcal{F}_t^0 := \sigma(\pi_s | s \leq t)$ with $\pi_s : C([0,\infty[)_0 \to \mathbb{R}$, $\pi_s(\omega) = \omega(s)$.

(ii) This solution is called strong unique*, if $\forall (X, W)$, $\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \geq 0}$, ξ_0, which fulfill (6.1.4)*

$$X = F(\xi_0, W)$$

("in other words F is uniquely determined") ($\implies$ weak uniqueness)

Remark 6.1.3. *$\mathcal{F}_t/\mathcal{F}_t$-measurability of $F(x,\cdot)$ implies: X adapted*

Example 6.1.4. *(i) The Ornstein-Uhlenbeck process is a strong (unique) solution of $dX = dW - \alpha X dt$ (cf. solution formula in Section 4.1)*

(ii) $dX = \alpha X dW + \beta X dt$ has as strong (unique) solution

$$X_t := X_0 \exp\left[\alpha W_t + (\beta - \frac{1}{2}\alpha^2)t\right]$$

(Heuristically: Itô $\implies$

$$\begin{aligned}\ln X_t &= \ln X_0 + \int_0^t \frac{1}{X_t} dX_t - \frac{1}{2}\int_0^t \frac{1}{X_t^2} \overbrace{d\langle X\rangle_t}^{=\alpha^2 X_t^2 d\langle W\rangle_t} \\ &= \ln X_0 + \alpha W_t + \beta t - \frac{1}{2}\alpha^2 t \\ \implies X_t &= X_0 \exp[\alpha W_t + (\beta - \frac{1}{2}\alpha^2)t]\end{aligned}$$

except for "X_t could become 0" $\approx$ proof of existence.)

Proof. strong solution: Check with Itô $\forall X_0$, W.
Uniqueness: Let (X, W, X_0) with (ii). Show with Itô:

$$X_t \exp\left[-\alpha W_t - \left(\beta - \frac{1}{2}\alpha^2\right)t\right] = X_0$$

☐

(iii) Tanaka's example (shows that one is possibly forced to look for a weak solution).

$$\begin{cases} X_0 &= 0 \\ dX &= sign(X)dW \end{cases} \text{, where } sign(x) := \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$$

($sign(x) \neq 0$, according to our definition, excludes the trivial solution $X \equiv 0$!)
Claim: $\exists!$ *weak solution.*

Proof. weak uniqueness: Let X be a solution $\implies$

$$\begin{aligned} X_t &= \int_0^t sign(X_s)dW_s \text{ is martingale} \\ \langle X\rangle_t &= \int_0^t \underbrace{(sign(X_s))^2}_{=1} ds = t. \end{aligned}$$

$\underset{\text{Levy}}{\implies}$ X is a Wiener process.
Existence: Let X be a Wiener process

$$\begin{aligned} \implies W_t &= \int_0^t sign(X_s)dX_s \quad \text{is a Wiener process} \\ \implies X_t &= \int_0^t \underbrace{sign(X_s)sign(X_s)}_{=1} dX_s \\ &= \int_0^t sign(X_s)dW_s \end{aligned}$$

(by Cor. 2.4.34) is a weak solution.
One can show that $\nexists$ strong solution (cf. [KS91]). ☐

6.2. Construction of solutions via transformation of the state space

Example 6.2.5. *(i) $X_t := W_t^2$ fulfills (according to Itô)*

$$dX_t = 2W_t dW_t + dt$$

Let

$$\tilde{W}_t := \int_0^t sign W_s dW_s \quad \text{Wiener process}$$

$$\implies 2W_t dW_t = 2|W_t| sign W_s dW_s = 2|W_t| d\tilde{W}_t = 2\sqrt{X_t} d\tilde{W}_t,$$

i.e. $dX_t = 2\sqrt{X_t} d\tilde{W}_t + dt$ has as weak solution $X_t := W_t^2$.

X large: strong diffusion! Although its drift is +1, X will be going back to 0 again and again (since it is the square of BM and therefore the law of the iterated log holds.)

(ii) ***Bessel process*** $||W||$

$X := ||W|| = \sqrt{\sum_{i=1}^n (W^i)^2}$, *$W = (W^1, ..., W^n)$, BM in $\mathbb{R}^n$ with start $W_0 = 0$.*

Let

$$Y := X^2 = \sum_{i=1}^n (W^i)^2 \quad (\implies X = \sqrt{Y})$$

$$\underset{\substack{multdim.\\ Itô}}{\implies} \quad dY = \sum 2W^i dW^i + ndt$$

and

$$d\langle Y\rangle_t = \sum 4(W^i)^2 dt = 4Y dt.$$

Hence $\tilde{W}_t := \int_0^t \frac{1}{2\sqrt{Y}} dY - \frac{nt}{2\sqrt{Y_t}} = \int_0^t \frac{1}{\sqrt{Y}} W^i dW^i$ is a local martingale with $\langle \tilde{W}\rangle_t = \int_0^t \frac{1}{4Y_s} \underbrace{d\langle Y\rangle_s}_{4Y_s ds} = t$, hence $\tilde{W}$ BM, since $\tilde{W}$ is a local martingale.

Apply Itô-formula to $X = \sqrt{Y}$ (ok for $n \geq 2$, since $0 \in \mathbb{R}^n$ polar, i.e. $Y_t > 0\ \forall t > 0$ P-a.s.) to get

$$\begin{aligned} dX &= \frac{1}{2\sqrt{Y}} dY + \frac{1}{2}\left(-\frac{1}{4} Y^{-\frac{3}{2}} d\langle Y\rangle\right) \\ &= d\tilde{W} + \frac{1}{2\sqrt{Y}} ndt - \frac{1}{8} Y^{-\frac{3}{2}} 4Y dt \\ &= d\tilde{W} + \frac{n-1}{2X} dt. \end{aligned}$$

That is, the Bessel process $X := ||W||$ is a weak solution of

$$dX = d\tilde{W} + \frac{n-1}{2X} dt \quad \text{(strongly drifting away from 0!)}$$

Theorem 6.2.6 (Zvonkin)**.** *Let $b : \mathbb{R} \to \mathbb{R}$ be Borel measurable, bounded. Then*

$$dX_t = dW_t + b(X_t) dt$$

has a strong solution.

Proof. **Idea**: Transform X into a local martingale $Y := \varphi(X)$ (for some to be found transformation $\varphi : \mathbb{R} \to \mathbb{R}$).
Ansatz: Let X be a solution of $dX = dW + b(X)\ dt$. Define

$$\begin{aligned} Y_t :&= \varphi(X_t) \\ \underset{It\tilde{o}}{\Longrightarrow}\ dY &= \varphi'(X)dX + \frac{1}{2}\varphi''(X)dt \\ &= \varphi'(X)dW + \{\varphi'(X)b(X) + \frac{1}{2}\varphi''(X)\}dt \end{aligned}$$

Choose φ such that $\{\varphi'(X)b(X) + \frac{1}{2}\varphi''(X)\} = 0$, so Y is a local martingale and we get rid of the (only) measurable b! I.e.

$$\begin{aligned} \varphi'' &= -2b\varphi' \\ \Longrightarrow\ \varphi'(y) &= C \cdot \exp\left[-\int_0^y 2b(z)dz\right] \quad (>0) \\ \Longrightarrow\ \varphi(x) :&= C\int_0^x \exp\left[-\int_0^y 2b(z)dz\right] dy + \tilde{C} \qquad \begin{matrix}\text{(strongly increasing, } C^1\text{)} \\ (\Longrightarrow \exists \varphi^{-1} : C^1)\end{matrix} \end{aligned}$$

Show: $\varphi(x) = \int_0^x \exp\left[-\int_0^y 2b(z)dz\right] dy$ is as desired.
Sketch: Then $dY = \varphi'(\underbrace{\varphi^{-1}(Y)}_{=X})dW$.
Let $\sigma(y) := \varphi'(\varphi^{-1}(Y))$, hence

$$dY = \sigma(Y)dW \tag{*}$$

Note that

$$\begin{aligned} \sigma(y) &= \exp\left[-\int_0^{\varphi^{-1}(y)} 2b(z)dz\right] \\ \Longrightarrow\ \sigma'(y) &= -\sigma(y)\ 2b(\varphi^{-1}(y)) \qquad \underbrace{\varphi^{-1}(y)'}_{=(\varphi'(\varphi^{-1}(y)))^{-1}=\sigma(y)^{-1}} \qquad = -2b(\varphi^{-1}(y)) \end{aligned}$$

Thus σ' globally bounded, hence σ globally Lipschitz continuous:

$$|\sigma(x) - \sigma(y)| \leq 2||b||_\infty |x-y|$$

Thus in $(*)$ we have more regular coefficients than in the original equation. According to the results in Section 6.4 below, $(*)$ has a unique strong solution Y (as σ is Lipschitz).

$$\Longrightarrow\ X = \varphi^{-1}(Y) \quad \text{(This even gives uniqueness!)}$$

Problem: φ^{-1} only C^1, but not C^2! Hence attention with Itô, but φ' is still weakly C^1.
Now rigorously: Solve $(*)$ and set

$$X := \varphi^{-1}(Y)$$

Then check by Itô that X solves $dX_t = dW_t + b_t dt$. ☐

Now more general:

$$dX = \sigma(X)dW + b(X)dt \tag{*}$$

6.3. Method of Lamperti-Doss-Sussmann

Lamperti ca. 1964
Doss-Sussmann end of the 70th
Surprisingly: One can solve $(*)$ **pathwise**!
For simplicity here only for $\sigma,\ b$ sufficiently smooth and for the Markoff case:

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt. \tag{6.3.5}$$

We want to rewrite (6.3.5) with respect to the **Stratonovich integral**: $\int \sigma(X)\circ dW$.

