Winter term 2022/2023
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Exercises to Stochastic Analysis

Sheet 2
Total points: 14
Submission before: Friday, 28.10.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Covariation, cf. Def. 1.3.14+Lem. 1.3.15). (1.5+1+1.5+1 Points)

The covariation *(not to be confused with the* covariance*!) is fundamental for the entire so-called* Itô-calculus *we aim to develop in this lecture, see for example Itô's product rule and the d-dimensional Itô formula.*

Let $X, Y : [0,\infty] \to \mathbb{R}$ be continuous with continuous quadratic variations $t \mapsto \langle X\rangle_t, \langle Y\rangle_t$ with respect to a common partition sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,\infty]$ with mesh $|\tau_n| \xrightarrow{n\to\infty} 0$. Prove the following assertions about the covariation $\langle X, Y\rangle$ as defined in Def. 1.3.14:

(i) $t \mapsto \langle X, Y\rangle_t$ exists and is continuous if and only if $t \mapsto \langle X+Y\rangle_t$ (the quadratic variation of $X+Y$ with respect to $(\tau_n)_{n\in\mathbb{N}}$) exists and is continuous. In this case, for all $t \geqslant 0$ the polarization identity

$$\langle X, Y\rangle_t = \frac{1}{2}\big(\langle X+Y\rangle_t - (\langle X\rangle_t + \langle Y\rangle_t)\big)$$

holds.

Now assume (i) holds and prove:

(ii) $t \mapsto \langle X, Y\rangle_t$ is of bounded variation.

(iii) $t \mapsto \langle X, Y\rangle_t$ is the distribution function of a signed measure on any $[0,T]$, $T > 0$.

(iv) The Cauchy-Schwarz-type inequality

$$|\langle X, Y\rangle_t| \leqslant \langle X\rangle_t^{\frac{1}{2}} \langle Y\rangle_t^{\frac{1}{2}}$$

holds for all $t \geqslant 0$. Conclude that if X or Y are BV, then $\langle X, Y\rangle_t = 0$ for all $t \geqslant 0$.

Hint: By now you know enough about BV functions to avoid calculations in (ii).

Problem 2 (Generalization of d-dim. Itô formula, cf. Prop. 1.3.21). (4 Points)

Show the following generalization of the d-dimensional Itô-formula: Let $m, n \geqslant 1$, $X = (X^i)_{1\leqslant i\leqslant m} : [0,\infty) \to \mathbb{R}^m$ and $Y = (Y^j)_{1\leqslant j\leqslant n} : [0,\infty) \to \mathbb{R}^n$ be continuous with continuous $\langle X^k, X^l\rangle, 1 \leqslant k, l \leqslant m$, and $Y^k, 1 \leqslant k \leqslant n$, of bounded variation, along a common sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ as in the beginning of Section 1.2. Then for $F \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^n)$

$$F(X_t, Y_t) - F(X_0, Y_0) = \int_0^t (\nabla_x F(X_s, Y_s), dX_s) + \frac{1}{2}\sum_{k,l=1}^m \int_0^t \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s, Y_s)\, d\langle X^k, X^l\rangle_s + \sum_{k=1}^n \int_0^t \frac{\partial F}{\partial y_k}(X_s, Y_s)\, dY_s^k,$$

where $\nabla_x F(x,y) = (\frac{\partial F}{\partial x_i}(x,y))_{1 \leqslant i \leqslant m}$, and

$$\int_0^t (\nabla_x F(X_s, Y_s), dX_s) := \lim_{n\to\infty} \sum_{t_i \in \tau_n, t_i \leqslant t} (\nabla_x F(X_{t_i}, Y_{t_i}), X_{t_{i+1} - X_{t_i}})$$

Problem 3 (Time-dependent Itô-formula: Geometric Bm, cf. Prop. 1.3.23+Exa.1.3.25). (3 Points)

One primary goal of this lecture is to solve stochastic differential equations (SDEs), *which are formally written as* $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$, *where* $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *and* B *is a Brownian motion. The solution* X *is a stochastic process. Before solving this equation, one first needs to make sense of all appearing terms, in particular of* $\sigma(t, X_t)dB_t \approx$ " $\int_t^{t+h} \sigma(s, X_s)dB_s$ " *for small* $h \ll 1$. *As we have seen, so far we can only give meaning to such integrals for very special integrands. One of the most fundamental SDEs is given by* $b(t,x) = \beta x$ *and* $\sigma(t,x) = \alpha x$, *where* $\alpha, \beta \in \mathbb{R}$. *Its solution is called* Geometric Brownian motion *and will be studied later on. However, already now we can solve this important SDE "pathwise", as the following exercise shows.*

Let $B = (B_t)_{t \geqslant 0}$ be a continuous Brownian motion with $B_0 = 0$ and G_0 a $\mathbb{R}$-valued random variable, defined on the same probability space as B. Use the time-dependent Itô-formula to prove that $G(\omega, t) := G_0 \exp\big(\alpha B(\omega)_t + (\beta - \frac{1}{2}\alpha^2)t\big)$ solves the stochastic differential equation

$$\begin{cases} dG(t) &= \beta G(t)dt + \alpha G(t) dB_t, \\ G(0) &= G_0 \end{cases}$$

in the sense that for almost every $\omega \in \Omega$ we have

$$G(\omega, t) - G_0(\omega) = \alpha \int_0^t G(\omega, s) dB_s(\omega) + \beta \int_0^t G(\omega, s) ds, \quad \forall t \geqslant 0.$$

Problem 4 (Continuous local martingales). (2 Points)

In Section 1.4. we will study continuous local martingales, which arise as admissible integrators for stochastic integrals. The following definition can also be found at the beginning of Section 1.4., but no content from Section 1.4. is needed for this exercise.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space such that each $\mathcal{F}_t$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets, and let $T : \Omega \to [0, \infty]$. A *continuous local martingale up to* T is a map $X : \{(t, \omega) \in]0, \infty) \times \Omega : t < T(\omega)\} \cup (\{0\} \times \Omega) \to \mathbb{R}$ such that there is a sequence $T_1 \leqslant T_2 \leqslant \ldots \leqslant T$, $T_n : \Omega \to [0, \infty]$, such that $T_n < T$ on $\{T > 0\}$, with the following properties:

(i) T_n is an $(\mathcal{F}_t)$-stopping time for all $n \in \mathbb{N}$

(ii) $\sup_{n \in \mathbb{N}} T_n = T$ $\mathbb{P}$-a.s.

(iii) $(X_{t \wedge T_n})_{t \geqslant 0}$ is a martingale.

Prove that every continuous (global) $(\mathcal{F}_t)$-martingale is a continuous local martingale.