Exercise sheet 10 on Introduction to Stochastic Analysis

Exercise 1. *For $\alpha \in (0,1)$ let $f_\alpha : [0,1] \to \mathbb{R}$ be given by $f_\alpha(t) := t^\alpha$.*

(i) Decide for which α the function f_α is an element of the Cameron-Martin space H defined in Theorem 4.3.11.

(ii) Show directly that for $\alpha < 1/2$ the law P_{f_α} (see again Theorem 4.3.11) is not equivalent to P_0. (Find an almost sure property of the Brownian motion that is not preserved by adding f_α.)

Exercise 2. *Prove that the mapping $F : \mathcal{C}([0,1])_0 \to \mathbb{R}$ given by*

$$F(\omega) = \omega(1) \cdot \omega\Big(\frac{1}{2}\Big), \quad \omega \in \mathcal{C}([0,1])_0,$$

is Frechet differentiable and compute its Frechet derivative and its Malliavin derivative.

Exercise 3. *(i) Let $F : \mathcal{C}([0,1])_0 =: E \to \mathbb{R}$ be given by*

$$F(\omega) := \max_{0 \le s \le 1} \omega(s), \quad \omega \in E.$$

Let $\omega \in E$ be such that ω has a unique maximum point t_0. Prove that for any $\eta \in E$ it holds

$$\lim_{\lambda \to 0} \frac{F(\omega + \lambda\eta) - F(\omega)}{\lambda} = \eta(t_0).$$

This is called Gâteaux differentiability of F.

(ii) Use the result from (i) in order to prove that F as given there is not Frechet differentiable at any ω as given in (i).

Hint to (i): For showing the limit for $\lambda \downarrow 0$ prove that for each $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that for each $0 < \lambda \le \lambda_0$ it holds

$$\eta(t_0) \le \max_{t \in [0,1]} \frac{\omega(t) - F(\omega)}{\lambda} + \eta(t) \le \max_{t \in B_\varepsilon(t_0)} \eta(t).$$

For treating $\lambda \uparrow 0$ replace η by $-\eta$.

Hint to (ii): If it would be Frechet differentiable, the Frechet derivative would coincide with the Gateaux derivative $\eta \mapsto \eta(t_0)$. Try the sequence η_n given by $\eta_n(t) := -2\max\{0, \omega(t) - \omega(t_0) + 1/n\}$ to get a contradiction.