Summer term 2023
Prof. Dr. Michael Röckner and Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 2
Total points: 12
Submission before: Friday, 21.04.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Let $(U, \langle \cdot, \cdot \rangle_U)$ be a separable Hilbert space.

Problem 1. (2+2+2 Points)

1. Assume that μ is a probability measure on $(U, \mathcal{B}(U))$ such that
$$\int_U e^{i\langle u,v\rangle_U} \mu(dv) = e^{i\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U} \quad \forall u \in U. \tag{1}$$
for some $m \in U$ and $Q \in L(U)$ symmetric, nonnegative, with finite trace. Prove that μ is a Gaussian measure on $(U, \mathcal{B}(U))$.

2. Let $g \in U$. Show that there exists a symmetric $Q \in L(U)$ such that
$$\langle Qh_1, h_2\rangle_U = \int_U \langle v, h_1\rangle_U \langle v, h_2\rangle_U \mu(dv) - \langle g, h_1\rangle_U \langle g, h_2\rangle_U \quad \forall h_1, h_2 \in U.$$

3. Assume that μ is a Gaussian measure on $(U, \mathcal{B}(U))$ and let $Q \in L(U)$ and $m \in U$ be as constructed in the proof of Theorem 2.1.2. Conclude from the proof of Theorem 2.1.2 that (1) holds.

Problem 2. (2 Points)

Prove Corollar 2.1.4 in the lecture notes.

Problem 3 (Fernique's theorem for Gaussian measures on Hilbert spaces)**.** (4 Points)

Let μ be a Gaussian measure on $(U, \mathcal{B}(U))$ with $\mu = N(0, Q)$. Prove that there exists $\alpha > 0$ such that
$$\int_U e^{\alpha ||v||_U^2} \mu(dv) < \infty.$$

Hint: Use Proposition 2.1.6.