Definition 6.3.7. *Assume X, Y are (local) semimartingales. Then the* Stratonovich integral *of X against Y is defined as*

$$\int_0^t X_s \circ dY_s := \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t,\ t \geq 0.$$

If we apply this to the semimartingales $\sigma(X)$ and W we obtain

$$\begin{aligned}\int_0^t \sigma(X_s)\circ dW_s &= \int_0^t \sigma(X_s)dW_s + \frac{1}{2}\langle \sigma(X), W\rangle_s \\ &= \int_0^t \sigma(X_s)dW_s + \frac{1}{2}\int_0^t \sigma'(X_s)d\langle X, W\rangle_s \\ \text{(since } d\sigma(X) = \sigma'(X)dX + \frac{1}{2}&\sigma''(X)d\langle X\rangle) \\ &\underset{(6.3.5)}{=} \int_0^t \sigma(X_s)dW_s + \frac{1}{2}\int_0^t \sigma'(X_s)\sigma(X_s)ds.\end{aligned}$$

or in differential form

$$\sigma(X)\circ dW = \sigma(X)dW + \frac{1}{2}(\sigma'\sigma)(X)dt.$$

Recall. In Section 1.2 we looked at another special case of Definition 6.3.7, namely for the semimartingales $f(X)$ and X (see p. 5) to get

$$\begin{aligned}\int_0^t f(X)\circ dX &= \int_0^t f(X)dX + \frac{1}{2}\langle f(X), X\rangle_t \\ &= \int_0^t f(X)dX + \frac{1}{2}\int_0^t f'(X)d\langle X\rangle\end{aligned}$$

(since $df(X) = f'(X)dX + \frac{1}{2}f''(X)d\langle X\rangle$.) Hence

$$dF(X) = F'(X)\circ dX.$$

Let $b^*(X_t) := \{b(X_t) - \frac{1}{2}\sigma'(X_t)\sigma(X_t)\}$, then (6.3.5) can be written as

$$dX_t = \sigma(X_t)\circ dW_t + b^*(X_t)dt \tag{6.3.6}$$

Let $\Phi(x,t)$ be the **flow** corresponding to the ODE

$$\dot{x} = \sigma(x)$$

i.e.

$$\begin{aligned}\Phi(x,0) &= x \\ \Phi_t := \frac{d}{dt}\Phi(x,t) &= \sigma(\Phi(x,t)) \\ \implies \Phi_{tt} := \frac{d^2}{dt^2}\Phi(x,t) &= \sigma'(\Phi(x,t))\frac{d}{dt}\Phi(x,t) = (\sigma'\sigma)(\Phi(x,t))\end{aligned}$$

Ansatz: $X_t = \varPhi(\xi_t, \eta_t)$ solves (6.3.6) with

$$\begin{aligned} \xi_0 &= X_0 \quad (= X_0(\omega)) \\ d\xi &= \dot{\xi} dt \quad (\text{in part. } t \mapsto \xi_t(\omega) \text{ is } C^1) \\ (*)\ d\eta_t &= dW_t + \underbrace{\tilde{b}^{X_0}}_{\text{to be det.!}}(\eta_t)dt, \ \eta_0 = 0 \end{aligned}$$

$$\begin{aligned} \xRightarrow[\text{2 dim. Itô}]{} dX_t &= \varPhi_x(\xi_t, \eta_t)d\xi_t + \varPhi_t(\xi_t, \eta_t)d\eta_t + \frac{1}{2}\varPhi_{tt}(\xi_t, \eta_t)dt \\ &= \underbrace{\sigma(X_t)dW_t + \frac{1}{2}(\sigma'\sigma)(X_t)dt}_{\sigma(X_t)\circ dW_t} + \underbrace{\left\{\varPhi_x(\xi_t, \eta_t)\dot{\xi} + \sigma(\varPhi(\xi_t, \eta_t))\tilde{b}^{X_0}(\eta_t)\right\} dt}_{\overset{!}{=} b^*(\underbrace{\varPhi(\xi_t, \eta_t)}_{X_t})} \end{aligned}$$

There are two possibilities to simplify :

$$\dot{\xi} = 0 \quad \text{ or } \quad \tilde{b}^{X_0} = 0$$

1st possibility: Lamperti

Let $\xi_t \equiv X_0$, thus $\dot{\xi} = 0$. Hence it should hold that

$$\tilde{b}^{X_0}(y) = \frac{b^*}{\sigma}(\varPhi(X_0, y)) \quad \forall y \in \mathbb{R}.$$

Then the problem is reduced to solve

$$\begin{aligned} d\eta_t &= dW_t + \frac{b^*}{\sigma}(\varPhi(X_0, \eta_t))dt \\ \eta_0 &= 0 \end{aligned} \qquad (*)$$

(ok, if (e.g.) $X_0 = x_0 \in \mathbb{R}$, and σ, b are such that $\frac{b^*}{\sigma}(\varPhi(x_0, \cdot))$ globally Lipschitz; cf. Section 6.4 below.)

$$\Longrightarrow \ X_t = \varPhi(X_0, \eta_t) \text{ solves (6.3.6).}$$

(strong solution, if $(*)$ has strong solution)

2nd possibility: Doss-Sussmann

$$\tilde{b}^{X_0} \equiv 0, \qquad \text{i.e.: } \eta_t = W_t$$

$$\Longrightarrow \ \dot{\xi}_t = \varPhi_x^{-1}(\xi_t, W_t(\omega))b^*(\varPhi(\xi_t, W_t(\omega)) \quad \forall \omega \text{ fixed, that is:}$$

$$\begin{aligned} \dot{\xi} &= F_\omega(\xi_t, t) \\ \xi_0(\omega) &= X_0(\omega) \quad , \ \omega \text{ fixed} \end{aligned}$$

Hence $\xi_t = \xi_t(\omega)$, but "smooth" in t for ω fixed. Thus the problem reduces to solve a family of certain differential equations. It follows:

$$X_t := \varPhi(\xi_t, W_t) \text{ solves (6.3.6)}$$

Special case for both:

$$dX = \sigma(X) \circ dW$$

(i.e. $b^* \equiv 0$)

1. Lamperti: $d\eta = dW$, hence $X_t = \Phi(X_0, W_t)$
2. Doss-Sussmann: $\dot{\xi}_t = 0$, $\xi_t \equiv X_0$, hence $X_t = \Phi(X_0, W_t)$

Example 6.3.8. $\sigma(x) = x^2$ *(and* $b^* \equiv 0$*)*

$$\implies \Phi(x,t) = \frac{1}{\frac{1}{x} - t}$$

According to the special case $X_t = \Phi(X_0, W_t)$ solves

$$dX_t = \sigma(X_t) \circ dW_t \quad \text{(Stratonovich-DE!)}$$

But explosion at time $t_0 := \inf\{t > 0 | W_t = \frac{1}{x_0}\}$.
Consider now for $\sigma(x) = x^2$

$$\begin{aligned} dX_t &= \sigma(X) dW \quad \text{(Itô-DE!)} \\ &= \sigma(X) \circ dW - \frac{1}{2} \underbrace{\sigma' \sigma(X)}_{2 \cdot X^3} dt \end{aligned}$$

Then the strong drift X^3 prevents an explosion, because due to Lamperti we get, with initial condition $X_0 = x_0$,

$$X_t = \frac{1}{\frac{1}{x_0} - \eta_t},$$

where

$$\begin{aligned} d\eta_t &= dW_t + \underbrace{\frac{b^*}{\sigma}}_{-id} (\Phi(x_0, \eta_t)) dt \\ &= dW_t + \frac{1}{\eta_t - \frac{1}{x_0}} dt, \end{aligned}$$

and $\eta_0 = 0$.
Solution: $\eta =$ Bessel process in $\mathbb{R}^3$ at $\frac{1}{x_0}$.
($\frac{1}{x_0}$ polar for BM in $\mathbb{R}^3$, hence also η doesn't hit $\frac{1}{x_0}$, thus (X_t) doesn't explode in finite time.)

6.4. Construction of strong solutions on Lipschitz conditions (Picard-Lindelöf method)

In this section let $d \in \mathbb{N}$ arbitrarily, $G : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d^2}$, $b : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ product measurable and so $\mathcal{F}_t$-adapted that $\exists K > 0$ with

$$\begin{aligned} &||G(\omega,t,x) - G(\omega,t,y)|| + ||b(\omega,t,x) - b(\omega,t,y)|| \le K||x-y|| \\ &\text{and } ||G(\omega,t,x)|| + ||b(\omega,t,x)|| \le K(1 + ||x||) \text{ for all } \omega \in \Omega,\ t \in \mathbb{R}_+,\ x, y \in \mathbb{R}^d. \end{aligned} \tag{6.4.7}$$

Let W be a given $\mathbb{R}^d$-valued Wiener process on $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)$.

Theorem 6.4.9. *For an arbitrary initial condition* $\xi_0 \in \mathcal{L}^2(\Omega; \mathbb{R}^d)$*,* $\mathcal{F}_0$*-measurable, there exists exactly one strong solution* X *of the stochastic differential equation*
(here: $G(\cdot, t, x) =: G(t,x)$*,* $b(\cdot, t, x) =: b(t,x)$*)*

$$dX_t = G(t, X_t) dW_t + b(t, X_t) dt,\ X_0 = \xi_0, \tag{6.4.8}$$

i.e., P-a.s.

$$X_t = \xi_0 + \int_0^t G(s, X_s) dW_s + \int_0^t b(s, X_s) ds \quad \forall t \geq 0$$

(in particular all integrals on the right hand side exist in the usual sense).
In particular, X has continuous sample paths.
Furthermore we have for every $T > 0$

$$E \left[\sup_{0 \leq t \leq T} ||X_t||^2 \right] \leq C \left(1 + E \left[||\xi_0||^2 \right] \right),$$

where the constant C only depends on K and T.

