

**Exercise sheet 9 on Introduction to Stochastic Analysis**

**Exercise 1.** Let  $(X_t)_{t \geq 0}$  be a  $d$ -dimensional Brownian motion. Let  $O$  be an orthogonal transformation of  $\mathbb{R}^d$ , i.e.  $O : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is linear and such that  $(Ox, Oy) = (x, y)$  for all  $x, y \in \mathbb{R}^d$ . Prove that  $(OX_t)_{t \geq 0}$  is a Brownian motion.

**Exercise 2.** Let  $(X_t)_{t \geq 0}$  be a Brownian motion on  $\mathbb{R}^d$ . (I.e. we have the canonical setting,  $P_x$  is the law of a continuous BM starting in  $x$ .) Let  $G$  be any open bounded set and  $\sigma_{G^c}$  be the first exit time of  $G$ .

- (i) Let  $f$  be a bounded measurable function, defined on the boundary  $\partial G$  of  $G$ . Let  $x \in G$  and let  $B_r(x)$  be the open ball with radius  $r$  centered in  $x$ . Choose  $r$  small enough such that  $B_r(x) \subset G$ . Let

$$u(x) := \mathbb{E}_x[f(X_{\sigma_{G^c}})].$$

Prove that

$$u(x) = \mathbb{E}_x[u(X_{\sigma_{B_r(x)^c}})].$$

- (ii) Let  $f, u$  be as in (i),  $x \in G$  and  $r$  as in (i). Show that

$$u(x) = \frac{1}{S_r} \int_{\partial B_r(x)} u(y) O(dy),$$

where  $O$  is the surface measure on the boundary of  $B_r(x)$  and  $S_r$  is the surface of  $\partial B_r(x)$ . From this one can conclude (not a part of the exercise) that  $u \in C^\infty(G)$  and  $\Delta u = 0$ .

- (iii) Situation as above. Let  $d = 3$ ,  $G = (0, 1)^3$  and let e.g.  $x = (0, x_2, x_3)$  with  $x_2, x_3 \in (0, 1)$ . Assume moreover that  $f$  is continuous in  $x$ . Prove that for any sequence  $G \ni y_n \rightarrow x$  it holds  $u(y_n) \rightarrow f(x)$ .

Hints: (i) is an application of the strong Markov property. (ii) follows from (i) and Exercise 1. One idea for (iii) could be the following: prove at first that  $P_{y_n}(\sigma_{G^c} < t, \sup_{s \in [0, t]} |X_s - X_0| < \varepsilon) \rightarrow P_0(\sup_{s \in [0, t]} |X_s - X_0| < \varepsilon)$ , and that the latter probability tends to 1 as  $t \rightarrow 0$ .