Exercises to Stochastic Analysis

Sheet 9
Total points: 18
Submission before: Friday, 16.12.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Local martingales need not be martingales). (1+2+2 Points)

We have dealt with local martingales throughout large parts of the lecture. One is led to ask: Is every local martingale actually a martingale? The answer is negative, as the following exercise shows. The process Y is also known as the inverse 3D Bessel process.

Let $B^i, i \in \{1,2,3\}$, be independent continuous standard Brownian motions, let $x \in \mathbb{R}^3 \backslash \{0\}$ and set

$$B^x = (B^x_t)_{t \geqslant 0},\ B^x_t := (B^1_t, B^2_t, B^3_t) + x,$$

i.e. B^x is a three-dim. Brownian motion shifted by x. Let $Y = (Y_t)_{t\geqslant 0}$ be defined by

$$Y_t = 0, \text{ if } B^x_t = 0 \text{ and } Y_t = |B^x_t|^{-1} \text{ else},$$

where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^3$.

(a) For $a \geqslant 0$, let $T^x_a := \inf\{t \geqslant 0 : |B^x_t| = a\}$. Use the identity $\mathbb{P}(T^x_a < T^x_b) = \frac{|x|^{-1}-b^{-1}}{a^{-1}-b^{-1}}$ for $0 < a < x < b$ to prove that $\mathbb{P}$-a.e. path of B^x never hits $0 \in \mathbb{R}^3$.

(b) Use (a) to show that Y is a continuous local martingale up to ∞.

(c) Prove that Y is not a martingale.

Hint for (c): It suffices to prove that $t \mapsto \mathbb{E}[Y_t]$ is not constant. $\mathbb{E}[Y_t]$ can be calculated by noting $Y_t = g(B^x_t)$, $g(x) = |x|^{-1}$, by the elementary change-of-variables rule for image measures and making use of spherical coordinates.

Problem 2 (The Brownian filtration, cf. beginning of Chapter 2.5). (1+2+2+2 Points)

The canonical filtration of a Brownian motion is often used in various fields of stochastic analysis. In this exercise, we investigate several basic important properties of this filtration and its right-continuous and augmented version.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $B = (B_t)_{t\geqslant 0}$ a Brownian motion,

$$\mathcal{F}^B_t := \sigma(B_s : 0 \leqslant s \leqslant t), \quad t \geqslant 0,$$

the natural Brownian filtration, $(\mathcal{F}^B_{t+})_{t\geqslant 0}$ its right-continuous version and, for

$$\mathcal{N} := \{N \in \mathcal{F} : \mathbb{P}(N) = 0\},$$

let $\mathcal{F}_t$ be the ($\mathbb{P}-$)augmented σ-algebra of $\mathcal{F}^B_{t+}$, i.e.

$$\mathcal{F}_t := \sigma(\mathcal{F}^B_{t+}, \mathcal{N}).$$

Prove:

(a) $(\mathcal{F}_t^B)_{t\geqslant 0}$ is not right-continuous.

(b) $\mathcal{F}_t = \cap_{\varepsilon>0}\sigma(\mathcal{F}_{t+\varepsilon}^B, \mathcal{N})$, i.e. the operations "taking right-continuous version" and "augment by zero-sets" commute. Conclude that $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous.

(c) B is an $(\mathcal{F}_t)$-Brownian motion, i.e. B is $(\mathcal{F}_t)$-adapted and $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \leqslant s \leqslant t$.

(d) If $M = (M_t)_{t\geqslant 0}$ is a right-continuous martingale wrt. $(\mathcal{F}_t^B)_{t\geqslant 0}$, it is also a martingale wrt. $(\mathcal{F}_t)_{t\geqslant 0}$. Is this specific to the particular filtration $(\mathcal{F}_t)_{t\geqslant 0}$ or is it true in general that if $(\mathcal{A}_t)_{t\geqslant 0}$ is a filtration and M is a right-continuous $(\mathcal{A}_t)$-martingale that M is also a martingale wrt. the right-continuous and augmented filtration of $(\mathcal{A}_t)_{t\geqslant 0}$?

Hint for (b): It may help to first prove the following: If $\mathcal{A} \subseteq \mathcal{F}$ is a sub-σ-algebra, then $\sigma(\mathcal{A}, \mathcal{N}) = \{F \in \mathcal{F} : \exists A \in \mathcal{A}, N \in \mathcal{N} : A\Delta F = N\}$, where $M\Delta N = (M\backslash N) \cup (N\backslash M)$ denotes the symmetric set difference.

Hint for (c): The independence of $B_t - B_s$ from $\sigma(B_u, u \leqslant s)$ can be obtained from the independent increments of B. Then, first show that $B_t - B_s$ is also independent of $\sigma(B_u, u \leqslant s, \mathcal{N})$. Afterwards, use the set of closed subsets $C \subseteq \mathbb{R}$ as a $\cap$-stable generator of $\mathcal{B}(\mathbb{R})$, use

$$\mathbb{1}_C \circ (B_t - B_s) = \mathbb{1}_{\{\mathrm{dist}(B_t - B_s, C) = 0\}} = \lim_{n\to\infty} \big((1 - n \cdot \mathrm{dist}(B_t - B_s, C)) \vee 0\big)$$

and the path-continuity of B.

Hint for (d): Backward martingale convergence theorem.

Problem 3 (Itô representation theorem and representation for Wiener functionals, cf. Thm. 2.5.42 and Cor.2.5.44)**.** ((1)+(1+1+1+2) Points)

In case you have not advanced to this point in the lecture, please familiarize yourself with Cor.2.5.44 for the second part of this exercise. You need not study the proof by yourself.

Let $B = (B_t)_{t\geqslant 0}$ be a continuous standard Brownian motion.

(i) Consider the martingale $M_t := B_t^2 - t, t \geqslant 0$. Show that M satisfies the assumptions of Itô's representation theorem and find the representing process H for M.

(ii) Let $t > 0$. Find a representing process H as in Cor.2.5.44 for the following "Wiener functionals":

  (a) B_t^2,

  (b) $B_{\sqrt{t}}$,

  (c) B_t^3,

  (d) $\sin(B_t)$.

  Hint for (d): Using Itô's formula, find a deterministic function f such that $t \mapsto f(t)\sin(B_t)$ is a continuous local martingale.