For the proof we need the following version of the Banach fixed-point theorem:

Theorem 6.4.10. *Let (E, d) be a complete metric space, $\Lambda : E \to E$ and $c_n \in [0, \infty[$, $n \geq 1$, with $\sum_{n \geq 1} c_n < +\infty$ fixed, such that*

$$d(\Lambda^n x, \Lambda^n y) \leq c_n d(x, y) \quad \forall x, y \in E, \ n \geq 1. \tag{6.4.9}$$

Then Λ has exactly one fixed-point x and we have for all $y \in E$

$$d(x, y) \leq (1 + \sum_{n \geq 1} c_n) d(\Lambda y, y).$$

Proof. Let $x_0 \in E$, $x_n := \Lambda^n x_0$. Then

$$\begin{aligned} d(x_{n+1}, x_n) = d(\Lambda^n x_1, \Lambda^n x_0) &\leq c_n \cdot d(x_1, x_0) \\ \underset{n>m}{\Longrightarrow} d(x_n, x_m) &\leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \leq (\sum_{k \geq m} c_k) d(x_1, x_0) \underset{m \nearrow \infty}{\to} 0. \\ \underset{E \text{ complete}}{\Longrightarrow} \exists x : &= \lim_{n \to \infty} x_n \text{ and } \Lambda x = \lim_{n \to \infty} \Lambda^{n+1} x_0 = x. \end{aligned}$$

If $\tilde{x} \in E$ is another fixed-point, we get

$$d(x, \tilde{x}) = d(\Lambda^n x, \Lambda^n \tilde{x}) \leq c_n d(x, \tilde{x}) \underset{n \to \infty}{\to} 0, \quad \text{i.e. } x = \tilde{x}.$$

If $y \in E$, we get

$$\begin{aligned} d(x, y) = \lim_{n \to \infty} d(\Lambda^n y, y) &\leq \overline{\lim_{n \to \infty}} \sum_{k=0}^{n-1} d(\Lambda^{k+1} y, \Lambda^k y) \\ &\leq \overline{\lim_{n \to \infty}} (1 + \sum_{k=1}^{n-1} c_k) d(\Lambda y, y) = (1 + \sum_{k \geq 1} c_k) d(\Lambda y, y). \end{aligned}$$

□

Proof of 6.4.9. Let $E := L^2(\Omega; C([0, T]; \mathbb{R}^d))$ and $|X|_T := E[\sup_{0 \leq t \leq T} ||X(t)||^2]^{\frac{1}{2}}$. Then $(E, |\cdot|_T)$ is complete, and for $X : \Omega \to C([0, T]; \mathbb{R}^d)$ define

$$(\Lambda X)(t) := \xi_0 + \underbrace{\int_0^t G(s, X(s)) dW_s}_{=: I_t} + \int_0^t b(s, X(s)) ds, \quad t \in [0, T].$$

1st step: $\Lambda E \subseteq E$.

Since $\Lambda X(\omega) \in C([0,T];\mathbb{R}^d)$, $E[\sup\limits_{0\le t\le T} I_t^2] \overset{\text{Doob}}{\le} 4E[I_T^2] \overset{\text{isometry}}{=} 4E[\int_0^T ||G(s,X(s)||^2 ds]$, hence

$$\begin{aligned}
|\Lambda X|_T &\le E\left[||\xi_0||^2\right]^{\frac{1}{2}} + 2E\left[\int_0^T ||G(s,X(s))||^2 ds\right]^{\frac{1}{2}} + E\left[\int_0^T ||b(s,X(s)||^2 ds\right]^{\frac{1}{2}} \\
&\le E\left|||\xi_0||^2\right]^{\frac{1}{2}} + 3KE\left[\int_0^T (1+||X(s)||)^2 ds\right]^{\frac{1}{2}} \\
&\le E\left[||\xi_0||^2\right]^{\frac{1}{2}} + 3KE\left[\int_0^T ||X(s)||^2 ds\right]^{\frac{1}{2}} + 3KT^{\frac{1}{2}} \\
&\le E\left[||\xi_0||^2\right]^{\frac{1}{2}} + 3KT^{\frac{1}{2}}(|X|_T+1) < \infty
\end{aligned}$$

2nd step: Λ fulfills (6.4.9), because

$$\begin{aligned}
&X, Y \in E \implies \\
&|\Lambda^{n+1}X - \Lambda^{n+1}Y|_t \\
&= E\left[\sup_{0\le r\le t}\left\|\int_0^r G(s,\Lambda^n X(s)) - G(s,\Lambda^n Y(s)) dW_s + \int_0^r b(s,\Lambda^n X(s)) - b(s,\Lambda^n Y(s)) ds\right\|^2\right]^{\frac{1}{2}} \\
&\le 2E\left[\int_0^t ||G(s,\Lambda^n X(s)) - G(s,\Lambda^n Y(s))||^2 ds\right]^{\frac{1}{2}} + E\left[\int_0^t ||b(s,\Lambda^n X(s)) - b(s,\Lambda^n Y(s))||^2 ds\right]^{\frac{1}{2}} \\
&\le 3KE\left[\int_0^t ||\Lambda^n X(s) - \Lambda^n Y(s)||^2 ds\right]^{\frac{1}{2}} \\
&\le 3K\left(\int_0^t |\Lambda^n X - \Lambda^n Y|_s^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

By iteration we get

$$\begin{aligned}
|\Lambda^n X - \Lambda^n Y|_T &\le 3^n K^n \left(\int_0^T \int_0^{t_1} \dots \int_0^{t_n - 1} |X-Y|_{t_n}^2 dt_n \dots dt_1\right)^{\frac{1}{2}} \\
&\le 3^n K^n \left(\frac{T^n}{n!}\right)^{\frac{1}{2}} |X-Y|_T. \qquad (6.4.10)
\end{aligned}$$

$\underset{6.4.10}{\Longrightarrow}$ $\exists!$ fixed-point $X \in E$, furthermore

$$\begin{aligned}
|X|_T &\le \left(1 + \sum_{n\ge 1} 3^n K^n \frac{T^{\frac{n}{2}}}{(n!)^{\frac{1}{2}}}\right) \cdot \left(E\left[||\xi_0||^2\right]^{\frac{1}{2}} + 3KT^{\frac{1}{2}}\right) \\
&\le C\left(1 + E\left[||\xi_0||^2\right]\right)^{\frac{1}{2}} \quad \text{for a constant } C = C(K,T).
\end{aligned}$$

(where the first inequality holds because of the last line of Theorem 6.4.10 with $y \equiv 0$ and step 1.)

3rd step: Let Y be an arbitrary continuous adapted process which fulfills (6.4.8), then $Y = X$. Let $S_n := \inf\{t \geq 0 | \, ||X_t|| \geq n \text{ or } ||Y_t|| \geq n\}$ $(\to +\infty, \ n \to +\infty)$

$$\begin{aligned}
&\implies X_{t\wedge S_n} - Y_{t\wedge S_n} = \int_0^{t\wedge S_n} (G(s,X_s) - G(s,Y_s))dW_s + \int_0^{t\wedge S_n} (b(s,X_s) - b(s,Y_s))ds \\
&\implies E\left[||X_{t\wedge S_n} - Y_{t\wedge S_n}||^2\right] \\
&\qquad \underset{\text{isometry}}{\leq} 2E\left[\int_0^{t\wedge S_n} ||G(s,X_s) - G(s,Y_s)||^2 ds\right] + 2E\left[\int_0^{t\wedge S_n} ||b(s,X_s) - b(s,Y_s)||^2 ds\right] \\
&\qquad \leq 4KE\left[\int_0^{t\wedge S_n} ||X_s - Y_s||^2 ds\right] \\
&\qquad \leq 4K \int_0^t E\left[||X_{s\wedge S_n} - Y_{s\wedge S_n}||^2\right] ds \qquad (6.4.11)
\end{aligned}$$

$$\underset{\substack{\text{Gronwall's}\\ \text{Lemma}}}{\implies} E\left[||X_{t\wedge S_n} - Y_{t\wedge S_n}||^2\right] = 0 \ \forall t \forall n \underset{\substack{n\nearrow\infty\\ \text{paths}\\ \text{are continuous}}}{\implies} X_t = Y_t \quad \forall t \leq S_n \ P\text{-a.s.} \ \forall n,$$

hence $X = Y$.

□

Lemma 6.4.11 (Gronwall). *Let $f : [0,T] \to \mathbb{R}$ continuous. If there exist constants $A \in \mathbb{R}$, $b \in]0,\infty[$ with*

$$f(t) \leq A + \int_0^t bf(s)ds, \quad t \in [0,T], \qquad (6.4.12)$$

then $f(t) \leq Ae^{bt}$, $t \in [0,T]$.

Proof. Without loss of generality $A = 0$, otherwise consider $f(t) - Ae^{bt}$, which fulfills (6.4.12) because

$$f(t) - Ae^{bt} \leq A \underbrace{(1 - e^{bt})}_{= -\int_0^t be^{bs}ds} + \int_0^t bf(s)ds = \int_0^t b(f(s) - Ae^{bs})ds.$$

But if $A = 0$, we have due to (6.4.12) for $g(t) := e^{-bt} \int_0^t f(s)ds$

$$g'(t) = -bg(t) + e^{-bt} f(t) \leq -bg(t) + bg(t) = 0, \ g(0) = 0$$

$\implies g(t) \leq 0 \ \forall t \in [0,T]$. Hence by (6.4.12) $f(t) \leq 0 \ \forall t \in [0,T]$. □

Lemma 6.4.12 (generalized Gronwall). *Let $f, b \in L^1([0,T])$, such that b is nonnegative, $f \cdot b \in L^1([0,T])$ and let $A : [0,T] \to \mathbb{R}$ be an increasing function, such that*

$$f(t) \leq A(t) + \int_0^t b(s)f(s)ds \quad \textit{for } dt - \textit{a.e. } t \in [0,T]. \qquad (6.4.13)$$

Then

$$\operatorname*{ess\,sup}_{s\in[0,t]} f(s) \leq A(t) e^{\int_0^t b(s)ds} \ \textit{for } dt - \textit{a.e. } t \in [0,T].$$

Proof.
Step 1 Assume A is constant.
Then without loss of generality $A = 0$; otherwise consider $f - Ae^{\int_0^\bullet b(s)ds}$, which fulfills (6.4.13) because for dt-a.e. $t \in [0,T]$

$$f(t) - Ae^{\int_0^t b(s)ds} \leq A \underbrace{\left(1 - e^{\int_0^t b(s)ds}\right)}_{-\int_0^t b(s)e^{\int_0^s b(r)dr}ds} + \int_0^t b(s)f(s)ds$$
$$= \int_0^t b(s)\left(f(s) - Ae^{\int_0^s b(r)dr}\right)ds.$$

But if $A = 0$, we have for $g(t) := e^{-\int_0^t b(s)ds} \int_0^t b(s)f(s)ds$ and dt-a.e. $t \in [0,T]$

$$g'(t) = -b(t)g(t) + e^{-\int_0^t b(s)ds}\, b(t) \underbrace{f(t)}_{\leq \int_0^t b(s)f(s)ds}$$
$$\leq 0.$$

Since $g(0) = 0$, it follows that $g \leq 0$ dt-a.e., hence by (6.4.13) (with $A = 0$) $f \leq 0$ dt-a.e.
Step 2 A increasing (not necessarily constant).
Let $t \in [0,T]$. Then by (6.4.13)

$$f(s) \leq A(t) + \int_0^s b(r)f(r)dr \text{ for } ds - \text{a.e. } s \in [0,t].$$

So, by Step 1, for ds-a.e. $s \in [0,t]$

$$f(s) \leq A(t)e^{\int_0^s b(r)dr}$$
$$\leq A(t)e^{\int_0^t b(r)dr}.$$

□

Remark 6.4.13. *(i) For the proof of uniqueness (more precisely in (6.4.11)) we only used that G, b are locally Lipschitz, i.e. Lipschitz continuous on compact sets (e.g. $\overline{B_n(0)}$). In other words: If G and b are locally Lipschitz, then the stochastic differential equation (6.4.8) has* ***at most one*** *strong solution.*

(ii) If $G \equiv 1$, then (6.4.8) can be solved directly pathwise deterministic. (Use transformation $x_t \to x_t - W_t$, but "new" b depends explicitly on time and ω) If $d = 1$, then (6.4.8) can be reduced via Lamperti to the case $G = 1$ if G, b are enough regular.

