SS 2021
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Exercises for Functional Analysis

Exercise 6
Submission date: Friday, 28.05.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let $p \in (1, \infty)$. *Let* $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Prove that* $f * g$ *is a continuous function.* *(4 Points)*

Hint: $C_c(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *and* $L^q(\mathbb{R})$.

Exercise 2.
Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ *with* $\varphi \geqslant 0$ *and* $\int \varphi = 1$. *Set*

$$\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon)$$

the Dirac sequence of φ. *Let* $f \in C(\mathbb{R}^n)$. *Prove that* $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon = f$ *converges uniformly on compact sets, i.e.*

$$\lim_{\varepsilon \to 0} \sup_{x \in K} |(f * \varphi_\varepsilon)(x) - f(x)| = 0$$

for all $K \subseteq \mathbb{R}^n$ *compact.* *(4 Points)*

Exercise 3.
Let $\Omega \subseteq \mathbb{R}$ *be open,* $p \in [1, \infty)$, $p' \in [1, \infty)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$. *Prove that for* $f \in H^{m,p}(\Omega)$ *and* $g \in H^{m,p'}(\Omega)$

$$fg \in H^{m,1}(\Omega)$$

and

$$\partial^\alpha (fg) = \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} f \partial^\beta g$$

hold. *(4 Points)*

Exercise 4.
Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$ *be open and* $\tau \colon \tilde{\Omega} \to \Omega$ *a* C^m*-Diffeomorphism such that the Jacobian matrix* $D\tau$ *and* $D\tau^{-1}$ *are bounded. Let* $f \in H^{m,p}(\Omega)$. *Prove that* $f \circ \tau \in H^{m,p}(\tilde{\Omega})$. *(4 Points)*
Hint: The weak derivative can be calculated via the chain rule.