WS 2011/12
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Exercise Sheet 1

Exercise sheet 1 on Introduction to Stochastic Analysis

Exercise 1. *Let $f \in \mathcal{C}(\mathbb{R})$ and let $X = (X_t)_{t\geq 0} : [0,\infty) \to \mathbb{R}$ be of bounded variation. For $\tau = \{t_0, \cdots, t_N\}$ with $0 = t_0 < \cdots < t_N < \infty$ and $t \geq 0$ define the Stieltjes sum*

$$\sum_{1\leq i<N} f_{t_i}\left(X_{t_{i+1}\wedge t} - X_{t_i \wedge t}\right) =: S_t(f,\tau).$$

(i) Prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tau = \{t_0, \cdots, t_N\}$, $\tau' = \{t'_0, \cdots, t'_{N'}\}$ are as above with $|\tau|, |\tau'| < \delta$ and $t_N, t'_{N'} \geq t$, then $|S_t(f,\tau) - S_t(f,\tau')| < \varepsilon$.

(ii) Conclude that for any sequence (τ_n) as above[1] with $t^{(n)}_{N_n} \to \infty$ and $|\tau^{(n)}| \to 0$ as $n \to \infty$ the limit

$$\int_0^t f_s \, dX_s := \lim_{n\to\infty} S_t(f,\tau_n)$$

exists and is independent of the sequence $(\tau_n)_n$.

(iii) Let $X_s := 5\mathbb{I}_{[\sqrt{2},\infty)}(s)$, $f \in \mathcal{C}(\mathbb{R})$. Compute $\int_0^t f(s)dX_s$.

Exercise 2. *Let $(X_t)_{t\geq 0}$ be a continuous Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$ and $(\tau_n)_n$ be an increasing sequence of subdivisions with $|\tau_n| \to 0$, $t_{N_n} \to \infty$. Use Theorem 1.1.4(i), which states that for all $t \geq 0$ it holds*

$$\sum_{t_i^{(n)}\in\tau_n, t_i^{(n)}\leq t} \left(X_{t^{(n)}_{i+1}} - X_{t^{(n)}_i}\right)^2 \to t \quad P\text{-a.s.},$$

in order to prove

$$P\Bigg(\sum_{t_i^{(n)}\in\tau_n, t_i^{(n)}\leq t} \left(X_{t^{(n)}_{i+1}} - X_{t^{(n)}_i}\right)^2 \to t, \quad \forall t \geq 0\Bigg) = 1.$$

Exercise 3. *Prove that the α-Integral defined in Definition 1.2.11 exists (under the assumptions that $f \in \mathcal{C}^1(\mathbb{R})$ and $\langle X\rangle$ exists for the particular sequence (τ_n)) and fulfills*

$$\alpha\text{-}\int_0^t f(X_s)\, dX_s = \int_0^t f(X_s) dX_s + \alpha \int_0^t f'(X_s)\, d\langle X\rangle_s.$$

[1] We always write $\tau_n = \{t^{(n)}_0, \cdots, t^{(n)}_{N_n}\}$, with $0 = t^{(n)}_0 < t^{(n)}_1 < \cdots < t^{(n)}_{N_n} < \infty$.