Theorem 6.4.14. *Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz. Then for $x_0 \in \mathbb{R}^d$*

$$\exists! \text{ solution of } X(t) = x_0 + W(t) + \int_0^t b(X(s))ds$$

on $[0,\xi[$, where $\xi =$ "explosion time", i.e. $\lim_{t \nearrow \xi(\omega)} |X(t)(\omega)| = +\infty$ if $\xi(\omega) < +\infty$. We have that $\xi \equiv +\infty$ if b is globally Lipschitz.

(Without proof).
(Idea: transformation $X_t \to X_t - W_t$, then argue pathwise. Then b depends explicitly on time and ω)

7. The martingale problem

Literature: [Str87]

7.1. Formulation, motivation

$\Omega := C([0,\infty);\mathbb{R}^N)$ (polish!), $\mathcal{M} := \mathcal{B}_\Omega$, $\mathcal{M}_t := \sigma(x(s)|0 \le s \le t)$, $t \ge 0$, $x(s)(\omega) := \omega(s)$,
$M_1(\Omega) :=$ all probability-measures on $(\Omega,\mathcal{M})$.
$S^+(\mathbb{R}^N) :=$ all non-negative definite symmetric $N \times N$ matrices.
Let $a : [0,\infty[\times\mathbb{R}^N \to S^+(\mathbb{R}^N)$, $b : [0,\infty[\times\mathbb{R}^N \to \mathbb{R}^N$ bounded, measurable. Define operator-valued map $t \in [0,\infty[\mapsto L_t$ by

$$L_t := \frac{1}{2}\sum_{i,j=1}^N a^{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^N b^i(t,x)\partial_{x_i} \tag{7.1.1}$$

with $\partial_{x_i} := \frac{\delta}{\delta_{x_i}}$, $1 \le i \le N$.

Definition 7.1.1. *$P \in M_1(\Omega)$ is called* solution of the martingale problem for (L_t) (with test function space $C_0^\infty(\mathbb{R}^N)$) with start in $(s,x) \in [0,\infty[\times\mathbb{R}^N$, *denoted by: $P \in M.P.((s,x);(L_t))$, if*

(i) $P[x(0)=x]=1$ (thus process $(s+t,x(t))_{t\ge0}$ starts in $(s, \underbrace{x(0)}_{=x\ P\text{-a.s.}})$),

(ii) $\varphi(x(t)) - \int_0^t [L_{s+u}\varphi](x(u))du$, $t \ge 0$, is an $(\mathcal{M}_t)$-martingale under P $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$.

For $\varphi \in C^2(\mathbb{R}^N)$ set

$$X_{s,\varphi}(t) := \varphi(x(t)) - \int_0^t [L_{s+u}]\varphi(x(u))du\ , \quad t \ge 0. \tag{7.1.2}$$

Example 7.1.2. *Suppose there exists a probability space $(\Omega,\mathcal{F},P)$ with filtration $(\mathcal{F}_t)$ and an $(\mathcal{F}_t)$-BM $(W(t))_{t\ge0}$ and an $(\mathcal{F}_t)$-adapted continuous process $X = X(t)$, $t \ge 0$, such that for some $x \in \mathbb{R}^N$*

$$X(t) = x + \int_0^t \sigma(s+u,\ X(u))dW(u) + \int_0^t b(s+u,X(u))du,\ t \ge 0,\ P\text{-a.s.}.$$

Let $\varphi \in C_0^2(\mathbb{R}^N)$. Then by the d-dimensional Itô-formula (see Proposition 1.3.21) with $(\ ,\ )_{\mathbb{R}^N} :=$ Euclidean inner product in $\mathbb{R}^N$ and $X = (X^1,\dots,X^N)$ for $t \ge 0$

$$\varphi(X(t)) = \varphi(x) + \int_0^t (\nabla\varphi(X(u)), dX(u))_{\mathbb{R}^N}$$
$$+ \frac{1}{2}\int_0^t \sum_{i,j=1}^N \frac{\partial^2\varphi}{\partial x_i\partial x_j}(X(u))d\langle X^i,X^j\rangle_u.$$

But since

$$\begin{aligned}\langle X^i, X^j\rangle_t &= \langle\sum_{\ell=1}^{N}\int_0^{\bullet}\sigma^{i\ell}(s+u,X(u))dW^{\ell}(u),\ \sum_{\ell'=1}^{N}\int_0^{\bullet}\sigma^{i\ell'}(s+u,X(u))dW^{\ell'}(u)\rangle_t\\ &= \int_0^t\sum_{\ell,\ell'=1}^{N}\sigma^{i\ell}(s+u,X(u))\sigma^{i\ell'}(s+u,X(u))d\underbrace{\langle W^{\ell},W^{\ell'}\rangle_u}_{\delta_{\ell,\ell'}du,}\end{aligned}$$

we obtain for $(a^{ij}) := \sigma\sigma^T$ *and* $t \geq 0$

$$\begin{aligned}\varphi(X(t)) = \varphi(x) &+ \int_0^t(\nabla\varphi(X(u)),\ \sigma(s+u,X(u))dW(u))_{\mathbb{R}^N}\\ &+ \int_0^t(\nabla\varphi(X(u)),\ b(s+u,X(u))_{\mathbb{R}^N}\,du\\ &+ \frac{1}{2}\int_0^t\sum_{i,j=1}^{N}a^{ij}(s+u,X(u))\frac{\partial^2\varphi}{\partial x_i\partial x_j}(X(u))du\\ &= \int_0^t\left[L_{s+u}\varphi\right](X(u))du\\ &+ \underbrace{\varphi(x)+\int_0^t(\nabla\varphi(X(u)),\ \sigma(s+u,X(u))dW(u))_{\mathbb{R}^N}}_{a\ martingale!}\end{aligned}$$

Hence $P\circ X^{-1} \in M.P.((s,x),(L_t))$.

Let $P \in M.P.((s,x);(L_t))$ be fixed and $\varphi \in C_0^{\infty}(\mathbb{R}^N)$. Then $X_{s,\varphi} \in Mart_c^2(P)$ (:= all $(\mathcal{M}_t)$-martingales under P in L^2 with P-a.s. continuous paths).

Lemma 7.1.3. $\forall\varphi,\psi \in C_0^{\infty}(\mathbb{R}^N)$

$$\langle X_{s,\varphi}, X_{s,\psi}\rangle_t = \int_0^t(\nabla\varphi, a\nabla\psi)_{\mathbb{R}^N}(s+u,x(u))du,\ t\geq 0,\quad P-a.s. \tag{7.1.3}$$

Proof. By polarization without loss of generality : $\varphi = \psi$. We have

$$\begin{aligned}
X_{s,\varphi}(t)^2 \quad & \underset{(7.1.2)}{=} \quad \varphi(x(t))^2 - 2\varphi(x(t)) \int_0^t [L_{s+u}\varphi](x(u))du + \left[\int_0^t [L_{s+u}\varphi](x(u))du\right]^2 \\
& \underset{(7.1.2)}{=} \quad \varphi^2(x(t)) - 2X_{s,\varphi}(t) \int_0^t [L_{s+u}\varphi](x(u))du - \left[\int_0^t [L_{s+u}\varphi](x(u))du\right]^2 \\
& \underset{\substack{(7.1.2)+ \\ \text{Itõ's product} \\ \text{formula}}}{=} \quad \underbrace{X_{s,\varphi^2}(t)}_{\text{mart.!}} + \int_0^t [L_{s+u}\varphi^2](x(u))du - \left[\int_0^t [L_{s+u}\varphi](x(u))du\right]^2 \\
& \qquad \underbrace{-2\int_0^t \int_0^u [L_{s+u'}\varphi](x(u'))du'\ dX_{s,\varphi}(u)}_{\text{mart.}} \\
& \qquad \underbrace{-\,2\int_0^t \underbrace{X_{s,\varphi}(u)}_{\substack{\overset{(7.1.2)}{=}\varphi(x(u)) \\ -\int_0^u [L_{s+u'}\varphi](x(u'))du'}} \quad d\left(\int_0^\bullet [L_{s+u'}\varphi](x(u'))du'\right)(u)}_{\underset{\substack{(\text{Itõ's}) \\ \text{product} \\ \text{formula}}}{=} \; 2\int_0^t [\varphi L_{s+u}\varphi](x(u))du - \left[\int_0^t [L_{s+u}\varphi](x(u))du\right]^2} \\
& = \quad \int_0^t \underbrace{[L_{s+u}\varphi^2 - 2\varphi L_{s+u}\varphi]}_{\overset{(*)}{=}\sum a^{ij}(s+u,\,\cdot\,)\partial_{x_i}\varphi\partial_{x_j}\varphi}(x(u))du + \text{ martingale} \\
& = \quad \int_0^t (\nabla\varphi, a\nabla\varphi)(s+u, x(u))du + \text{ martingale}
\end{aligned}$$

where $(*)$ holds because $\partial_{x_i}\partial_{x_j}\varphi^2 = 2\partial_{x_i}(\varphi\partial_{x_j}\varphi) = 2\varphi\partial_{x_i}\partial_{x_j}\varphi + 2(\partial_{x_i}\varphi)(\partial_{x_j}\varphi)$ and $\partial_{x_i}\varphi^2 = 2\varphi\partial_{x_i}\varphi$. □

Lemma 7.1.4. *(i) $\forall \varphi \in C^2(\mathbb{R}^N)$, $X_{s,\varphi} \in Mart_c^{loc}(P)$ (:=all continuous loc. $(\mathcal{M}_t)$-martingales until $+\infty$) under P with P-a.s. continuous paths), and (7.1.3) holds $\forall \varphi, \psi \in C^2(\mathbb{R}^N)$.*

