SS 2021
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Exercises for Functional Analysis

Exercise 2
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Exercise 1.
Let (X, d) be a metric space and $\Omega \subseteq X$ an open set. Prove that there exists a sequence of sets $(K_n)_{n\in\mathbb{N}}$ with the following properties:

- *K_n is increasing,*
- *$K_n \subseteq \mathring{K}_{n+1}$,*
- *K_n is closed and bounded,*
- *$\Omega = \bigcup_{n\in\mathbb{N}} K_n$,*
- *Let $K \subseteq \Omega$ be compact, then there is a $m \in \mathbb{N}$ such that $K \subseteq K_m$.*

(4 Points)

Exercise 2.
Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Set

$$C^0(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ continuous}\}.$$

Let $(K_m)_{m\in\mathbb{N}}$ be a sequence with the properties from Exercise 1. Set:

$$\rho(f) := \sum_{m=1}^{\infty} 2^{-m} \frac{\|f\|_{B(K_m)}}{1 + \|f\|_{B(K_m)}}.$$

Prove that $d(f, g) := \rho(f - g)$ is a metric on $C^0(\Omega)$.

(2 Points)

Prove that $(C^0(\Omega), d)$ is complete.

(2 Points)

Exercise 3.
Let V be a $\mathbb{R}$-vector space. We know that every scalar product s on V induces a norm via

$$\|x\| := \sqrt{s(x, x)}.$$

Conversely: Prove that every norm $\|\cdot\|$ on V for which the Parallelogram law (Parallelogram identity)

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \qquad x, y \in V$$

holds, is induced by a scalar product.

(4 Points)

Hint: Define $s(x, y) := \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$, $x, y \in V$ and show first with the Parallelogram law that $s(x, \frac{y_1+y_2}{2}) = \frac{1}{2}(s(x, y_1) + s(x, y_2))$ holds. Deduce that $s(x, \frac{y}{2}) = \frac{1}{2}s(x, y)$ and $s(x, y_1 + y_2) = s(x, y_1) + s(x, y_2)$ holds, and via induction conclude that $s(x, m2^{-n}y) = m2^{-n}s(x, y)$ for $n, m \in \mathbb{N}$. Infer from this that s is a scalar product.

Exercise 4.
Let $M\colon [0,\infty) \to [0,\infty)$ *be a continuous and convex function with* $M(t) = 0 \Leftrightarrow t = 0$. *The set* $\mathcal{L}_M(\mathbb{R})$ *is defined as the set of all measurable functions* $f\colon \mathbb{R} \to \mathbb{R}$ *such that there exists a* $c > 0$ *with*

$$\int_{\mathbb{R}} M\left(\frac{|f(t)|}{c}\right) \,\mathrm{d}t < \infty.$$

Consider the quotient space

$$L_M(\mathbb{R}) := \mathcal{L}_M(\mathbb{R}) / \{f \in \mathcal{L}_M(\mathbb{R}) \mid f = 0 \textit{ almost everywhere}\}.$$

For $f \in L_M(\mathbb{R})$ *we define*

$$\|f\|_M := \inf\left\{ c > 0 : \int_{\mathbb{R}} M\left(\frac{|f(t)|}{c}\right) \,\mathrm{d}t \leqslant 1 \right\}.$$

Prove that this is a norm on $L_M(\mathbb{R})$ *(also show that* $\|f\|_M < \infty$ *holds).*

(2 Points)

Prove that $(L_M(\mathbb{R}), \|\cdot\|_M)$ *is a Banach space.*

(2 Points)