Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 12
Total points: 16
Submission before: Friday, 30.06.2023, 12:00 noon

Problem 1 (Elementary inequalities). (2+1+1 Points)

(i) Show that every concave function $f : [0, \infty) \to \mathbb{R}$ with the property $f(0) \geqslant 0$ fulfills

$$f(a + b) \leqslant f(a) + f(b) \quad \forall a, b \in [0, \infty). \tag{I}$$

(ii) Let $a, b \in [0, \infty)$. Prove that for all $p \geqslant 1$

$$(a + b)^p \leqslant 2^{p-1}(a^p + b^p).$$

(iii) Find an exemplary convex function $f : [0, \infty) \to \mathbb{R}$ which does not fulfill (I).

Problem 2 (Prove the details). (2 Points)

Explain in detail, how one can conclude (4.1.11) and (4.1.12) in the lecture notes on the basis of what has been shown before.

Problem 3. (3+3 Points)

Suppose that $V = H = \mathbb{R}^d$ for an arbitrary $d \in \mathbb{N}$. Consider the situation of Theorem 4.2.4. Suppose A, B satisfy (H1)-(H4) with $\alpha := p$, $f \in L^{\frac{p}{2}}([0, T] \times \Omega; dt \otimes P)$ and $X_0 \in L^p(\Omega, \mathcal{F}_0, P; H)$ for some $p \geqslant 2$.

(i) Write $||X(t)||_H^p$ in terms of Itô's formula.

(ii) Use (i) to prove that

$$E\left(\sup_{t\in[0,T]} ||X(t)||_H^p\right) < \infty.$$

Problem 4. (4 Points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (E, d) be a Polish space. Let $R(\Omega; E)$ consist of all $\mathcal{F}/\mathcal{B}(E)$-measurable functions from Ω into E. Let $Z_n \in R(\Omega; E), n \in \mathbb{N}$. Then $(Z_n)_{n\in\mathbb{N}}$ converges in probability to some $Z \in R(\Omega; E)$ if and only if for every pair of subsequences $(Z_{n_k^1})_{k\in\mathbb{N}}, (Z_{n_k^2})_{k\in\mathbb{N}}$ there exists a subsequence $v_l := (Z_{n_{k_l}^1}, Z_{n_{k_l}^2}), l \in \mathbb{N}$, converging in distribution to a random element $v \in R(\Omega; E) \times R(\Omega; E)$ supported on the diagonal $\{(x, y) \in E \times E : x = y\}$.