(ii) If $\bar{x}(t) := x(t) - \int_0^t b(s+u, x(u))du$, $t \geq 0$, then $\bar{x}_i \in Mart_c^2(P)$, $1 \leq i \leq N$ (even $\bar{x}_i \in L^q(P)$ $\forall q \in]0,\infty[$) and

$$(\langle \bar{x}_i, \bar{x}_j \rangle(t))_{1\leq i,j\leq N} = \int_0^t a(s+u, x(u))du, \quad t \geq 0 \ P-a.s. \tag{7.1.4}$$

(iii) If $f \in C^{1,2}([0,\infty[\times\mathbb{R}^N)$ (i.e. C^1 in t, C^2 in x), then

$$\begin{aligned}
& f(t,x(t)) - f(0,x) - \int_0^t [(\partial_u + L_{s+u})f](u,x(u))du \\
& = \underbrace{\sum_{i=1}^N \int_0^t \partial_{x_i} f(u,x(u)) d\bar{x}_i(u)}_{\textit{loc. mart.}} =: \int_0^t [\nabla_x f](u,x(u)) \cdot d\bar{x}(u),\ t \geq 0, \quad P-a.s.
\end{aligned}$$

Proof. (i) : Clear (with trivial approximation) $X_{s,\varphi} \in Mart_c^2(P)$ $\forall \varphi \in C_0^2(\mathbb{R}^N)$. Now let $\varphi \in C^2(\mathbb{R}^N)$ and $n \in \mathbb{N}$. Define

$$\sigma_n := \inf\{t \geq 0 | \, |x(t)| \geq n\},$$

and take $\eta_n \in C_0^\infty(\mathbb{R}^N)$ with $\eta_n = 1$ on $B_{n+1}(0)$. Then $\sigma_n \uparrow \infty$ and $X_{s,\varphi}(\cdot \wedge \sigma_n) = X_{s,\eta_n \cdot \varphi}(\cdot \wedge \sigma_n) \in Mart_c^2(P)$, hence $X_{s,\varphi} \in Mart_c^{loc}(P)$ and (7.1.3) is still true.

(ii) : (i) with $\varphi(x) = x_i$, implies that $\bar{x}_i \in Mart_c^{loc}(P)$, $1 \leq i \leq N$, and (7.1.3) holds with $\psi(x) = x_j$ which implies (7.1.4). By Cor. 1.4.32 it follows that $\bar{x}_i$ is a square integrable martingale $\forall 1 \leq i \leq N$, since by (7.1.4) $\langle \bar{x}_i \rangle_t$ is bounded for every $t \geq 0$.
Now we need

Lemma 7.1.5. *Let* $X \in Mart_c^{l1oc}(P)$ *and* $\sigma \leq \tau$ *stopping times with* $\langle X \rangle(\tau) - \langle X \rangle(\sigma) \leq A$ *for some* $A \in]0,\infty[$. *Then*

$$P[\sup_{\sigma \leq t \leq \tau} |X(t) - X(\sigma)| \geq R] \leq 2\exp(-R^2/2A) \quad \forall R \geq 0.$$

In particular $\forall q \in]0,\infty[$ $\exists C_q \in [1,\infty[$ *with*

$$E_p\left[\sup_{\sigma \leq t \leq \tau} |X(t) - X(\sigma)|^q\right] \leq C_q A^{\frac{q}{2}}.$$

Proof. [Str87, p.59, Lemma (3.10)] □

Continuing with proof of 7.1.4 (ii): 7.1.5 $\Longrightarrow$ $\forall 0 \leq s < T$

$$P\left[\sup_{s \leq t \leq T} ||\bar{x}(t) - \bar{x}(s)|| \geq R\right] \leq 2 \cdot N \exp\left[-\frac{R^2}{2AN(T-s)}\right] \tag{7.1.5}$$

with $A := \sup_{t,y} ||a(t,y)||_{oper.norm}$. In particular $\forall q \in]0,\infty[$ $\exists C(q,N) \in [1,\infty[$ with

$$| \sup_{s \leq t \leq T} ||\bar{x}(t) - \bar{x}(s)|| \, |_{L^q(P)} \leq C(q,N)(A(T-s))^{\frac{1}{2}} \tag{7.1.6}$$

In particular $\bar{x}_i \in L^q(P)$ $\forall 1 \leq i \leq N$.

(iii) Follows directly with (ii) from Itô's formula (for the time-depending case).

□

Remark 7.1.6. $a \equiv 0 \underset{\substack{(7.1.4) \\ \& \ 2.2.8(iii)}}{\Longrightarrow} \bar{x}(t) = \bar{x}(0) = x(0) = x$. *Thus*

$$x(t) = x + \int_0^t b(s+u, x(u))du \ , \ t \geq 0 \quad P - a.s.$$

Hence (P-a.s.) deterministic case included.

Theorem 7.1.7. *Let* $a : [0,\infty[\times\mathbb{R}^N \to S^+(\mathbb{R}^N)$, $b : [0,\infty[\times\mathbb{R}^N \to \mathbb{R}^N$ *bounded, measurable and* $P \in M.P.((s,x),L_t))$ *for an* $(s,x) \in [0,\infty[\times\mathbb{R}^N$. *Supposing* $\exists \sigma : [0,\infty[\times\mathbb{R}^N \to Hom(\mathbb{R}^d;\mathbb{R}^N)$ *(=* $d \times N$ *matrices over* $\mathbb{R}$*) bounded, measurable with* $a = \sigma\sigma^T$.

Then $\exists$ *d*$-$*dim. BM* $((\beta(t))_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, Q)$ *on a probability space* $(\bar{\Omega}, \mathcal{F}, Q)$ *and continuous,* $(\mathcal{F}_t)_{t\geq 0}$*-progressiv measurable map* $X : [0,\infty[\times\tilde{\Omega} \to \mathbb{R}^N$*, such that*

$$X(t) = x + \int_0^t \sigma(s+u, X(u))d\beta(u) + \int_0^t b(s+u, X(u))du,\ t \geq 0, \quad Q-a.s.$$

and

$$P = Q \circ X^{-1}.$$

(i.e. briefly: $dX(t) = \sigma(s+t, X(t))d\beta(t) + b(s+t, X(t))dt,\ X(0) = x$*). If particularly* $a(t,y) > 0$ $\forall (t,y) \in [0,\infty[\times\mathbb{R}^N$ *and* $\sigma := a^{\frac{1}{2}}$ *(as pos. s.a. operator uniquely determined!), then one can take:* $\tilde{\Omega} := \Omega$, $Q := P$, $X(\cdot) = x(\cdot)$ *and*

$$\beta(t) := \int_0^t \sigma^{-1}(s+u, x(u))d\bar{x}(u),\ t \geq 0,\ \mathcal{F}_t := \mathcal{M}_t,\ \mathcal{F} := \mathcal{M},$$

with $\bar{x}(t) := x(t) - \int_0^t b(s+u, x(u))du,\ t \geq 0$. *(martingale according to 7.1.4(ii)!)*

Proof. [Str87, Theorem(2.6), p.91 ff.] ☐

Exercise 7.1.8: (i) Show: $P \in M.P.((s,x),(L_t)) \Leftrightarrow P[x(0) = x] = 1$ and $\bar{x}_i \in Mart_c^{loc}(P)$ $\forall 1 \leq i \leq N$ and (7.1.4) holds.

(ii) Show: What we have done so far, can be generalized to the following situation: Let $a : [0,\infty[\times\mathbb{R}^N\times\Omega \to S^+(\mathbb{R}^N)$, $b : [0,\infty[\times\mathbb{R}^N\times\Omega \to \mathbb{R}^N$ bounded, $(\mathcal{M}_t)$-progressiv measurable and (L_t) as in (7.1.1) with these a, b. Define $M.P(x;(L_t))$ as the set of all $P \in M_1(\Omega)$ with $P[x(0) = x] = 1$ and

$$\varphi(x(t)) - \int_0^t [L_u\varphi](x(u))du,\ t \geq 0,$$

is an $(\mathcal{M}_t)$-martingale under P $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$.
Define $\bar{x}(t) := x(t) - \int_0^t b(u)du$. Show:

(a) $P \in M.P.(x;(L_t)) \iff P[x(0) = x] = 1$ and $\bar{x}_i \in Mart_c^l oc(P)$ $\forall 1 \leq i \leq N$ and $(\langle \bar{x}_i, \bar{x}_j\rangle)_{i,j} = \int_0^\bullet a(u)du)$ P-a.s.

(b) $P \in M.P.(x;(L_t)) \implies$ 7.1.4(iii), (7.1.5), (7.1.6) hold.

(iii) If a, b in (7.1.1) are independent of t, then the martingale problem in 7.1.1 is time independent. Every time dependent martingale problem becomes time independent via the following trick ("space time homogenization"): Set $\tilde{\Omega} := C([0,\infty[;\mathbb{R}^{N+1}) \equiv C([0,\infty);\mathbb{R}^1) \times \Omega$. Show: $P \in M.P.((s,x),(L_t)) \iff \tilde{P} \in M.P.(\tilde{x}, \tilde{L})$, where $\tilde{x} := (s,x)$ $(\in [0,\infty) \times \mathbb{R}^N)$, $\tilde{L} := \partial_t + L_t$, $\tilde{P} := \sigma_{s+\cdot} \otimes P$ and $\sigma_{s+\cdot} \in M_1\left(C\left([0,\infty[;\mathbb{R}^1\right)\right)$ is Dirac measure on path $t \mapsto s+t$.

Definition 7.1.9. *The martingale problem for* (L_t) *in (7.1.1) (with test function space* $C_0^\infty(\mathbb{R}^N)$*) is called* "well-posed"*, if* $\forall (s,x) \in [0,\infty[\times\mathbb{R}^N$ $\#M.P.((s,x),(L_t)) = 1$.

Theorem 7.1.10. *Suppose the martingale problem for* (L_t) *in (7.1.1) (with* $C_0^\infty(\mathbb{R}^N)$*) is "well-posed". Let* $\{P_{s,x} | (s,x) \in [0,\infty[\times\mathbb{R}^N\}$ *be the corresponding family of solutions. Then:*

(i) $(s,x) \mapsto P_{s,x}$ *is measurable from* $[0,\infty[\times\mathbb{R}^N$ *to* $M_1(\Omega)$ *(with weak topology).*

(ii) Suppose we have

$$x \mapsto \int_0^T a(t,x)dt, \quad x \mapsto \int_0^T b(t,x)dt \text{ continuous } \forall T > 0. \tag{7.1.7}$$

Then $(s,x) \mapsto P_{s,x}$ *continuous from* $[0,\infty[\times\mathbb{R}^N$ *to* $M_1(\Omega)$.

