Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 11
Total points: 13+3*
Submission before: Friday, 23.06.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1. (4 Points)

Consider for e.g. $u \in C((0,1);\mathbb{R})$ the operator

$$Au := \Delta u - u^3,$$

where Δ is the Laplace operator given in the lecture notes. Find a suitable Gelfand triple to prove that A satisfies (H1)-(H3) in the lecture notes.

Problem 2. (3+3+3 Points)

Prove Exercise 4.2.3 in the lecture notes.

Problem 3. (3* Points)

Two essential properties of Lebesgue measure λ^d on $\mathbb{R}^d, 1 \leqslant d < \infty$, are

1. (Translation invariance) $\forall x \in \mathbb{R}^d$: $\lambda^d \circ T_x^{-1} = \lambda^d$, where $\mathbb{R}^d \ni y \mapsto T_x(y) := x + y$;
2. (Finiteness and positivity on balls) $\forall R > 0, x \in \mathbb{R}^d$: $0 < \lambda^d(B_R(x)) < \infty$.

Let $(U, \mathcal{B}(U))$ be a Hilbert space such that $\dim(U) = \infty$. Then there is no nonnegative measure on $(U, \mathcal{B}(U))$ such that analogous statements 1. and 2. hold.

Hint: Construct an infinite family of disjoint balls that are inside a larger ball. Then use 1. in order to find a contradiction to 2.