Winter term 2022/2023
Prof. Dr. Michael Röckner and Dr. Marco Rehmeier

Exercises to Stochastic Analysis

Sheet 5
Total points: 16
Submission before: Friday, 18.11.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Proof of Proposition 1.5.34). (3 Points)

In the situation of the proof of Proposition 1.5.34, prove that (1.5.16) implies $X_t - X_s$ is $\mathcal{N}(0, t-s)$-distributed and that $X_t - X_s$ is independent of $\mathcal{F}_s$. How exactly does this imply independence of

$$X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}$$

for all $0 \leqslant t_1 < \cdots < t_n < \infty$ as claimed at the end of the proof?

Problem 2 (Covariation reflects independence of Brownian motions). (5+3 Points)

We already know that for two independent Brownian motions X^1, X^2, their covariation is zero, i.e. $\langle X^1, X^2 \rangle_t = 0$ for all $t \geqslant 0$ a.s. for any sequence of partitions with the usual conditions. In this exercise, we consider the converse implication.

(a) Let X, Y be continuous standard Brownian motions on a common filtered probability space with a right-continuous filtration $(\mathcal{F}_t)_{t \geqslant 0}$ such that $\mathcal{F}_0$ contains all zero sets, and assume $X_t - X_s$ and $Y_t - Y_s$ are independent of $\mathcal{F}_s$ for all $t \geqslant s \geqslant 0$ (by the proof of Lévy's characterization, this is true if and only if X and Y are $(\mathcal{F}_t)$-martingales). Prove: If $\langle X, Y \rangle_t = 0$ for all $t \geqslant 0$ a.s., then X and Y are independent.

(b) Let $d \geqslant 1$. A stochastic process $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a *d-dim. standard Brownian motion*, if $X = (X^1, \dots, X^d)$ and $\{X^i, i \leqslant d\}$ is an independent family of one-dim. Brownian motions.

Use part (a) to show under the same assumptions on the filtration as above: If $X^1, \dots, X^d$ are Brownian motions on a common probability space such that $X^i_t - X^i_s$ is independent of $\mathcal{F}_s$ for all $t \geqslant s \geqslant 0$, $i \leqslant d$ and

$$\langle X^i, X^j \rangle_t = 0 \quad \forall t \geqslant 0, \quad i \neq j,$$

then $X = (X^1, \dots, X^d)$ is a d-dim. standard Brownian motion.

Plan for the exercise: Independence of stochastic processes means independence of $\sigma(X)$ and $\sigma(Y)$ with $X, Y : \Omega \to C(\mathbb{R}_+, \mathbb{R})$, the latter space equipped with the σ-algebra generated by the canonical projections. First show that the collection of sets $\{X_{t_n} - X_{t_{n-1}} \in A_n, \dots, X_{t_1} - X_{t_0} \in A_1\}$ with $0 \leqslant t_i$ not necessarily increasing, $n \in \mathbb{N}, A_i \in \mathcal{B}(\mathbb{R})$ is a $\cap$-stable generator of $\sigma(X)$, and argue why it suffices to prove independence of the families $\{X_{t_n} - X_{t_{n-1}}, \dots, X_{t_1} - X_{t_0}, Y_{t_n} - Y_{t_{n-1}}, \dots, Y_{t_1} - Y_{t_0}\}$ for ***increasing*** *$0 \leqslant t_1 < \cdots < t_n$. To prove the independence of each such family, prove it is a Gaussian family and all covariances within the family are 0 (cf. Ch.9 of last semester's lecture).*

Concerning the covariances, you should first prove that each linear combination $\alpha X+\beta Y$, $\alpha, \beta \in \mathbb{R}$, is (up to a constant factor $c = c(\alpha, \beta)$) a Brownian motion (Lévy's characterization might be helpful), and then argue by a straightforward calculation. For the Gaussianity, you should use again that $\alpha X + \beta Y$ is up to a constant factor a Brownian motion and that for this Brownian motion, the proof of Lévy's characterization yields that its increments are independent from the respective past σ-algebras. Then you can use the characterization of Gaussianity in Prop.9.1.3. of last semester's lecture and conclude by a suitable iterative argument.
For part (b) you can iterate the argument from (a) in a suitable sense to obtain that each family

$$\{X^1_{t_n} - X^1_{t_{n-1}}, \dots, X^1_{t_1} - X^1_{t_0}, X^2_{t_n} - X^2_{t_{n-1}}, \dots, X^2_{t_1} - X^2_{t_0}, \dots, X^d_{t_n} - X^d_{t_{n-1}}, \dots, X^d_{t_1} - X^d_{t_0}\}$$

is Gaussian and all appearing covariances are 0.
If you need further assistance, you may contact me (Marco Rehmeier) for further thoughts until Wednesday morning.

Problem 3 (L^p-stability of martingales). (2 Points)

A particularly helpful property of the class of martingales (wrt. a common filtration on a common probability space) is that it has good stability properties, *i.e. the martingale property is preserved by taking limits in suitable senses. One such instance is proved in this exercise.*

Prove Proposition 2.1.4. (i).

Problem 4 ("Taking squares is additive for martingale increments"). (3 Points)

Prove the identity

$$\mathbb{E}\left[\left(V_t^{(m,n)} - V_t^{(n)}\right)^2\right] = 4\mathbb{E}\left[\sum_{u\in\tau_n}\sum_{\substack{u\leqslant s<u'\\ s\in\tau_n\cup\tau_m}} (M_s - M_u)^2(M_{s'\wedge t} - M_{s\wedge t})^2\right]$$

within the proof of the "Claim" within Case 1 of the proof of Proposition 2.2.8 (i).