Proof. [Str87, p.85 ff.] ☐

Theorem 7.1.11 (Existence)**.** *Let (L_t) be as in (7.1.1), and (7.1.7) holds. Then*

$$\#M.P.((s,x),(L_t)) \geq 1 \quad \forall (s,x) \in [0,\infty[\times \mathbb{R}^N.$$

Proof. [Str87, p. 84 (1.20) & (1.21)] ☐

Theorem 7.1.12 (Uniqueness)**.** *Let (L_t) be as in (7.1.1). Suppose the following condition holds: $\forall (s,x) \in [0,\infty[\times \mathbb{R}^N$ we have*

$$P, Q \in M.P.((s,x);(L_t)) \implies P \circ x(t)^{-1} = Q \circ s(t)^{-1} \quad \forall t \geq 0. \tag{7.1.8}$$

Then

$$\#M.P.((s,x);(L_t)) \leq 1 \quad \forall (s,x) \in [0,\infty[\times \mathbb{R}^N.$$

7.1.11 is useful for applications, 7.1.12 not. To get a sufficient condition for (7.1.8) we need some preparations:

Definition 7.1.13. *Let $\mathcal{F}$ be a set of bounded, measurable functions on a measurable space $(E,\mathcal{B})$. $\mathcal{F}$ is called* determining set, *if $\forall \mu,\nu \in M_1(E)$*

$$\int \varphi d\mu = \int \varphi d\nu \quad \forall \varphi \in \mathcal{F} \implies \mu = \nu.$$

Theorem 7.1.14. *Suppose there exisists a determining set $\mathcal{F} \subset C_b(\mathbb{R}^N)$, such that one of the following conditions holds:*

(i) $\forall T > 0,\ \varphi \in \mathcal{F}\ \exists u_{T,\varphi} \in \overset{(*)}{C_b^{1,2}}([0,T[\times \mathbb{R}^N)$ *with*

$$(\partial_t + L_t)u_{T,\varphi} = 0 \text{ on } [0,T[\times \mathbb{R}^N$$

and

$$\lim_{t\uparrow T} u_{T,\varphi}(t,x) = \varphi(x) \quad \forall x \in \mathbb{R}^N$$

(automatically unifom in x because of $(*)$).

(ii) $\forall T > 0,\ \varphi \in \mathcal{F}\ \exists \tilde{u}_{T,\varphi} \in C_b^{1,2}([0,T[\times \mathbb{R}^N)$ *with*

$$(\partial_t + L_t)\tilde{u}_{T,\varphi} = \varphi \text{ on } [0,T[\times \mathbb{R}^N$$

and

$$\lim_{t\uparrow T} \tilde{u}_{T,\varphi}(t,x) = 0 \quad \forall x \in \mathbb{R}^N$$

(automatically uniform in x).

Then $\#M.P.((s,x);(L_t)) \leq 1 \quad \forall (s,x) \in [0,\infty[\times \mathbb{R}^N$.
More precisely, in case (i): $\forall (s,x) \in [0,\infty[\times \mathbb{R}^N$ and $\forall P \in M.P.((s,x);(L_t))$

$$E_P[\varphi(x(T-s))] = u_{T,\varphi}(s,x) \quad \forall T > s \tag{7.1.9}$$

Thus (7.1.8) is fulfilled.
More precisely, in case (ii): $\forall (s,x) \in [0,\infty[\times \mathbb{R}^N$ and $\forall P \in M.P.((s,x);(L_t))$

$$E_P\left[\int_0^{T-s} \varphi(x(u))du\right] = -\tilde{u}_{T,\varphi}(s,x) \quad \forall T > s \tag{7.1.10}$$

Thus (7.1.8) is fulfilled.

Proof. Suppose (i) holds: Let $T > s$. Then according to 7.1.4(iii) with $f := u_{T,\varphi}(\cdot + s, \cdot)$ $\forall t \in [0, T - s[$

$$E_P\left[u_{T,\varphi}(t + s, x(t))\right] = u_{T,\varphi}(s, x).$$

With $t \uparrow T - s$ (7.1.9) follows , hence (7.1.8) holds and the assertion is implied by 7.1.12. Suppose (ii) holds: Let $T > s$. Then according to 7.1.4(iii) with $f := \tilde{u}_{T,\varphi}(\cdot + s, \cdot)$ $\forall t \in [0, T - s[$

$$E_P\left[\tilde{u}_{T,\varphi}(t + s, x(t))\right] - \tilde{u}_{T,\varphi}(s, x) = E_P\left[\int_0^t \varphi(x(u))du\right].$$

With $t \uparrow T - s$ (7.1.10) follows, thus $\forall a, b \in \mathbb{R}_+,\ a < b$

$$\int_a^b E_P[\varphi(x(u))du = -\tilde{u}_{b+s,\varphi}(s, x) + \tilde{u}_{a+s,\varphi}(s, x)$$

Hence by continuity $u \mapsto E_P[\varphi(x(u))]$ is uniquely determined on $\mathbb{R}_+$. Thus (7.1.8) holds and 7.1.12 implies the assertion. □

Finally we want to prove 7.1.12. Therefore we need:

Theorem 7.1.15. *Let $P \in M.P.((s, x), (L_t))$ and τ a stopping time. Then:*

(i) $\mathcal{M}_\tau = \sigma(\{x(t \wedge \tau) | t \geq 0\})$. In particular $\mathcal{M}_\tau$ is countably generated.

(ii) Let $\mathcal{A}$ be a countably generated sub-σ-algebra of $\mathcal{M}_\tau$, such that

$$\omega \mapsto x(\tau(\omega), \omega) \quad \mathcal{A} - \text{measurable}.$$

Let $\omega \mapsto P_\omega$ be a regular conditional probability distribution of P given $\mathcal{A}$ (i.e.: $(\omega, A) \mapsto P_\omega[A]$, $\omega \in \Omega$, $A \in \mathcal{M}$, is stochastic kernel on $(\Omega, \mathcal{M})$ such that:

a) $\forall A \in \mathcal{M}:\ P[A|\mathcal{A}](\omega) = P_\omega[A]$ for $P - a.a.\omega \in \Omega$.

b) $\forall A \in \mathcal{M}:\ \omega \mapsto P_\omega(A)$ is $\mathcal{A}$-measurable and ("regular")

$$P_\omega[A] = 1_A(\omega) \text{ for all } \omega \in \Omega,\ \text{ if } A \in \mathcal{A}.$$

(exists, since $\mathcal{A}$ is countably generated) i.e.: $P_{\omega \upharpoonright \mathcal{A}} = \epsilon_\omega$).

Then $\exists P$-null set $\Lambda \in \mathcal{A}$ such that: $\forall \omega \in \Lambda^c$

$$P_\omega \circ \theta^{-1}_{\tau(\omega)} \in M.P.((s + \tau(\omega),\ x(\tau(\omega), \omega); (L_t))$$

(where for $t \geq 0:\ \theta_t : \Omega \to \Omega,\ \theta_t(\omega) = \omega(\cdot + t)$.)

Proof. cf. [Str87, III.(1.13), p.78, and III.(1.14), p. 79]. □

Proof of 7.1.12. Let $P, Q \in M.P.((s, x), (L_t))$.
To show: $(*)$ $P \circ (x(t_1), ..., x(t_n))^{-1} = Q \circ (x(t_1), ..., x(t_n))^{-1}$ $\forall\ 0 \leq t_1 < ... < t_n$ and **all** $n \in \mathbb{N}$.
Induction over n:
$n = 1$: Ok, according to assumption.
$n \mapsto n + 1$: Suppose $(*)$ holds for $n \in \mathbb{N}$ and all $0 \leq t_1 < ... < t_n$. Let $0 \leq t_1 < ... < t_n < t_{n+1}$.
Let $\mathcal{A} =: \sigma(\{x(t_1), ..., x(t_n)\})$ and $\omega \mapsto P_\omega$ resp. $\omega \mapsto Q_\omega$ regular conditional probability distributions of P resp. Q given $\mathcal{A}$.
According to 7.1.14(ii) $\exists$ a P-null set $\Lambda_1 \in \mathcal{A}$ resp. a Q-null set $\Lambda_2 \in \mathcal{A}$ such that

$$P_\omega \circ \theta^{-1}_{t_n},\ Q_\omega \circ \theta^{-1}_{t_n} \in M.P.((s + t_n, x(t_n, \omega)); (L_t))$$

$\forall \omega \in \Lambda_1^c \cap \Lambda_2^c$.

Hence for $\Lambda := \Lambda_1 \cup \Lambda_2$ we have, according to assumption (7.1.8), $\forall \omega \in \Lambda^c$

$$\begin{aligned} P_\omega \circ x(t_{n+1})^{-1} &= (P_\omega \circ \theta_{t_n}^{-1}) \circ x(t_{n+1} - t_n)^{-1} \\ &\underset{(7.1.8)}{=} (Q_\omega \circ \theta_{t_n}^{-1}) \circ x(t_{n+1} - t_n)^{-1} \\ &= Q_\omega \circ x(t_{n+1})^{-1}. \qquad (**) \end{aligned}$$

In addition, due to the induction hypothesis $P = Q$ on $\mathcal{A}$, hence $P[\Lambda] = Q[\Lambda] = 0$, and thus $\forall\ B_1, ..., B_{n+1} \in \mathcal{B}(\mathbb{R}^N)$

$$\begin{aligned} &P[x(t_1) \in B_1, ..., x(t_{n+1}) \in B_{n+1}] \\ &= E_P[\underbrace{1_{B_1}(x(t_1))...1_{B_n}(x(t_n))}_{\in \mathcal{A}} \underbrace{E_{P_\cdot}[1_{B_{n+1}}(x(t_{n+1}))]}_{\mathcal{A}-\text{measurable}}] \\ &= E_Q[1_{\Lambda^c} 1_{B_1}(x(t_1))...1_{B_n}(x(t_n)) \underbrace{E_{P_\cdot}[1_{B_{n+1}}(x(t_{n+1}))]}_{\underset{(**)}{=} E_{Q_\cdot}[1_{B_{n+1}}(x(t_{n+1}))]}] \\ &\underset{\text{analogously}}{=} Q[x(t_1) \in B_1, ..., x(t_{n+1}) \in B_{n+1}]. \end{aligned}$$

□

A. Time Transformation

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ a right-contiuous, complete filtration and Y a P-a.s. continuous locale martingale up to ∞ (with respect to $(\mathcal{F}_t)$) such that $Y_0 \equiv 0$. For simplicity let

$$\langle Y\rangle_\infty (:= \lim_{t\to\infty} \langle Y\rangle_t) = \infty \quad P\text{-a.s.}.$$

Define the "inverse" of $\langle Y\rangle_t$ by

$$C_t := \inf \underbrace{\{s > 0 | \langle Y\rangle_s > t\}}_{\neq\emptyset}.$$

Theorem A.0.1. *Let $\langle Y\rangle_\infty = \infty$ P-a.s. and $W_t := Y_{C_t}$, $t \geq 0$. Then, W is a $(\mathcal{G}_t)_{t\geq 0} := \mathcal{F}_{C_t}$-Brownian motion (cf. 1.5.34) and $Y_t = W_{\langle Y\rangle_t}$ for all $t \geq 0$.*

For the proof we need the following two lemmas.

Lemma A.0.2. *(i) The map $t \mapsto C_t$ is increasing and right-continuous.*

(ii) C_t is an $(\mathcal{F}_s)$-stopping time for all t.

(iii)

$$\langle Y\rangle_{C_t} = t \quad P\text{-a.s. } \forall t.$$

(iv) $t \leq C_{\langle Y\rangle_t}$. (Note that in general case $t \neq C_{\langle Y\rangle_t}$!)

Proof. (iii): Without loss of generality $\langle Y\rangle$ is continuous everywhere. Then $\langle Y\rangle_{C_t} \geq t$, since $t \mapsto \langle Y\rangle_t$ is (right-)continuous.
Assumption: $\langle Y\rangle_{C_t} \geq t + \varepsilon$ for $\varepsilon > 0$.
Then, since $t \mapsto \langle Y\rangle_t$ is continuous, there exists a $\delta > 0$ such that $\langle Y\rangle_{C_t - \delta} \geq t + \frac{\varepsilon}{2} > t$. Hence, $C_t \leq C_t - \delta$. Since this is impossible it follows that $\langle Y\rangle_{C_t} < t + \varepsilon$ for all $\varepsilon > 0$.
(i): Obviuously $t \mapsto C_t$ is increasing. It remains to show that

$$\lim_{u \searrow t} C_u \leq C_t.$$

("$\geq$" is clear since C_k is increasing.)
Let $\varepsilon > 0$. Then $\langle Y\rangle_{C_t+\varepsilon} > t$. Hence, there exists a $\delta > 0$ such that

$$\langle Y\rangle_{C_t+\varepsilon} > u \quad \forall u \in [t, t+\delta].$$

Thus,

$$C_u \leq C_t + \varepsilon \quad \forall u \in [t, t+\delta].$$

(ii): We have

$$\{C_t < u\} = \{\langle Y\rangle_u > t\} \quad \forall u, t, \tag{A.0.1}$$

because "$\subset$" is clear and, if $\langle Y\rangle_u > t$, then there exists an $\varepsilon > 0$ such that $\langle Y\rangle_{u-\varepsilon} > t$ and, therefore, $C_t \leq u - \varepsilon < u$. But

$$\{\langle Y\rangle_u > t\} \in \mathcal{F}_u.$$

Hence, by (A.0.1)

$$\{C_t \leq u\} \in \mathcal{F}_{u+} = \mathcal{F}_u.$$

(iv): If $C_{\langle Y\rangle_t} < t$, then by (A.0.1) we would get $\langle Y\rangle_t > \langle Y\rangle_t$. Therefore, $C_{\langle Y\rangle_t} \geq t$. □

Note that $\mathcal{G}_t := \mathcal{F}_{C_t}$, $t \geq 0$ (filtration, since C_t is increasing!).

Lemma A.0.3. *(i) $(\mathcal{G}_t)_{t\geq 0}$ is right-continuous and complete.*

(ii) $\langle Y\rangle_t$ is a stopping time with respect to $(\mathcal{G}_s)_{s\geq 0}$ for all t.

Proof. (ii) is clear by (A.0.1).
(i): Right-continuity: Let $A \in \bigcap_{\varepsilon>0} \mathcal{F}_{C_{t+\varepsilon}}$ (in particular, $A \in \mathcal{F}_{C_{t+\frac{1}{n}}} \forall n$). Hence, for all s

$$A \cap \{C_{t+\frac{1}{n}} < s\} \in \mathcal{F}_s \quad \forall n,\, s,$$

therefore,

$$A \cap \underbrace{\bigcup_n \{C_{t+\frac{1}{n}} < s\}}_{=\{C_t<s\}} \in \mathcal{F}_s \quad \forall s,$$

thus $A \cap \{C_t \leq s\} \in \mathcal{F}_{s+} = \mathcal{F}_s$ $\forall s$, and $A \in \mathcal{F}_{C_t}$.
Completeness: Let $A_0 \in \mathcal{F}_{C_t}$ with $P[A_0] = 0$ and $A \subset A_0$. Then for all s

$$A \cap \{C_t \leq s\} \in \mathcal{F}_s$$

as a subset of an $\mathcal{F}_s$-measurable P-zero set $A_0 \cap \{C_t \leq s\}$, since $\mathcal{F}_s$ is complete. Hence, $A \in \mathcal{F}_{C_t}$. □

Proof of A.0.1 . Since Y is P-a.s. continuous and (C_t) is right-continuous, (W_t) is P-a.s. right-continuous.
Step 1. $t \mapsto W_t = Y_{C_t}$ is P-a.s. continuous:
It suffice to show that (cf. Exercises)

$$P[Y_u = Y_t \quad \text{for } t \leq u \leq \sigma_t \quad \forall t \geq 0] = 1, \tag{A.0.2}$$

where

$$\sigma_t := \inf\{s > t | \langle Y\rangle_s > \langle Y\rangle_t\},$$

hence,

$$\langle Y\rangle_u = \langle Y\rangle_t \text{ for } u \in [t, \sigma_t].$$

(A.0.2) is sufficient, since then Y is constant on the interval, where $\langle Y\rangle$ is constant and by definition this is the case on $[C_{t-}, C_t]$. Therefore, $Y_{C_t} = Y_{C_{t-}}$. For (A.0.2) it remains to show that for all $r \in \mathbb{Q}^+$

$$P[Y_u = Y_r \text{ for } r \leq u \leq \sigma_r\ \forall r \in \mathbb{Q}^+] = 1, \tag{A.0.3}$$

because, if $t \geq 0$ with $t < \sigma_t$, then for all $r \in [t, \sigma_t] \cap \mathbb{Q}^+$ $\sigma_r = \sigma_t$ and (A.0.3) implies (A.0.2), since Y is P-a.s. (right-)continuous.
Let $(T'_n)_{n\in\mathbb{N}}$ be a localizing sequence for Y. Then

$$T_n := \inf\{t > 0 | |Y_t| > n\} \wedge T'_n,\ n \in \mathbb{N},$$

is again a localizing sequence. Fix $r \in \mathbb{Q}^+$. For $n \in \mathbb{N}$ set

$$N_t^{(n)} := Y_{(r+t)\wedge\sigma_r\wedge T_n} - Y_{r\wedge T_n},\ t \geq 0,$$

$$\tilde{\mathcal{F}}_t := \mathcal{F}_{t+r},\ t \geq 0.$$

Then $N^{(n)}$ is a continuous bounded martingale with respect to $(\tilde{\mathcal{F}}_t)$, since by the stopping theorem $\forall s \leq t$

$$E[N_t^{(n)} | \tilde{\mathcal{F}}_s] = E[Y_{(r+t)\wedge\sigma_r\wedge T_n} - Y_{r\wedge T_n} | \mathcal{F}_{r+s}] = Y_{(r+s)\wedge\sigma_r\wedge T_n} - Y_{r\wedge T_n} = N_s^{(n)},$$

since σ_r is a stopping time with respect to $(\tilde{\mathcal{F}}_t)$. Additionally,

$$\langle N^{(n)}\rangle_t = \langle Y\rangle_{(r+t)\wedge\sigma_r\wedge T_n} - \langle Y\rangle_{r\wedge T_n} = 0 \quad \forall t \geq 0.$$

Hence,

$$E[(N_t^{(n)})^2] = E[\langle N^{(n)}\rangle_t] = 0 \quad \forall t \geq 0,$$

thus

$$N_t^{(n)} = 0 \quad P\text{-a.s..}$$

Letting $n \to \infty$ implies

$$Y_{(r+t)\wedge\sigma_r} - Y_r = 0 \quad P\text{-a.s. } \forall t \geq 0,$$

which yields (A.0.3).

Step 2. *W is a local martingale (up to ∞) with respect to $(\mathcal{G}_t)$:*
By A.0.2(ii) C_t is an $(\mathcal{F}_s)$-stopping time. For $t \geq 0$

$$E[\langle Y^{C_t}\rangle_s] = E[\langle Y\rangle_{C_t\wedge s}] \leq E[\langle Y\rangle_{C_t}] = t.$$

Therefore, by Corollary 1.4.32 Y^{C_t} is a martingale and

$$E[(Y_s^{C_t})^2] \leq \liminf_n E[Y^2_{s\wedge C_t\wedge T_n}] = \liminf_n E[\langle Y\rangle_{s\wedge C_t\wedge T_n}] \leq t.$$

Hence, $(Y_s^{C_t})_{s\geq 0}$ and $((Y_s^{C_t})^2)_{s\geq 0}$ are uniformly integrable.
Moreover, $C_{\langle Y\rangle_{T_n}}$ is a $(\mathcal{F}_t)$-stopping time, because:

$$\begin{aligned}
\{C_{\langle Y\rangle_{T_n}} < u\} &\overset{(\text{A.0.1})}{=} \{\langle Y\rangle_u > \langle Y\rangle_{T_n}\} \\
&= \bigcup_{r\in\mathbb{Q}} \{\langle Y\rangle_u > r\} \cap \{r \geq \langle Y\rangle_{T_n}\} \\
&\overset{(\text{A.0.1})}{=} \bigcup_{r\in\mathbb{Q}} \{C_r < u\} \cap \{C_r \geq T_n\} \\
&= \bigcup_{r\in\mathbb{Q}} \underbrace{\{C_r < u\}}_{\in\mathcal{F}_u} \cap \underbrace{\{C_r \wedge u \geq T_n\}}_{\in\mathcal{F}_{C_r\wedge u\wedge T_n}\subset\mathcal{F}_u}
\end{aligned}$$

Thus,

$$\{C_{\langle Y\rangle_{T_n}} \leq u\} \in \mathcal{F}_{u+} = \mathcal{F}_u.$$

Furthermore, $\langle Y\rangle_{T_n}$ is a stopping time with respect to $(\mathcal{G}_t)$, since

$$\{\langle Y\rangle_{T_n} \leq u\} \overset{(\text{A.0.1})}{=} \{C_u \geq T_n\} \in \mathcal{F}_{T_n\wedge C_u} \subset \mathcal{F}_{C_u} = \mathcal{G}_k.$$

Hence, for all $t > s$ (since $C_{t\wedge\langle Y\rangle_{T_n}} = C_t \wedge C_{\langle Y\rangle_{T_n}}$, because (C_t) is increasing)

$$\begin{aligned}
E[W_{t\wedge\langle Y\rangle_{T_n}}|\mathcal{G}_s] &= E[W_{t\wedge\langle Y\rangle_{T_n}}|\mathcal{G}_s] = E[Y_{C_{t\wedge\langle Y\rangle_{T_n}}}|\mathcal{G}_s] \\
&= E[Y_{C_t\wedge C_{\langle Y\rangle_{T_n}}}|\mathcal{F}_{C_s}] = E[Y^{C_t}_{C_t\wedge C_{\langle Y\rangle_{T_n}}}|\mathcal{F}_{C_s}]
\end{aligned}$$

Since $(Y_s^{C_t})^{\wedge C_{\langle Y\rangle_{T_n}}}$ is an uniformly integrable martingale (note that C_s is not necessarily bounded), it follows that

$$= Y^{C_t}_{C_s\wedge C_{\langle Y\rangle_{T_n}}} \overset{s\leq t}{=} Y_{C_s\wedge C_{\langle Y\rangle_{T_n}}} = W_{s\wedge\langle Y\rangle_{T_n}}.$$

Hence, W is a local martingale with localising sequence $(\langle Y\rangle_{T_n})_{n\in\mathbb{N}}$ (up to ∞).
Step 3. $\langle W\rangle_t = t\ \forall t$. In particular, W is a Brownian motion:
Since by L^2-martingale convergence theorem $((Y_s^{C_t})^2)_{s\geq 0}$ is uniformly integrable and $\langle Y^{C_t}\rangle_s = \langle Y\rangle_{C_t\wedge s} \leq \langle Y\rangle_{C_t} = t \quad \forall s$, it follows that $(Y^{C_t})^2 - \langle Y^{C_t}\rangle$ is an uniformly integrable martingale. Hence, (though C_t, C_s are not bounded) by the stopping theorem we get that for all $t > s$

$$\begin{aligned}E[W_t^2 - t|\mathcal{G}_s] &= E[Y_{C_t}^2 - \langle Y\rangle_{C_t}|\mathcal{F}_{C_s}]\\ &= E[(Y_{C_t}^{C_t})^2 - \langle Y^{C_t}\rangle_{C_t}|\mathcal{F}_{C_s}] = (Y_{C_s}^{C_t})^2 - \langle Y^{C_t}\rangle_{C_s}\\ &= Y_{C_s}^2 - \langle Y\rangle_{C_s} = W_s^2 - s.\end{aligned}$$

Therefore, $(W_t^2 - t)_{t\geq 0}$ is a continuous martingale with respect to $(G)_{t\geq 0}$. By the Doob-Meyer decomposition it follows that $\langle W\rangle_t = t$ for all t and by 1.4.32 that $(W_t)_{t\geq 0}$ is a martingale. By Lévy's characterization theorem 1.5.34 W_t is a Brownian motion.
Step 4. $Y_t = W_{\langle Y\rangle_t}$ for all $t \geq 0$:
We have

$$W_{\langle Y\rangle_t} = Y_{C_{\langle Y\rangle_t}}$$

and since $s \mapsto \langle Y\rangle_s$ is increasing and continuous

$$C_{\langle Y\rangle_t} = \inf\{s > 0|\langle Y\rangle_s > \langle Y\rangle_t\} = \inf\{s > t|\langle Y\rangle_s > \langle Y\rangle_t\} = \sigma_t.$$

Therefore, $\langle Y\rangle$ is constant on $[t, C_{\langle Y\rangle_t}]$. By (A.0.2) we get that Y is constant on $[t, C_{\langle Y\rangle_t}]$, thus, $W_{\langle Y\rangle_t} = Y_{C_{\langle Y\rangle_t}} = Y_t$.

Remark A.0.4. *We have supposed that* $P[\langle Y\rangle_\infty = \infty] = 1$. *(This has been necessary as the counter example*

$$\Omega := \{\omega\},\ Y \equiv 0$$

shows.) Basically, the theorem also holds for the case, where $P[\langle Y\rangle_\infty < \infty] > 0$. *But, one possibly has to enlarge* Ω.
Construction of the enlarged Wiener space:
Let $(W_t')_{t\geq 0}$ *be a Wiener process on* $(\Omega', \mathcal{F}', P')$ *with respect to* $(\mathcal{F}_t')$. *Set*

$$\tilde{\Omega} := \Omega \times \Omega',$$

$$\tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}',$$

$$\tilde{P} := P \times P',$$

$$C_t := \begin{cases} \inf\{s|\langle Y\rangle_s > t\} & \text{on } \{\langle Y\rangle_\infty > t\},\\ \infty & \text{on } \{\langle Y\rangle_\infty \leq t\}.\end{cases}$$

Let

$$\hat{\mathcal{F}}_t := \sigma(\mathcal{F}_{C_t\wedge s}|s \geq 0),$$

$$\mathcal{G}_t := \hat{\mathcal{F}}_t \times \mathcal{F}_t',$$

$$W_t := \begin{cases} Y_{C_t} & \text{on } \{\langle Y\rangle_\infty > t\},\\ W_t' - W_{\langle Y\rangle_\infty}' + \langle Y\rangle_\infty & \text{on } \{\langle Y\rangle_\infty \leq t\}.\end{cases}$$

Then W *is a Wiener process on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *with respect to* $(\mathcal{G}_t)_{t\geq 0}$, $\langle Y\rangle_t$ *is a* $(\mathcal{G}_t)$*-stopping time and we have*

$$Y_t = W_{\langle Y\rangle_t}.$$

For the proof cf. [IW89, Chapter II, Theorem 7,21].

$\square$

Literaturverzeichnis

[Arn06] Vladimir I. Arnold, *Ordinary differential equations*, Universitext, Springer-Verlag, Berlin, 2006, Translated from the Russian by Roger Cooke, Second printing of the 1992 edition. MR MR2242407

[Bau78] Heinz Bauer, *Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie*, revised ed., Walter de Gruyter, Berlin-New York, 1978, de Gruyter Lehrbuch. MR MR514208 (80b:60001)

[BH86] Jürgen Bliedtner and Wolfhard Hansen, *Potential theory: an analytic and probabilistic approach to balayage*, Springer-Verlag, Universitext (1979), 1986.

[Cho69a] Gustave Choquet, *Lectures on analysis. Vol. I: Integration and topological vector spaces*, Edited by J. Marsden, T. Lance and S. Gelbart, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR MR0250011 (40 #3252)

[Cho69b] ______, *Lectures on analysis. Vol. II: Representation theory*, Edited by J. Marsden, T. Lance and S. Gelbart, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR MR0250012 (40 #3253)

[CW90] K. L. Chung and R. J. Williams, *Introduction to stochastic integration*, second ed., Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 1990. MR MR1102676 (92d:60057)

[dB03] G. de Barra, *Measure theory and integration*, Horwood Publishing Series. Mathematics and Its Applications, Horwood Publishing Limited, Chichester, 2003, Revised edition of the 1981 original. MR MR2052644 (2004k:28002)

[IW89] Nobuyuki Ikeda and Shinzo Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam, 1989. MR MR1011252 (90m:60069)

[KL99] Claude Kipnis and Claudio Landim, *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 320, Springer-Verlag, Berlin, 1999. MR MR1707314 (2000i:60001)

[Kry] Nicolai V. Krylov, *http://www.math.umn.edu/~krylov/*.

[KS91] I. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus*, Graduate texts in mathematics, Springer, 1991.

[LS72] R. Sh. Lipster and A. N. Shiryayev, *Statistics of conditionally Gaussian random sequences*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory (Berkeley, Calif.), Univ. California Press, 1972, pp. 389–422. MR MR0400377 (53 #4211)

[MR92] Zhi Ming Ma and Michael Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992. MR MR1214375 (94d:60119)

[Nua06] David Nualart, *The Malliavin calculus and related topics*, second ed., Probability and its Applications (New York), Springer-Verlag, Berlin, 2006. MR MR2200233 (2006j:60004)

[Oks03] Bernt Oksendal, *Stochastic differential equations*, sixth ed., Universitext, Springer-Verlag, Berlin, 2003, An introduction with applications. MR MR2001996 (2004e:60102)

[Röc04] M. Röckner, *Einführung in die lineare Funktionalanalysis (Vorlesungsskript)*, 2004.

[Röc11] ______, *Wahrscheinlichkeitstheorie (Vorlesungsskript)*, 2011.

[RS80] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis. MR MR751959 (85e:46002)

[RW87] L. C. G. Rogers and David Williams, *Diffusions, Markov processes, and martingales. Vol. 2*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1987, Itô calculus. MR MR921238 (89k:60117)

[RY99] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR MR1725357 (2000h:60050)

[Str87] Daniel W. Stroock, *Lectures on stochastic analysis: diffusion theory*, London Mathematical Society Student Texts, vol. 6, Cambridge University Press, Cambridge, 1987. MR MR884711 (88d:60203)

[SV06] Daniel W. Stroock and S. R. Srinivasa Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition. MR MR2190038 (2006f:60005)

[vWW90] Heinrich von Weizsäcker and Gerhard Winkler, *Stochastic integrals*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1990, An introduction (Url: http://www.mathematik.uni-kl.de/∼wwwstoch/literatur.html). MR MR1062600 (91i:60150)

[Wat84] S. Watanabe, *Lectures on stochastic differential equations and Malliavin calculus*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 73, Published for the Tata Institute of Fundamental Research, Bombay, 1984, Notes by M. Gopalan Nair and B. Rajeev. MR MR742628 (86b:60113)

[Wei87] Joachim Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, 1987. MR MR923320 (89b:47070)